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# Approximate solutions for an advection-diffusion problem, via a new modified Galerkin method

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The notion of approximate inertial manifold (AIM) has shown it's usefulness in the construction of approximate solutions for a class of parabolic PDEs generating dissipative dynamical systems. Thus, by the use of AIMs, the so-called non-linear Galerkin methods and post processed non-linear Galerkin methods were conceived, improving the well-known Galerkin method. These new methods proved to yield much lower errors at the same dimension of the projection space than the classical Galerkin method. In a previous paper we presented a new modified Galerkin method, related to the non-linear and post-processed Galerkin methods, but different of these. Our method leads to accurate approximate solutions using very low dimensional projection spaces. In the present paper we use this method for an advection-diffusion problem.

## 1. Introduction

In [12] we presented a new modified Galerkin method for the construction of approximate solutions for the two-dimensional Navier-Stokes equations. The method is related to the nonlinear and to the post-processed Galerkin methods but makes use of the so-called "induced trajectories" instead of the approximate inertial manifolds as do the above cited methods. In order to place our method in the context of the modified Galerkin methods, we remind here the main ideas on which the nonlinear and the post-processed Galerkin methods rely.

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### 1.1. Approximate inertial manifolds for evolution problems

We consider a class of nonlinear evolution equations of parabolic type, that can be written as abstract problems in a Hilbert space, as follows

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} + Au + R(u) &= f, && (1) \\ u(0) &= u_0, && (2) \end{aligned}$$

where $u$ is a function of time with values in a Hilbert space$\mathcal{H}$ (whose definition comprises the boundary value conditions imposed to equation (1)). We assume that the problem is dissipative, in the sense that there is a sphere in $\mathcal{H}$ that contains every trajectory from a certain moment of time on.

The operator $A$ is a linear operator, defined on a subspace$D(A)$ of $\mathcal{H}$, self-adjoint, positive definite, with compact inverse, while $R : D(R) \subset D(A) \to \mathcal{H}$ is a nonlinear operator. If $R$ contains differential operators, they must be of lower order than $A$. We assume that $f \in \mathcal{H}$ and $R$ is such that the Cauchy problem (1)–(2) has an unique solution on $[0, T]$.

An approximate inertial manifold (a.i.m.) is a finite dimensional, at least Lipschitz manifold in the space $\mathcal{H}$ (the phase space of the considered problem), with the property that all the trajectories of the dynamical system enter a narrow neighborhood of the manifold at a certain moment and never leave the neighborhood after. Even if it has not the invariance property, an a.i.m. is important because, if the problem has a global attractor, this is contained in the narrow neighborhood mentioned in the definition.

The notion of a.i.m. appeared in the context of the theory of inertial manifolds, as approximations of these, or as substitutes, when the existence of an inertial manifold could not be proved. From among the papers devoted to a.i.m.s we cite: [4], [14], [16], [17], [18], [22], [24], [25], [26], [27].

A.i.m.s found very interesting applications in the construction of approximate solutions (the numerical integration) of the nonlinear evolution problems.

### 1.2. Methods of numerical integration for the nonlinear parabolic equations, based on a.i.m.s

In the hypotheses assumed on the operator $A$, it follows that it has positive eigenvalues:

$$0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ...\lambda_n \to \infty$$

The eigenfunctions of $A$ form a total (orthonormal) system for $\mathcal{H}$. The first $m$ eigenvalues, the corresponding eigenfunctions and the subspace spanned by these are considered. We denote by $P$ the orthogonal projection operator on this subspace and we set $Q = I - P$ (where $I$ is the identity application on $\mathcal{H}$). The equation (1) is projected on the subspaces $PH$ and $QH$. By denoting $p = Pu$, $q = Qu$, we obtain

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}t} + Ap + PR(p+q) &= Pf,\\ \frac{\mathrm{d}q}{\mathrm{d}t} + Aq + QR(p+q) &= Qf.\end{aligned}$$

If in the first equation $q$ is neglected in the presence of $p$, the Galekin approximation of (1),

$$\frac{\mathrm{d}p}{\mathrm{d}t} + Ap + PR(p) = Pf, \tag{3}$$

is obtained. The solution of the problem (3) with the initial condition

$$p(0) = Pu_0 \tag{4}$$

is the Galerkin approximation of the solution of (1)–(2). In order to estimate the error of this approximate solution, the number

$$\delta = \lambda_1/\lambda_{m+1} \tag{5}$$

is used. For many problems of interest it is proved that the $\mathcal{H}$ norm of the difference between $u(t)$ and it's Galerkin approximation is of the order of $\delta^a$, with $a > 0$ depending on the considered problem. For the two-dimensional Navier-Stokes [7] equations $a = 1$ (in the hypothesis $f \in \mathcal{H}$) . The problem (3)–(4) is equivalent to a system of ordinary differential equations for the coordinates of $p(t)$ along the eigenfunctions that span $P\mathcal{H}$. The definition of $\delta$ shows that the greater will be $m$, (hence the dimension of $PH$), the smaller will be the error.

In the construction of the Galerkin equation, the $q$ component of the solution (that is proved to be small for large times) is approximated with 0. The nonlinear (and/or post-processed) Galerkin methods of approximation rely on the idea that $q(t)$ is better approximated by using some a.i.m.s.

## Families of a.i.m.s used in the generation of the nonlinear Galerkin methods and of the postprocessed Galerkin methods

Among the various types of a.i.m.s, those defined in [4], [26], [27] for the two-dimensional Navier-Stokes equation, generated new numerical integration methods, based on the Galerkin method. They form a family of manifolds from among the first, $\mathcal{M}_0$, is defined in [4], the following two, $\mathcal{M}_1$, $\mathcal{M}_2$, are defined in [26], and the following, $\mathcal{M}_n$, of a superior order, are defined in [27]. They are constructed as the graphs of some applications $\Phi_n$ defined on $PH$ and having values in $QH$.

Similar a.i.m.s may be (and were) defined for many other problems of the form (1)–(2). The main property of these a.i.m.s, on which their use in the construction of the numerical methods is based, is the following: the distance (in the norm of $\mathcal{H}$) between $q(t)$ and the images of $p(t)$ on the a.i.m. $\mathcal{M}_n$ is of the order of $\delta^{a(n)}$ where $a(n)$ is increasing with $n$. That is

$$|q(t) - \Phi_n(p(t))| < C\delta^{a(n)}. \tag{6}$$

As an example, for the two-dimensional Navier-Stokes problem it is proved [26], [27] that $a(n) = (n+3)/2$. Since about the norm of $q(t)$ only the fact of being of the order of $\delta$ is known, it is clear that any of the above a.i.m.s provides a better approximation of $q(t)$ than the so-called plane manifold $q = 0$.

### The nonlinear Galerkin methods

Starting from the ideas on a.i.m.s presented above, in [19], [2] the nonlinear Galerkin method (NL Galerkin method) is defined. In this method, in the $P$ projection of the equation, in the argument of the nonlinear term, $q$ is approximated by some $\Phi_n(p(t)), n \geq 0$. Thus the equation:

$$\frac{\mathrm{d}p}{\mathrm{d}t} + Ap + PR(p + \Phi_n(p)) = Pf, \tag{7}$$

is considered, with the initial condition (4). Let $p_n(.)$ be the solution of this problem. The approximate solution is defined as

$$u_n(t) = p_n(t) + \Phi_n(p_n(t)).$$

More precisely, in [19] the case $n = 0$ is considered, and in [2] the case $n > 1$ is considered. For all the problems considered in the context of nonlinear Galerkin problems, it is proved that the error is of the order of $\delta^{b(n)}$, where $b(n)$ is increasing with $n$ [19], [21].

E.g. for the Navier-Stokes equations it is proved in [3] that $b(n) = (n+3)/2$.

### The Post-processed Galerkin methods

In [7] another method for the approximation of the solution is proposed, based also upon the Galerkin method and making use of a.i.m.s. Let $p_0(.)$ be the solution of (3). Then, only at the right end side of the time interval $[0, T]$, that is in $T$, the value of $\Phi_1(p_0(t))$ is computed, and the approximate solution in $T$ is defined as

$$u_1(T) = p_0(T) + \Phi_1(p_0(T)).$$

This method is named post-processed Galerkin method (PP Galerkin method) because the solution of the Galerkin problem is corrected (processed) after finishing the numerical integration of the Galerkin problem, by using the first a.i.m. of the sequence cited in 1.2.1, hence post-processed. The error of this approximate solution (i.e. the $\mathcal{H}$ norm of the difference between the approximate and the exact solution) is less than that of the Galerkin method. Thus, for the two-dimensional Navier-Stokes equations, it is proved in [7] to be of the order of $\delta^{5/4}$. Another estimate is given in [8], where the error of the PP Galerkin method is found to be less than $CL_n^2\delta^{3/2}$, where $L_n = 1 + log\frac{\lambda_n}{\lambda_1}$. The appearance of the factor $L_n^2$, that increases with $n$, makes this latter estimate to be not necessarily better than the former.

The next idea appeared in the literature [21] was to post-process the NL Galerkin method of the preceding section. More precisely, the equation (7) is considered, it is integrated on all the time interval $[0, T]$ , then $\Phi_{n+1}(p_n(T))$, is computed, and the approximate solution in $T$ is defined as

$$u_n(T) = p_n(T) + \Phi_{n+1}(p_n(T)).$$

This method is called the nonlinear post-processed Galerkin method (NL-PP Galerkin method).

### 1.3. A method proposed by us

In [12] we proposed a method for the construction of approximate solutions for the two-dimensional Navier-Stokes equations, method that has its roots both in the NL and the PP Galerkin methods, but is different of both these methods. It is a method structured on several levels, each level consisting of a modified Galerkin problem for the approximation of $\mathbf{p}$, followed by a computation of an approximation of $\mathbf{q}$. Trying to find a short name for this method we found the name "repeatedly - adjusted and post-processed" (R-APP) Galerkin method as justified.

In computing the approximations of $\mathbf{q}$, the method makes use, instead of the a.i.m.s., of some functions connected to the a.i.m.s, functions that are approximations of the "induced trajectories" defined in [26]. More precisely, in [26] the construction of the family of a.i.m.s is based upon that of the family of induced trajectories. Hence we use a more basic notion than that of a.i.m. and this brings some simplifications to the calculus.

In this paper we use our method for a problem modelling the advection-diffusion of a substance in a Newtonian fluid in the framework of the Fickean law of diffusion, with periodic boundary conditions.

## 2. The problem, the functional framework

The Fick-ean diffusion of a substance in a Newtonian fluid is modelled by the equations

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p &= \mathbf{f}(t), && (8)\\ \operatorname{div}\mathbf{u} &= 0, && (9)\\ \frac{\partial c}{\partial t} - D\Delta c + \mathbf{u}\cdot\nabla c &= h(t), && (10)\\ \mathbf{u}(0) &= \mathbf{u}_0, && (11)\\ c(0) &= c_0, && (12) \end{aligned}$$

where $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ is the velocity of the fluid, $\mathbf{x} \in \Omega$, $\mathbf{u}(., \mathbf{x}) : [0, \infty) \to \mathbb{R}^2$, $c = c(t, \mathbf{x})$ is the concentration of the diffused substance, $c(., \mathbf{x}) : [0, \infty) \to \mathbb{R}$, $\nu > 0$ is the kinematic viscosity, $D > 0$ is the constant diffusion coefficient.

We consider $\Omega = (0,l) \times (0,l)$ and we impose periodic boundary conditions. We assume that $\mathbf{f}(.)$ is an analytic function of time with values in $\left[L^2_{per}(\Omega)\right]^2$, and $h(.)$ is an analytic function of time with values in $L^2_{per}(\Omega)$.

Regarding the dependence on $t$ of the functions $\mathbf{f}(.)$ and $h(.)$, the most realistic hypothesis is that of periodicity in time. Anyway, we assume that these functions are bounded: there is a number $M_{\mathbf{f}} > 0$ such that $|\mathbf{f}(t)| \leq M_{\mathbf{f}}$, and there is a number $M_h > 0$ such that $|h(t)| \leq M_h$ for every $t > 0$.

**An example of concrete problem** modeled by the equations above is that of a lake or sea with a population periodically distributed that generates a certain substance that is harmful (e.g. by the decomposition of the dead bodies). There are some other populations (or some processes) that contribute to the consumption of this substance. Both the generation and the consumption of the harmful substance are modeled by the function $h$. We suppose that the water is subjected to some periodic in time forces (as those generated by winds or tide). These contribute to the diffusion of the substance. The problem is whether the concentration of the noxious substance becomes (during this process) greater than a certain danger limit.

As is usual in the study of the Navier-Stokes equations with periodic boundary conditions, we assume that [28], [23]

$$\overline{\mathbf{f}} = \frac{1}{l^2} \int_\Omega \mathbf{f}(\mathbf{x})\,\mathrm{d}x = 0, \tag{13}$$

and that the pressure is a periodic function on $\Omega$. For simplicity we will assume also that the average of the velocity over the periodicity cell is zero.

The velocity $\mathbf{u}$ is looked for in the space $\mathcal{H}_1 = \Big\{ \mathbf{v} \in \left[L^2_{per}(\Omega)\right]^2;\ \operatorname{div}\mathbf{v} = \mathbf{0}, \overline{\mathbf{v}} = 0 \}$. The scalar product in $\mathcal{H}_1$ is $(\mathbf{u},\mathbf{v}) = \int_\Omega (u_1 v_1 + u_2 v_2)\,dx$, (where $\mathbf{u} = (u_1,u_2)$, $\mathbf{v} = (v_1,v_2)$). The induced norm is denoted by $|\ \ |$.

We consider the average of the concentration on the periodicity cell,

$$\overline{c}(t) = \frac{1}{l^2}\int_\Omega c(t,\mathbf{x})\,\mathrm{d}x.$$

By taking the average of the equation for $c$ and by using the assumption of periodicity, we find

$$\frac{\mathrm{d}\overline{c}}{\mathrm{d}t} = \overline{h},$$

where $\overline{h}$ is the average of $h$ over $\Omega$. By denoting $\widetilde{c}(t,\mathbf{x}) = c(t,\mathbf{x}) - \overline{c}(t,\mathbf{x})$, the equation

$$\frac{\partial \widetilde{c}}{\partial t} - D\Delta\widetilde{c} + \mathbf{u}\cdot\nabla\widetilde{c} = \widetilde{h}$$

follows for $\widetilde{c}$, where $\widetilde{h} = h - \overline{h}$. The function $\widetilde{c}$ is looked for in the space $\mathcal{H}_2 = \left\{c \in L^2_{per}(\Omega),\ \overline{c} = 0\right\}$, endowed with the scalar product on $L^2(\Omega)$, denoted also by $(\ ,\ )$.

We also need the spaces $\mathcal{V}_1 = \left\{\mathbf{u} \in \left[H^1_{per}(\Omega)\right]^2,\ div\,\mathbf{u} = \mathbf{0}, \overline{\mathbf{u}} = 0\right\}$, with the scalar product $((\mathbf{u},\mathbf{v})) = \sum_{i,j=1}^2 \left(\frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j}\right)$ and $\mathcal{V}_2 = \left\{c \in H^1_{per}(\Omega),\ \overline{c} = 0\right\}$, with the scalar product given by $(c_1, c_2) = (\nabla c_1, \nabla c_2)$. The norms in both these spaces are denoted by $\| \, \|$.

The classical variational formulation of the Navier-Stokes equations [28] leads to the abstract equation

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = \mathbf{f}\,(t) \quad \text{in } \mathcal{V}_1{}', \tag{14}$$

$$\mathbf{u}\,(0) = \mathbf{u}_0, \;\; \mathbf{u}_0 \in \mathcal{H}_1, \tag{15}$$

while the abstract equation for $\widetilde{c}$ is

$$\frac{\mathrm{d}\widetilde{c}}{\mathrm{d}t} - D\Delta\widetilde{c} + \mathbf{u}\cdot\nabla\widetilde{c} = \widetilde{h}\,(t)\,, \tag{16}$$

$$\widetilde{c}\,(0) = \widetilde{c}_0, \;\; \widetilde{c}_0 \in \mathcal{H}_2. \tag{17}$$

We take $\mathbf{A} = -\boldsymbol{\Delta}$, $A = -\Delta$, and observe that $\mathbf{A}$ is defined on $D(\mathbf{A}) = \mathcal{H}_1 \cap \left[H^2\,(\Omega)\right]^2$, while $A$ is defined on $D(A) = \mathcal{H}_2 \cap H^2\,(\Omega)$.

We shall use the notations $\mathbf{B}(\mathbf{u},\mathbf{v}) = (\mathbf{u}\cdot\nabla)\mathbf{v}$, $\mathbf{b}\,(\mathbf{u},\mathbf{v},\mathbf{w}) = (\mathbf{B}(\mathbf{u},\mathbf{v}),\mathbf{w})$, $B(\mathbf{u},c) = \mathbf{u}\nabla c$, $b(\mathbf{u},c,c_1) = (B(\mathbf{u},c),c_1)$. The inequalities

$$|\mathbf{B}\,(\mathbf{u},\mathbf{v})| \leq c_1\,|\mathbf{u}|^{\frac{1}{2}}\,|\boldsymbol{\Delta}\mathbf{u}|^{\frac{1}{2}}\,\|\mathbf{v}\|\,, \qquad (\forall) \;\; \mathbf{u} \in D(\mathbf{A}), \;\; \mathbf{v} \in \mathcal{V}, \tag{18}$$

$$|\mathbf{B}\,(\mathbf{u},\mathbf{v})| \leq c_2\,\|\mathbf{u}\|\,\|\mathbf{v}\|\left[1 + \ln\left(\frac{|\boldsymbol{\Delta}\mathbf{u}|^2}{\lambda_1\,\|\mathbf{u}\|^2}\right)\right]^{\frac{1}{2}}, \; (\forall) \;\; \mathbf{u} \in D(\mathbf{A}), \;\; \mathbf{v} \in \mathcal{V}, \tag{19}$$

hold [10], [28], [26], as well as the following [23]

$$|\mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{w})| \leq c_3\,|\mathbf{u}|^{\frac{1}{2}}\,\|\mathbf{u}\|^{\frac{1}{2}}\,\|\mathbf{v}\|\,|\mathbf{w}|^{\frac{1}{2}}\,\|\mathbf{w}\|^{\frac{1}{2}}\,, \; (\forall)\,\mathbf{u},\mathbf{v},\mathbf{w} \in \mathcal{V}, \tag{20}$$

$$|\mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{w})| \leq c_4\,|\mathbf{u}|^{\frac{1}{2}}\,\|\mathbf{u}\|^{\frac{1}{2}}\,\|\mathbf{v}\|^{\frac{1}{2}}\,|\Delta\mathbf{v}|^{\frac{1}{2}}\,|\mathbf{w}|\,, \; (\forall)\,\mathbf{u} \in \mathcal{V}, \;\; \mathbf{v} \in \mathbf{D}\,(\mathbf{A})\,, \mathbf{w} \in \mathcal{H}. \tag{21}$$

Similar inequalities can be proved for $B(\mathbf{u},c)$ and for $b(\mathbf{u},c,c_1)$.

We remind also the properties

$$\mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{w}) = -\mathbf{b}(\mathbf{u},\mathbf{w},\mathbf{v}), \tag{22}$$

$$\mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{v}) = 0, \tag{23}$$

that hold for periodic boundary conditions and are true also for the trilinear form $b(\mathbf{u},c,c_1)$ (for periodic boundary conditions).

## 3. Existence of the solutions

The flow of the incompressible, viscous fluid in which the diffusion takes place is not affected by the substance that is diffused. Hence, for the problem (14), (15) we have the classical existence and uniqueness results for the equations Navier-Stokes in $\mathbb{R}^2$.

**Theorem 1** [28]. a) *If* $\mathbf{u}_0 \in \mathcal{H}_1$, $\mathbf{f}$ *is analytical in time and for every* $t \geq 0$, $\mathbf{f}(t,\cdot) \in \mathcal{H}_1$, *then the problem* (14), (15) *has an unique solution* $\mathbf{u}$ *defined on* $[0,T]$ *for every* $T > 0$, *that is analytical in time and such that for every* $t \in [0,T]$, $\mathbf{u}(t,\cdot) \in \mathcal{V}_1$.

b) *If, in addition to the hypotheses in* a), $\mathbf{u}_0 \in \mathcal{V}_1$, $\mathbf{f}(t,\cdot) \in \mathcal{V}_1$, *then* $\mathbf{u}(t,\cdot) \in D(\mathbf{A})$ *for every* $t \in [0,T]$.

By using the Galerkin-Faedo method we can easily prove the following theorem.

**Theorem 2.** a) *In the conditions* a) *of Theorem 1 and if* $h$ *is analytical in time and for every* $t \geq 0$, $h(t,\cdot) \in \mathcal{H}_2$, *and* $c_0 \in \mathcal{H}_2$, *then there is an unique solution* $\widetilde{c}$ *of the problem (16)–(17), defined on* $[0,\, T], \forall T > 0$, *analytical in time and such that* $\widetilde{c}(t,\cdot) \in \mathcal{H}_2, \forall t \in [0,\, T]$.

b) *In the conditions* b) *of Theorem 1 and if* $c_0 \in \mathcal{V}_2$, $h(t,\cdot) \in \mathcal{V}_2$ *then* $\widetilde{c}(t,\cdot) \in D(A)$, $\forall t \in [0,\, T]$.

## 4. Some bounds of the various norms of the solution

It was proved [1], [25], that the semi-dynamical system generated by the autonomous Navier-Stokes equations ($\mathbf{f}$ independent of $t$) is dissipative, in the sense that there is an absorbing ball in $\mathcal{H}_1$ for it. A similar result can be proved for our problem, by using the fact that $\mathbf{f}(.)$ and $h(.)$ are bounded. That is, there is a $\rho_0 > 0$ such that for every $R > 0$, there is a $t_0(R) > 0$ with the property that for every $\mathbf{u}_0 \in \mathbf{B}(0,R) \subset \mathcal{H}_1$, we have $\mathbf{u}(t,.,\mathbf{u}_0) \in \mathbf{B}(0,\rho_0) \subset \mathcal{H}_1$, for $t > t_0(R)$. We can also prove, as in [25], some similar estimates for the $\mathcal{V}_1$ norm and, as in [24], for the $\mathbf{D}(\mathbf{A})$ norm of of $\mathbf{u}(t,.,\mathbf{u}_0)$. I.e. there is a $\rho_1 > 0$ such that $\mathbf{u}_0 \in \mathbf{B}(0,R) \subset \mathcal{H}_1$ implies $\|\mathbf{u}(t,.,\mathbf{u}_0)\| \leq \rho_1$ for $t > t_1(R)\,(\geq t_0(R))$, respectively there is a $\rho_2 > 0$ such that $\mathbf{u}_0 \in \mathbf{B}(0,R) \subset \mathcal{H}_1$ implies $|\mathbf{A}\mathbf{u}(t,.,\mathbf{u}_0)| \leq \rho_2$ for $t > t_2(R) \geq t_1(R)$.

By using the same techniques as in the proof of the dissipativity of $\mathbf{u}$, the dissipativity of the component $\widetilde{c}$ of the solution $(\mathbf{u},\widetilde{c})$ both in $\mathcal{H}_2$ and $\mathcal{V}_2$ can be proved. By using the fact that, representing a concentration, $c$ is such that $0 \leq c(0,\mathbf{x}) \leq 1$, and thus $|c_0| \leq l$, it follows that the following result is true (where $\widetilde{c}(t,.,\mathbf{u}_0,c_0)$ is the $c$ component of the solution of :

**Theorem 3.** a) *There is an* $\eta_0 > 0$ *with the property for every* $R > 0$ *there is a* $t_{c0}(R) > 0$ *such that for* $|\mathbf{u}_0| \leq R$,

$$|\widetilde{c}(t,.,\mathbf{u}_0,c_0)| \leq \eta_0,\ t \geq t_{c0}(R,R_c)\,.$$

b) *There is an* $\eta_1 > 0$ *with the property that for every* $R > 0$ *there is a* $t_{c1}(R) > 0$ *such that*

$$\|\widetilde{c}(t,.,\mathbf{u}_0,c_0)\| \leq \eta_1$$

*for* $|\mathbf{u}_0| \leq R,\ t \geq t_{c1}(R,R_c)$,

c) *There is an* $\eta_2 > 0$ *with the property that for every* $R > 0$ *there is a* $t_{c2}(R) > 0$ *such that*

$$|\Delta\widetilde{c}\,(t,.,\mathbf{u}_0,c_0)| \le \eta_2$$

*for* $|\mathbf{u}_0| \le R,\ t \ge t_{c2}\,(R,R_c)\,.$

## 5. The decomposition of the spaces, the projected equations

The eigenvalues of the two operators $\mathbf{A}$ and $A$ have the same form

$$\lambda_{j_1,j_2} = \frac{4\pi^2}{l^2}\left(j_1^2 + j_2^2\right),\ j_1,\ j_2 \in \mathbb{N},\ \ j_1 \cdot j_2 \neq 0. \tag{24}$$

To each eigenvalue $\lambda_{j_1,j_2}$ above, the following eigenfunctions of $\mathbf{A}$ correspond [26]:

$$\mathbf{w}^1_{j_1,j_2} = \frac{\sqrt{2}}{l}\frac{(j_2,-j_1)}{|\mathbf{j}|}\sin\left(2\pi\frac{j_1x_1+j_2x_2}{l}\right), \mathbf{w}^2_{j_1,j_2} = \frac{\sqrt{2}}{l}\frac{(j_2,j_1)}{|\mathbf{j}|}\sin\left(2\pi\frac{j_1x_1-j_2x_2}{l}\right),$$
$$\mathbf{w}^3_{j_1,j_2} = \frac{\sqrt{2}}{l}\frac{(j_2,-j_1)}{|\mathbf{j}|}\cos\left(2\pi\frac{j_1x_1+j_2x_2}{l}\right), \mathbf{w}^4_{j_1,j_2} = \frac{\sqrt{2}}{l}\frac{(j_2,j_1)}{|\mathbf{j}|}\cos\left(2\pi\frac{j_1x_1-j_2x_2}{l}\right),$$

where $\mathbf{j} = (j_1,j_2)\,,\ |\mathbf{j}| = \left(j_1^2+j_2^2\right)^{\frac{1}{2}}$. The set of all these eigenfunctions form a total system for $\mathcal{H}_1$.

Similarly for $A$ (the scalar Laplace operator), to each eigenvalue (24) the following eigenfunctions

$$\begin{aligned} w^1_{j_1,j_2} &= \frac{\sqrt{2}}{l}\sin\left(2\pi\frac{j_1x_1+j_2x_2}{l}\right),\ w^2_{j_1,j_2} = \frac{\sqrt{2}}{l}\sin\left(2\pi\frac{j_1x_1-j_2x_2}{l}\right), \\ w^3_{j_1,j_2} &= \frac{\sqrt{2}}{l}\cos\left(2\pi\frac{j_1x_1+j_2x_2}{l}\right),\ w^4_{j_1,j_2} = \frac{\sqrt{2}}{l}\cos\left(2\pi\frac{j_1x_1-j_2x_2}{l}\right), \end{aligned}$$

correspond. The set of all these eigenfunctions form a total system for $\mathcal{H}_2$.

We fix a $m \in \mathbb{N}$ and we consider the set $\Gamma_m$ of the eigenvalues $\lambda_{j_1,j_2}$ having $0 \le j_1, j_2 \le m$. We set

$$\begin{aligned} \lambda &: = \lambda_{1,0} = \lambda_{0,1} = \frac{4\pi^2}{l^2}, \\ \Lambda &: = \lambda_{m+1,0} = \lambda_{0,m+1} = \frac{4\pi^2}{l^2}(m+1)^2, \\ \delta &: = \frac{\lambda}{\Lambda} = \frac{1}{(m+1)^2}. \end{aligned}$$

We denote by $\mathbf{P}$ the projection operator on $\mathcal{L}\left(\left\{\mathbf{w}^i_{j_1,j_2},\ 0 \le j_1,\ j_2 \le m\right\}\right) \subset \mathcal{H}_1$, and by $\mathbf{Q}$ the projection operator on the orthogonal complement of this subspace in $\mathcal{H}_1$ (here $\mathcal{L}(a,b,c,...)$ represents the subspace spanned by the vectors $a,b,c,...$). Analogously, we denote by $P$ the projection operator on $\mathcal{L}\left(\left\{w^i_{j_1,j_2},\ 0\leq j_1,\ j_2\leq m\right\}\right)\subset\mathcal{H}_2$, and by $Q$ the projection operator on the orthogonal complement of this space in $\mathcal{H}_2$.

For the solution $\mathbf{u}$ of the Navier-Stokes equations, we write

$$\mathbf{p}=\mathbf{Pu},\ \mathbf{q}=\mathbf{Qu},$$

and, for the concentration $c$ of the diffused substance, we define

$$c_p=P\widetilde{c},\ c_q=Q\widetilde{c}.$$

With these notations, the projections of the equations (14), (16) by the projection operators defined above can be written as

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}-\nu\Delta\mathbf{p}+\mathbf{PB}(\mathbf{p}+\mathbf{q}) &= \mathbf{Pf}, && (25)\\
\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}-\nu\Delta\mathbf{q}+\mathbf{QB}(\mathbf{p}+\mathbf{q}) &= \mathbf{Qf}, && (26)\\
\frac{\mathrm{d}c_p}{\mathrm{d}t}-D\Delta c_p+PB\left(\mathbf{p}+\mathbf{q},c_p+c_q\right) &= Ph, && (27)\\
\frac{\mathrm{d}c_q}{\mathrm{d}t}-D\Delta c_q+QB\left(\mathbf{p}+\mathbf{q},c_p+c_q\right) &= Qh. && (28)
\end{aligned}$$

## 6. Estimates for the "small components" of the unknown functions

We assume in the following that $|\mathbf{u}_0|\leq R$.

In [12] we improved the estimates given in [4] for the norm of $\mathbf{q}$. More precisely, we proved that there is a certain moment $t_3(R)$, such that

$$\begin{aligned}
|\mathbf{q}(t)| &\leq K_0\delta,\ \ \|\mathbf{q}(t)\|\leq K_1\delta^{\frac{1}{2}},\\
|\mathbf{q}'(t)| &\leq K_2\delta,\ \ |\mathbf{\Delta q}(t)|\leq K_3,\ \ \text{for } t\geq t_3(R),
\end{aligned}$$

where $K_i$ depend only on the data $\nu$, $D$, $\mathbf{f}$, $h$, $l$ (and not on $m$ as in [4], where each coefficient contained a factor $L^{1/2}$, with $L=\ln(1+2m^2)$).

A similar result is true for $c_q$ :

**Theorem 4.** *There is a moment $t_{c3}\geq t_{c2}$, depending on $R$, such that for every $t\geq t_{c3}$ the following inequalities hold:*

$$\begin{aligned}
|c_q(t)| &\leq J_0\delta, && \|c_q(t)\|\leq J_1\delta^{1/2}, && (29)\\
\left|c'_q(t)\right| &\leq J_2\delta, && |\Delta c_q(t)|\leq J_3, && (30)
\end{aligned}$$

*where $J_0,\ J_1,\ J_2, J_3$ are independent on $m$.*

*Proof.* We shall frequently use below the inequalities:

$$\begin{aligned} \|c_q\| &\geq \Lambda^{\frac{1}{2}} |c_q|, \\ |\Delta c_q| &\geq \Lambda |c_q|, \end{aligned} \tag{31}$$

easy to prove by considering the Fourier series of $c_q$.

We take the scalar product of eq. (16) with $-\Delta c_q$

$$\left(\frac{\mathrm{d}c_q}{\mathrm{d}t}, -\Delta c_q\right) + D(\Delta c_q, \Delta c_q) - (\mathbf{u}\nabla\widetilde{c}, \Delta c_q) = -\left(Q\widetilde{h}, \Delta c_q\right).$$

From here, by using the inequality for $b(.,.,.)$ analogous to (21), we find

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|c_q\|^2 + D|\Delta c_q|^2 &\leq |b(\mathbf{p},\widetilde{c},\Delta c_q)| + |b(\mathbf{q},\widetilde{c},\Delta c_q)| + \left|\left(Q\widetilde{h},\Delta c_q\right)\right| \\ &\leq c_3|\mathbf{p}|^{\frac{1}{2}}\|\mathbf{p}\|^{\frac{1}{2}}\|\widetilde{c}\|^{\frac{1}{2}}|\Delta\widetilde{c}|^{\frac{1}{2}}|\Delta c_q| + \\ &\quad + c_3|\mathbf{q}|^{\frac{1}{2}}\|\mathbf{q}\|^{\frac{1}{2}}\|\widetilde{c}\|^{\frac{1}{2}}|\Delta\widetilde{c}|^{\frac{1}{2}}|\Delta c_q| + \left|Q\widetilde{h}\right||\Delta c_q| \\ &\leq c_3\rho_0^{1/2}\rho_1^{1/2}\eta_1^{1/2}\eta_2^{1/2}|\Delta c_q| + C\delta^{3/4}\eta_1^{1/2}\eta_2^{1/2}|\Delta c_q| + \\ &\quad + \left|Q\widetilde{h}\right||\Delta c_q| \\ &\leq c_3^2\rho_0\rho_1\eta_1\eta_2\frac{1}{D} + \frac{D}{4}|\Delta c_q|^2 + C^2\delta^{3/2}\eta_1\eta_2\frac{1}{D} + \\ &\quad + \frac{D}{4}|\Delta c_q|^2 + \frac{1}{D}|Qh|^2 + \frac{D}{4}|\Delta c_q|^2. \end{aligned}$$

From here,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|c_q\|^2 + \frac{D\Lambda}{4}\|c_q\|^2 \leq c_3^2\rho_0\rho_1\eta_1\eta_2\frac{1}{D} + C^2\delta^{3/2}\eta_1\eta_2\frac{1}{D} + \frac{1}{D}|Qh|^2,$$

and, with $C_1 = \frac{2}{D}(c_3^2\rho_0\rho_1\eta_1\eta_2 + C^2\delta^{3/2}\eta_1\eta_2 + |Qh|^2)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|c_q\|^2 + \frac{D\Lambda}{2}\|c_q\|^2 \leq C_1.$$

By applying the usual Gronwall Lemma, we obtain

$$\|c_q\|^2(t) \leq \frac{2C_1}{D\Lambda} + \|c_q(t_{c2})\|^2 \mathrm{e}^{-D\Lambda\left(t-t_{c_2}\right)/2} \leq \frac{2C_1}{D\Lambda} + \eta_1^2\mathrm{e}^{-D\Lambda\left(t-t_{c_2}\right)/2},$$

thus, for $t_{c3} \geq t_{c2}$ such that $t \geq t_{c3}$ implies $\eta_1^2\mathrm{e}^{-D\Lambda\left(t-t_{c_2}\right)/2} \leq \frac{2C_1}{D\Lambda}$, the assertion $(29)_2$ follows, with $J_1 = 2\sqrt{\frac{C_1}{D\lambda}}$.

Then, by using (31), $(29)_1$ is also obtained.

In the hypothesis that $h(.)$ is analytic as function of time, the analiticity of $c$ and, as consequences the analicity of $c_p$ and $c_q$ follow. More than that, these functions are the restrictions to the time real axis of some analytic functions of complex variable defined on a neighborhood of the real axis. Then, by using the Cauchy integral formula, the estimate $(30)_1$ can be obtained (as a similar estimate for $|\mathbf{q}'|$ is obtained in [26]).

Finally, by writing

$$D\Delta c_q = \frac{\mathrm{d}c_q}{\mathrm{d}t} + QB\left(\left(\mathbf{p}+\mathbf{q}\right)\nabla\left(c_p + c_q\right)\right) - Qf,$$

and by using the inequalities already proved, the inequality $(30)_2$ is proved. □

## 7. The construction of the approximate solutions by the R-APP Galerkin method

In [12] we gave an algorithm for the construction of a sequence of approximate solutions for the two-dimensional Navier-Stokes equations with periodic boundary conditions. There the volum force $\mathbf{f}$ was taken constant in time. As long as $\mathbf{f}$ is analytic in time the conclusions of the cited paper are still valid. In the problem (14), (16), once the velocity of the fluid is known, the equation in $c$ can be regarded as a linear equation. We however treat the problem as a coupled, nonlinear problem, and a treatment similar to that used for the Navier-Stokes equations will be applied to it. A sequence of approximations for $\mathbf{u}$, $c$ will be obtained.

As the construction of approximate solutions for the Navier-Stokes is [12], that of approximations for $c$ is structured on several levels.

### 7.1. The first level

Let $\mathbf{p}_0\left(t,\mathbf{x}\right)$ be the solution of the equation (the Galerkin approximation of the Navier-Stokes equations):

$$\begin{aligned} \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} - \nu\boldsymbol{\Delta}\mathbf{p} + \mathbf{PB}\left(\mathbf{p}\right) &= \mathbf{Pf}, \\ \mathbf{p}(0) &= \mathbf{Pu}_0, \end{aligned} \tag{32}$$

and define

$$\mathbf{q}_0(t) = \boldsymbol{\Phi}_0\left(\mathbf{p}_0(t)\right),$$

where $\boldsymbol{\Phi}_0 : \mathbf{P}\mathcal{H} \to \mathbf{Q}\mathcal{H}$ is the function whose graph is the first a.i.m. $\mathcal{M}_0$, defined in [4], for the Navier-Stokes equations, that is

$$\boldsymbol{\Phi}_0\left(\mathbf{p}\right) = \left(\nu\mathbf{A}\right)^{-1}\left[\mathbf{Qf} - \mathbf{QB}\left(\mathbf{p}\right)\right]. \tag{33}$$

Let $c_{p0}$ be the solution of the problem:

$$\frac{\mathrm{d}c_p}{\mathrm{d}t} - D\Delta c_p + P\left(\mathbf{p}_0\nabla c_p\right) = Ph, \tag{34}$$

and

$$c_{q0} = \Psi_{c,0}\left(\mathbf{p}_0, c_{p0}\right),$$

where $\Psi_{c,0}$ is the corresponding to $c$ component of the a.i.m. of the coupled problem (14)–(16) (that was studied in [13]), that is

$$c_{q0} = (DA)^{-1}\left[Qh - Q\left(\mathbf{u}_0 \nabla c_{p0}\right)\right]. \tag{35}$$

The approximate solution for the coupled problem is

$$\begin{aligned} \mathbf{u}_0(t) &= \mathbf{p}_0(t) + \mathbf{q}_0(t), \\ \widetilde{c}_0(t) &= c_{p0}(t) + c_{q0}(t). \end{aligned}$$

**Remarks**

**1.** This first level is different from the PP Galerkin method only in the fact that we compute $\mathbf{q}_0$ and $c_{q0}$ at every moment of time $t$. In practice, in the numerical scheme relying on our method, $\mathbf{q}_0$ and $c_{q0}$ must be evaluated at every node of the time grid on the interval $[0,\ T]$.

**2.** In [26] a family of functions $\{\mathbf{u}_{j,m}(.)\}_{j\geq 0}$ called "induced trajectories" is defined. The first induced trajectory of the family is $\mathbf{u}_{0,m}(t) = \mathbf{p}(t) + \mathbf{q}_{0,m}(t)$, where $\mathbf{p}(t)$ is the projection of the exact solution, and

$$\mathbf{q}_{0,m}(t) = (\nu\mathbf{A})^{-1}\left[\mathbf{Qf}(t) - \mathbf{QBp}(t)\right]. \tag{36}$$

On this definition relies that of the first a.i.m., $\mathcal{M}_0$. At this level, our approximate solution may be regarded as being defined via the first a.i.m. as well as via the first induced trajectory.

### 7.2. The second level

We define, as in [12], $\mathbf{p}_1$ as the solution of the equation:

$$\begin{aligned} \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} - \nu\boldsymbol{\Delta}\mathbf{p} + \mathbf{PB}\left(\mathbf{p} + \mathbf{q}_0\right) &= \mathbf{Pf}, \\ \mathbf{p}(0) &= \mathbf{Pu}_0. \end{aligned} \tag{37}$$

Then we set, also following [12],

$$\begin{aligned} \mathbf{q}_1(t) = {} & (\nu\mathbf{A})^{-1}\left[\mathbf{Qf}(t) - \mathbf{QB}\left(\mathbf{p}_1(t)\right) - \mathbf{QB}\left(\mathbf{p}_1(t), \mathbf{q}_0(t)\right) - \right. \\ & \left. - \mathbf{QB}\left(\mathbf{q}_0(t), \mathbf{p}_1(t)\right)\right]. \end{aligned} \tag{38}$$

The approximate solution for (14)–(15) at this level is

$$\mathbf{u}_1(t) = \mathbf{p}_1(t) + \mathbf{q}_1(t). \tag{39}$$

For the approximation of $c$ we consider the equation

$$\frac{\mathrm{d}c_p}{\mathrm{d}t} - D\Delta c_p + P\left(\left(\mathbf{p}_1 + \mathbf{q}_0\right)\nabla\left(c_p + c_{q0}\right)\right) = Ph, \tag{40}$$

and $c_{p1}$ its solution. Then, we define

$$\begin{aligned} c_{q1}(t) = \; & -(D\Delta)^{-1}\left[Qh(t) - QB(\mathbf{p}_1(t), c_{p1}(t)) - \right. \\ & \left. -QB(\mathbf{p}_1(t), c_{q0}(t)) - QB(\mathbf{q}_0(t), c_{p1}(t))\right]. \end{aligned} \tag{41}$$

The approximation of $\widetilde{c}$ at this level is

$$\widetilde{c}_1(t) = c_{p1}(t) + c_{q1}(t). \tag{42}$$

**Remarks**

**1.** In the equation for the approximation of $\mathbf{p}(t)$, we consider the argument of $\mathbf{B}$ of the form $\mathbf{p} + \mathbf{q}_0$ and not of the form $\mathbf{p} + \mathbf{\Phi}_0(\mathbf{p})$ as in the NL Galerkin method. This is an essentially different approach, since $\mathbf{q}_0$ is known from the preceding level of the method. Thus, by rearranging the terms, we obtain a differential equation of the same degree of difficulty as the Galerkin approximation for the Navier-Stokes equation.

**2.** We remark that $\mathbf{u}_1$is an approximation of the induced trajectory $\mathbf{u}_{1,m}$ defined in [26]. This is defined as $\mathbf{u}_{1,m}(t, \mathbf{x}) = \mathbf{p}(t, \mathbf{x}) + \mathbf{q}_{1,m}(t, \mathbf{x})$, where $\mathbf{p}(t, .)$ is the $\mathbf{P}$ projection of the exact solution, and

$$\begin{aligned} \mathbf{q}_{1,m}(t) \;=\; & (\nu\mathbf{A})^{-1}\left[\mathbf{Qf}(t) - \mathbf{QB}(\mathbf{p}(t)) - \mathbf{QB}\left(\mathbf{p}(t), \mathbf{q}_{0,m}(t)\right) - \right. \\ & \left. -\,\mathbf{QB}(\mathbf{q}_{0,m}(t), \mathbf{p}(t))\right], \end{aligned} \tag{43}$$

with $\mathbf{q}_{0,m}$ given by (36). Since, as we proved in [12], $\mathbf{p}_1(t)$ is an approximation for $\mathbf{p}(t)$, it can be seen that $\mathbf{q}_1$ is an approximation of $\mathbf{q}_{1,m}$, hence $\mathbf{u}_1(.)$ is an approximation of the induced trajectory $\mathbf{u}_{1,m}(.)$.

Similarly, "the induced trajectories" for the $c$ component of the solution can be defined (this was done in [13]) and we can see that $c_1$ is an approximation of the second of these induced trajectories.

The induced trajectories are set at the basis of the definition of a.i.m.s in [26]. That is why the use of induced trajectories instead of a.i.m.s for the construction of the approximations of $\mathbf{q}$ and $c_q$ leads to some simplifications in the computations in our method, when compared to the NL Galerkin method.

### 7.3. The k$^{th}$ level

We assume that for a $k \geq 2$, $\mathbf{q}_{k-2}$, $\mathbf{q}_{k-1}$ and $c_{q,k-2}$, $c_{q,k-1}$ were defined. We consider the equation

$$\begin{aligned} \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} - \nu\mathbf{\Delta}\mathbf{p} + \mathbf{PB}\left(\mathbf{p} + \mathbf{q}_{k-1}\right) &= \mathbf{Pf}, \\ \mathbf{p}(0) &= \mathbf{Pu}_0, \end{aligned} \tag{44}$$

and we denote by $\mathbf{p}_k$ its solution.

Then we define $\mathbf{q}_k$ as

$$\begin{aligned}\mathbf{q}_k &= (\nu\mathbf{A})^{-1}\left[\mathbf{Qf}-\mathbf{QB}(\mathbf{p}_k)-\mathbf{QB}(\mathbf{p}_k,\mathbf{q}_{k-1})-\right.\\ &\left.-\mathbf{QB}(\mathbf{q}_{k-1},\mathbf{p}_k)-\mathbf{QB}(\mathbf{q}_{k-2},\mathbf{q}_{k-2})-\mathbf{q}'_{k-2}\right]. \qquad (45)\end{aligned}$$

Naturally, the corresponding approximate solution of (14)–(15) is defined by

$$\mathbf{u}_k(t)=\mathbf{p}_k(t)+\mathbf{q}_k(t).$$

For the approximation of $c$, we consider the equation

$$\frac{\mathrm{d}c_p}{\mathrm{d}t}-D\Delta c_p+P\left(\left(\mathbf{p}_k+\mathbf{q}_{k-1}\right)\nabla\left(c_p+c_{q,k-1}\right)\right)=Ph,$$

denote by $c_{p,k}$ its solution and define

$$\begin{aligned}c_{q,k} &= (DA)^{-1}\left[Qh-QB\left(\mathbf{p}_k,c_{p,k}\right)-QB\left(\mathbf{p}_k,c_{q,k-1}\right)-\right.\\ &\left.-QB\left(\mathbf{q}_{k-1},c_{p,k}\right)-QB(\mathbf{q}_{k-2},c_{q,k-2})-c'_{q,k-2}\right].\end{aligned}$$

The approximation of $\widetilde{c}$ is defined as

$$\widetilde{c}_k(t)=c_{p,k}(t)+c_{q,k}(t). \qquad (46)$$

## 8. The error of the R-APP Galerkin method

In [12] we proved the results in the following

**Theorem 5**. *The functions* $\mathbf{p}_k(t),\ \mathbf{q}_k(t),\ k\geq 0,$ *defined in the previous section, satisfy the inequalities :*

$$\begin{aligned}|(\mathbf{p}-\mathbf{p}_k)(t)| &\leq C\delta^{5/4+k/2}, \qquad (47)\\ |(\mathbf{q}-\mathbf{q}_k)(t)| &\leq C\delta^{3/2+k/2}, \qquad (48)\end{aligned}$$

*for $t$ large enough.*

In the proof of this results, as well as in the proof of the similar one for $c$, we need the following result that is similar to Lemma 1 of [7]. Here $\widehat{b}^i_{j,l}$ denotes the coordinate of $b\in\mathcal{H}_2$ with respect to the eigenfunction $w^i_{j,l}$

**Lemma**. *Let* $G(s)=\sum\limits_{j,l}\left(\sum\limits_{i=1}^{4}\widehat{G}^i_{j,l}(s)\,w^i_{j,l}\right)$ *and suppose that*

$$\left|\widehat{\mathbf{G}}^i_{j,l}(s)\right|\leq c^i_{j,l},\quad for\quad 0\leq j,l\leq m,\ 1\leq i\leq 4.$$

*Then*

$$\left|\int_0^t \mathrm{e}^{-(t-s)DA}PG(s)ds\right|\leq\frac{1}{D}\left[\sum_{j,k\leq m}\sum_{i=1}^{4}\frac{\left(c^i_{j,l}\right)^2}{\lambda^2_{j,l}}\right]^{\frac{1}{2}}. \qquad (49)$$

Now we can state and prove

**Theorem 6**. *The functions $c_{p,k}$ and $c_{q,k}$, $k \geq 0$, defined in the previous section, satisfy the inequalities*

$$
\begin{aligned}
|(c_p - c_{p,k})(t)| &\leq C\delta^{5/4+k/2}, & (50)\\
|(c_q - c_{q,k})(t)| &\leq C\delta^{3/2+k/2} & (51)
\end{aligned}
$$

*for every $k \geq 0$ and for $t$ large enough.*

*Proof.* The proof is inductive. In the following, $C$ represents a generic coefficient (depending on the data of the problem and not on $m$).

**k=0.** The function $c_{p0}$ satisfies the relation

$$
\frac{\mathrm{d}c_{p0}}{\mathrm{d}t} - D\Delta c_{p0} + P\left(\mathbf{p}_0 \nabla\left(c_{p0}\right)\right) = Ph,
$$

relation that we subtract from equation (27), to obtain

$$
\frac{\mathrm{d}(c_p - c_{p0})}{\mathrm{d}t} - D\Delta(c_p - c_{p0}) + P\left(\mathbf{u}\nabla\left(c_p + c_q\right)\right) - P\left(\mathbf{p}_0 \nabla\left(c_{p0}\right)\right) = 0.
$$

We use here the method of [7] to estimate the norm of $c_p - c_{p0}$.

By using the semigroup of linear operators of infinitesimal generator $DA$, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tDA}\left(c_p - c_{p0}\right)(t) &= -\mathrm{e}^{tDA}\left\{P\left(\mathbf{p}\nabla c_p\right) - P\left(\mathbf{p}_0 \nabla c_{p0}\right) + \right.\\
&\quad \left. + P\left(\mathbf{q}\nabla c_p + \mathbf{u}\nabla c_q\right)\right\},
\end{aligned}
$$

or, by using the notations settled in Section 2, and by arranging the terms,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tDA}\left(c_p - c_{p0}\right)(t) &= -\mathrm{e}^{tDA}\left\{PB\left(\mathbf{p} - \mathbf{p}_0, c_p\right) + PB\left(\mathbf{p}_0, c_p - c_{p0}\right) + \right.\\
&\quad \left. + PB\left(\mathbf{q}, c_p\right) + PB\left(\mathbf{u}, c_q\right)\right\}.
\end{aligned}
$$

We integrate between the initial and the current moment, $t$, to obtain

$$
\begin{aligned}
\left(c_p - c_{p0}\right)(t) &= -\int_0^t \mathrm{e}^{-(t-s)DA}\left\{PB\left(\mathbf{p} - \mathbf{p}_0, c_p\right) + PB\left(\mathbf{p}_0, c_p - c_{p0}\right)\right\}\mathrm{d}s - \\
&\quad - \int_0^t \mathrm{e}^{-(t-s)DA}\left\{PB\left(\mathbf{q}, c_p\right) + PB\left(\mathbf{u}, c_q\right)\right\}\mathrm{d}s.
\end{aligned}
$$

The inequalities

$$
\left|\mathbf{A}^{-\delta}\mathbf{B}\left(\mathbf{u}, \mathbf{v}\right)\right| \leq \left\{
\begin{array}{l}
C\left|\mathbf{A}^{1-\delta}\mathbf{u}\right|\left|\mathbf{v}\right| \leq C\left|\mathbf{A}^{1/2}\mathbf{u}\right|\left|\mathbf{v}\right|, \\
C\left|\mathbf{u}\right|\left|\mathbf{A}^{1-\delta}\mathbf{v}\right| \leq C\left|\mathbf{u}\right|\left|\mathbf{A}^{1/2}\mathbf{v}\right|,
\end{array}\right. \quad \delta \in (1/2, 1) \qquad (52)
$$

from [1] are valid also for the bilinear application $B(.,.)$ and, in conjunction with the inequality

$$\left|A^{\delta}\mathrm{e}^{-tDA}\right| \leq Ct^{-\delta}\mathrm{e}^{-\frac{D\lambda}{2}t}, \tag{53}$$

from [11], they lead us to

$$\begin{aligned} &\left|\left(c_p - c_{p0}\right)(t)\right| \leq \textstyle\int_0^t C\,(t-s)^{-\delta}\,\mathrm{e}^{-\frac{D\lambda}{2}(t-s)}\left|\left(c_p - c_{p0}\right)(s)\right|\mathrm{d}s + \\ &+\left|\textstyle\int_0^t \mathrm{e}^{-(t-s)DA}\left[PB\left(\mathbf{p}-\mathbf{p}_0, c_p\right) + PB\left(\mathbf{q}, c_p\right) + PB\left(\mathbf{u}, c_q\right)\right](s)\,\mathrm{d}s\right|. \end{aligned}$$

A form of Gronwall inequality ([11], Lemma 7.1.1) implies

$$\begin{aligned} &\left|\left(c_p - c_{p0}\right)(t)\right| \leq \\ &\leq C\max_{0\leq t\leq T}\left|\textstyle\int_0^t \mathrm{e}^{-(t-s)DA}\left[PB\left(\mathbf{p}-\mathbf{p}_0, c_p\right) + PB\left(\mathbf{q}, c_p\right) + PB\left(\mathbf{u}, c_q\right)\right](s)\,\mathrm{d}s\right|. \end{aligned}$$

The constant $C$ above is of the order of $\mathrm{e}^T$, where $[0,T]$ is the time interval on which we work.

The idea is to find estimates for the coordinates of the terms in the brackets and then to use the Lemma from the beginning of the section (as in [7]). We thus find (by denoting $\lambda_{k,0} = \lambda_k$):

$$\begin{aligned} \left|PB\,(\widehat{\mathbf{p}-\mathbf{p}}_0, c_p)^{i}_{j,k}\right| &= \left|\left(PB\left(\mathbf{p}-\mathbf{p}_0, c_p\right), w^i_{j,k}\right)\right| \leq \\ &\leq \left|\mathbf{p}-\mathbf{p}_0\right|\left\|c_p\right\|\left|w^i_{j,k}\right|_{L^\infty(\Omega)} \leq C\eta_1\delta^{5/4}, \\ \left|P\widehat{B\,(\mathbf{q}, c}_p)^{i}_{j,k}\right| &\leq C\delta\left(\frac{1}{\lambda^{\frac{1}{2}}_{m-j+1}} + \frac{1}{\lambda^{\frac{1}{2}}_{m-k+1}}\right), \\ \left|P\widehat{B\,(\mathbf{p}, c}_q)^{i}_{j,k}\right| &\leq C\delta^{\frac{1}{2}}\left(\frac{1}{\lambda_{m-j+1}} + \frac{1}{\lambda_{m-k+1}}\right), \\ \left|P\widehat{B\,(\mathbf{q}, c}_q)^{i}_{j,k}\right| &\leq C\left|\mathbf{q}\right|\left\|c_q\right\| \leq C\delta^{3/2}, \end{aligned}$$

(since $\left|w^i_{j,k}\right|_{L^\infty(\Omega)} \leq 1$). From these estimates by using (49), and the inequalities

$$\begin{aligned} \sum_{j,k\leq m}\lambda^{-2}_{j,k} &\leq \widetilde{C}, \\ \sum_{j,k\leq m}\frac{1}{\lambda^2_{j,k}\lambda^2_{m-j+1}} &\leq \frac{C}{(m+1)^3} = C\delta^{3/2}, \end{aligned}$$

proved in [7], it follows that

$$\left|\int_0^t \mathrm{e}^{-(t-s)DA}\left[PB\left(\mathbf{p}-\mathbf{p}_0, c_p\right) + PB\left(\mathbf{q}, c_p\right) + PB\left(\mathbf{u}, c_q\right)\right](s)\,\mathrm{d}s\right| \leq C\delta^{5/4}.$$

Thus we obtained

$$|c_p - c_{p0}| \leq C\delta^{5/4}. \tag{54}$$

Let us remark that the preceding inequality implies

$$\begin{aligned} |c_{p0}| &\leq \eta_0 + C\delta^{5/4} := \eta_0', \\ \|c_{p0}\| &\leq \eta_1 + C\delta^{3/4} := \eta_1', \\ |\Delta c_{p0}| &\leq \eta_2 + C\delta^{1/4} := \eta_2'. \end{aligned} \tag{55}$$

The component $c_q$ of the solution satisfies

$$D\Delta c_q = QB\left(\mathbf{u}, c_p + c_q\right) - Qh + \frac{\mathrm{d}c_q}{\mathrm{d}t} \tag{56}$$

By subtracting from this the relation

$$D\Delta c_{q0} = QB\left(\mathbf{u}_0, c_{p0}\right) - Qh,$$

that is equivalent with the definition relation of $c_{q0}$, we obtain

$$\begin{aligned} |D\Delta\left(c_q - c_{q0}\right)| &\leq |QB\left(\mathbf{u}, c_p + c_q\right) - QB\left(\mathbf{u}_0, c_{p0}\right)| + \left|\frac{\mathrm{d}c_q}{\mathrm{d}t}\right| \\ &\leq |QB\left(\mathbf{u}, c_p\right) - QB\left(\mathbf{u}_0, c_{p0}\right)| + |QB\left(\mathbf{u}, c_q\right)| + \left|\frac{\mathrm{d}c_q}{\mathrm{d}t}\right| \\ &\leq |QB\left(\mathbf{u}, c_p - c_{p0}\right)| + |QB\left(\mathbf{u} - \mathbf{u}_0, c_{p0}\right)| + \\ &\quad + |QB\left(\mathbf{u}, c_q\right)| + \left|\frac{\mathrm{d}c_q}{\mathrm{d}t}\right|. \end{aligned}$$

From here, by using the analogous of (18) for $B(.,.)$ and that of (21) for $b(.,.,.)$, we find

$$\begin{aligned} |D\Delta\left(c_q - c_{q0}\right)| &\leq c_1 |\mathbf{u}|^{\frac{1}{2}} |\boldsymbol{\Delta}\mathbf{u}|^{\frac{1}{2}} \|c_p - c_{p0}\| + c_1 |\mathbf{u}|^{\frac{1}{2}} |\boldsymbol{\Delta}\mathbf{u}|^{\frac{1}{2}} \|c_q\| + \left|\frac{\mathrm{d}c_q}{\mathrm{d}t}\right| \\ &\quad + c_4 |\mathbf{u} - \mathbf{u}_0|^{\frac{1}{2}} \|\mathbf{u} - \mathbf{u}_0\|^{\frac{1}{2}} \|c_{p0}\|^{\frac{1}{2}} |\Delta c_{p0}|^{\frac{1}{2}} \\ &\leq C\varrho_0^{1/2}\rho_2^{1/2}\delta^{3/4} + C\varrho_0^{1/2}\rho_2^{1/2}\delta^{1/2} + C\delta + C\delta{\eta_1'}^{1/2}\delta{\eta_2'}^{1/2}. \end{aligned}$$

Then we get

$$|\Delta\left(c_q - c_{q0}\right)| \leq C\delta^{1/2}$$

and the following inequalities are obtained as consequences

$$\|c_q - c_{q0}\| \leq C\delta, \tag{57}$$

$$|c_q - c_{q0}| \leq C\delta^{3/2}. \tag{58}$$

We assumed that $\mathbf{f}(.)$ and $h(.)$ are analytic in time. Then it can be proved, by using the method of [1], [28], that $\mathbf{u}_0$ and $c_{p0}$ are analytical in time, and more than that, they are the restrictions of some analytical functions of complex variable, defined in some neighborhoods of the real axis. Then, by using the Cauchy representation formula, we can prove that

$$\left|\frac{\mathrm{d}c_q}{\mathrm{d}t}-\frac{\mathrm{d}c_{q0}}{\mathrm{d}t}\right|\leq C\delta^{3/2}.$$

**k=1**. We subtract the relation satisfied by $c_{p1}$, that is

$$\frac{\mathrm{d}c_{p1}}{\mathrm{d}t}-D\Delta c_{p1}+P\left(\left(\mathbf{p}_1+\mathbf{q}_0\right)\nabla\left(c_{p1}+c_{q0}\right)\right)=Ph,$$

from the equation (25) to obtain

$$\frac{\mathrm{d}\left(c_p-c_{p1}\right)}{\mathrm{d}t}-D\Delta\left(c_p-c_{p1}\right)+P\left(\mathbf{u}\nabla\left(c_p+c_q\right)\right)-P\left(\left(\mathbf{p}_1+\mathbf{q}_0\right)\nabla\left(c_{p1}+c_{q0}\right)\right)=0.$$

We use again the method of [7]. We write the above relation in the equivalent form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tDA}\left(c_p-c_{p1}\right) &= \mathrm{e}^{tDA}\left\{PB\left(\mathbf{u},c_p\right)-PB\left(\mathbf{p}_1+\mathbf{q}_0,c_{p1}\right)+\right.\\ &\quad\left.+PB\left(\mathbf{u},c_q\right)-PB\left(\mathbf{p}_1+\mathbf{q}_0,c_{q0}\right)\right\},\end{aligned}$$

and we integrate after conveniently arranging the terms:

$$\begin{aligned}\left(c_p-c_{p1}\right)(t) &= \int_0^t \mathrm{e}^{-DA(t-s)}\left\{PB\left(\mathbf{u},c_p-c_{p1}\right)+PB\left(\mathbf{u}-\left(\mathbf{p}_1+\mathbf{q}_0\right),c_{p1}\right)\right\}(s)\,\mathrm{d}s-\\ &\quad-\int_0^t \mathrm{e}^{-DA(t-s)}\left\{PB\left(\mathbf{u},c_q-c_{q0}\right)+PB\left(\mathbf{u}-\left(\mathbf{p}_1+\mathbf{q}_0\right),c_{q0}\right)\right\}(s)\,\mathrm{d}s.\end{aligned}$$

As we did for **k=0**, we use the inequalities (52), (53) to find

$$\begin{gathered}\left|\left(c_p-c_{p1}\right)(t)\right|\leq\int_0^t C\left(t-s\right)^{-\delta}\mathrm{e}^{-\frac{D\lambda}{2}(t-s)}\left|\left(c_p-c_{p1}\right)(s)\right|\mathrm{d}s+\\ +\left|\int_0^t \mathrm{e}^{-(t-s)DA}\left[PB\left(\mathbf{u}-\left(\mathbf{p}_1+\mathbf{q}_0\right),c_{p1}\right)+PB\left(\mathbf{u},c_q-c_{q0}\right)\right](s)\,\mathrm{d}s+\right.\\ \left.+\int_0^t \mathrm{e}^{-(t-s)DA}\left[PB\left(\mathbf{u}-\left(\mathbf{p}_1+\mathbf{q}_0\right),c_{q0}\right)\right](s)\,\mathrm{d}s\right|.\end{gathered}$$

Then, the Gronwall type Lemma from [11], cited above, implies
$\left|\left(c_p-c_{p1}\right)(t)\right|\leq$

$$\begin{aligned}&\leq C\max_{0\leq t\leq T}\left|\int_0^t \mathrm{e}^{-(t-s)DA}\left[PB\left(\mathbf{u}-\left(\mathbf{p}_1+\mathbf{q}_0\right),c_{p1}\right)+PB\left(\mathbf{u},c_q-c_{q0}\right)\right](s)\,\mathrm{d}s+\right.\\ &\left.+\int_0^t \mathrm{e}^{-(t-s)DA}\left[PB\left(\mathbf{u}-\left(\mathbf{p}_1+\mathbf{q}_0\right),c_{q0}\right)\right](s)\,\mathrm{d}s\right|.\end{aligned}$$

We now estimate the coordinates of the terms in the brackets

$$\begin{aligned}
\left|PB\left(\mathbf{u}-\widehat{(\mathbf{p}_1+\mathbf{q}_0)},c_{p1}\right)_{j,k}^{i}\right| &\leq \left|\mathbf{u}-(\mathbf{p}_1+\mathbf{q}_0)\right|\left\|c_{p1}\right\|\left|w_{j,k}^{i}\right|_{L^\infty(\Omega)} \leq C\eta_1\delta^{3/2},\\
\left|PB\left(\widehat{\mathbf{p},c_q}-c_{q0}\right)_{j,k}^{i}\right| &\leq C\delta\left(\frac{1}{\lambda_{m-j+1}}+\frac{1}{\lambda_{m-k+1}}\right),\\
\left|PB\left(\widehat{\mathbf{q},c_q}-c_{q0}\right)_{j,k}^{i}\right| &\leq \left|\mathbf{q}\right|\left\|c_q-c_{q0}\right\| \leq C\delta^2,\\
\left|PB\left(\mathbf{u}-\widehat{(\mathbf{p}_1+\mathbf{q}_0)},c_q\right)_{j,k}^{i}\right| &\leq C\left|\mathbf{u}-(\mathbf{p}_1+\mathbf{q}_0)\right|\left\|c_{q0}\right\| \leq C\delta^2.
\end{aligned}$$

The method used at the level **k=0**, (borrowed from [7]), leads us to the conclusion

$$|c_p-c_{p1}| \leq C\delta^{7/4}.$$

The inequalities (55) are true also for $c_{p1}$.

The definition relation of $c_{q1}$ is equivalent with

$$D\Delta c_{q1} = QB(\mathbf{p}_1,c_{p1})+QB(\mathbf{p}_1,c_{q0})+QB(\mathbf{q}_0,c_{p1})-Qh.$$

We subtract this relation from (56) to find, after grouping the terms, and taking the norm

$$\begin{aligned}
|D\Delta(c_q-c_{q1})| \leq\; & |QB(\mathbf{p}-\mathbf{p}_1,c_p)|+|QB(\mathbf{p}_1,c_p-c_{p1})|+|QB(\mathbf{p}-\mathbf{p}_1,c_q)|+\\
&+|QB(\mathbf{p}_1,c_q-c_{q0})|+|QB(\mathbf{q}-\mathbf{q}_0,c_p)|+\\
&+|QB(\mathbf{q}_0,c_p-c_{p1})|+|QB(\mathbf{q},c_q)|+\left|\frac{\mathrm{d}c_q}{\mathrm{d}t}\right|.
\end{aligned}$$

We use the inequalities similar to (18), (21) and the inequalities already proved above to get

$$\begin{aligned}
|D\Delta(c_q-c_{q1})| \leq\; & c_4\left|\mathbf{p}-\mathbf{p}_1\right|^{\frac{1}{2}}\left\|\mathbf{p}-\mathbf{p}_1\right\|^{\frac{1}{2}}\left\|c_p\right\|^{\frac{1}{2}}\left|\Delta c_p\right|^{\frac{1}{2}}+\\
&+c_4\left|\mathbf{p}_1\right|^{\frac{1}{2}}\left\|\mathbf{p}_1\right\|^{\frac{1}{2}}\left\|c_p-c_{p1}\right\|^{\frac{1}{2}}\left|\Delta(c_p-c_{p1})\right|^{\frac{1}{2}}+\\
&+c_4\left|\mathbf{p}-\mathbf{p}_1\right|^{\frac{1}{2}}\left\|\mathbf{p}-\mathbf{p}_1\right\|^{\frac{1}{2}}\left\|c_q\right\|^{\frac{1}{2}}\left|\Delta c_q\right|^{\frac{1}{2}}+\\
&+c_1\left|\mathbf{p}_1\right|^{\frac{1}{2}}\left|\boldsymbol{\Delta}\mathbf{p}_1\right|^{\frac{1}{2}}\left\|c_q-c_{q0}\right\|+\\
&+c_4\left|\mathbf{q}-\mathbf{q}_0\right|^{\frac{1}{2}}\left\|\mathbf{q}-\mathbf{q}_0\right\|^{\frac{1}{2}}\left\|c_p\right\|^{\frac{1}{2}}\left|\Delta c_p\right|^{\frac{1}{2}}+\\
&+c_4\left|\mathbf{q}\right|^{\frac{1}{2}}\left\|\mathbf{q}\right\|^{\frac{1}{2}}\left\|c_p-c_{p1}\right\|^{\frac{1}{2}}\left|\Delta(c_p-c_{p1})\right|^{\frac{1}{2}}+\\
&+c_4\left|\mathbf{q}\right|^{\frac{1}{2}}\left\|\mathbf{q}\right\|^{\frac{1}{2}}\left\|c_q\right\|^{\frac{1}{2}}\left|\Delta c_q\right|^{\frac{1}{2}}+\left|\frac{\mathrm{d}c_q}{\mathrm{d}t}\right|\\
\leq\; & C\eta_1^{1/2}\eta_2^{1/2}\delta^{7/8}\delta^{5/8}+C\rho_0^{1/2}\rho_1^{1/2}\delta^{5/8}\delta^{3/8}+C\delta^{7/8}\delta^{5/8}\delta^{1/4}+\\
&+C\left(\rho_0'\rho_2'\right)^{1/2}\delta+C\eta_1^{1/2}\eta_2^{1/2}\delta^{3/4}\delta^{1/4}+\\
&+C\delta^{1/2}\delta^{1/4}\delta^{5/8}\delta^{3/8}+C\delta^{1/2}\delta^{1/4}\delta^{1/4}+C\delta,
\end{aligned}$$

from where

$$
\begin{aligned}
|\Delta (c_q - c_{q1})| &\leq C\delta, \\
\|c_q - c_{q1}\| &\leq C\delta^{3/2}, \\
|c_q - c_{q1}| &\leq C\delta^2.
\end{aligned}
$$

With the same argument as that used at the level **k=0** we can prove that

$$
\left| \frac{\mathrm{d}c_q}{\mathrm{d}t} - \frac{dc_{q1}}{\mathrm{d}t} \right| \leq C\delta^2.
$$

**Induction step**. We suppose that for every $j < k$ the following inequalities hold

$$
\begin{aligned}
|c_p - c_{p,j}| &\leq C\delta^{5/4+j/2}, \\
|\Delta (c_q - c_{q,j})| &\leq C'\delta^{(1+j)/2}, \\
\left| \frac{\mathrm{d}c_q}{\mathrm{d}t} - \frac{\mathrm{d}c_{q,j}}{\mathrm{d}t} \right| &\leq C\delta^{(3+j)/2}.
\end{aligned}
$$

We subtract the relation

$$
\frac{\mathrm{d}c_{p,k}}{\mathrm{d}t} - D\Delta c_{p,k} + P\left(\left(\mathbf{p}_k + \mathbf{q}_{k-1}\right) \nabla \left(c_{p,k} + c_{q,k-1}\right)\right) = Ph,
$$

that is satisfied by $c_{p,k}$, from the equation (25), and obtain

$$
\frac{\mathrm{d}\left(c_p - c_{p,k}\right)}{\mathrm{d}t} - D\Delta\left(c_p - c_{p,k}\right) + P\left(\mathbf{u}\nabla\left(c_p + c_q\right)\right) - P\left(\left(\mathbf{p}_k + \mathbf{q}_{k-1}\right) \nabla \left(c_{p,k} + c_{q,k-1}\right)\right) = 0.
$$

From here, by following the same path as the levels **k=0** and **k=1**, we succesively obtain

$$
\left(c_p - c_{p,k}\right)(t) = \int_0^t \mathrm{e}^{-DA(t-s)} \left\{PB\left(\mathbf{u}, c_p - c_{p,k}\right) + PB\left(\mathbf{u} - \left(\mathbf{p}_k + \mathbf{q}_{k-1}\right), c_{p,k}\right)\right\}(s)\,\mathrm{d}s -
$$

$$
- \int_0^t \mathrm{e}^{-DA(t-s)} \left\{PB\left(\mathbf{u}, c_q - c_{q,k-1}\right) + PB\left(\mathbf{u} - \left(\mathbf{p}_k + \mathbf{q}_{k-1}\right), c_{q,k-1}\right)\right\}(s)\,\mathrm{d}s,
$$

then

$$
\left|\left(c_p - c_{p,k}\right)(t)\right| \leq \int_0^t C\left(t-s\right)^{-\delta} \mathrm{e}^{-\frac{D\lambda}{2}(t-s)} \left|\left(c_p - c_{p,k}\right)(s)\right| \mathrm{d}s +
$$

$$
+ \left| \int_0^t \mathrm{e}^{-(t-s)DA} \left[PB\left(\mathbf{u} - \left(\mathbf{p}_k + \mathbf{q}_{k-1}\right), c_{p,k}\right) + PB\left(\mathbf{u}, c_q - c_{q,k-1}\right)\right](s)\,\mathrm{d}s \right.
$$

$$
\left. + \int_0^t \mathrm{e}^{-(t-s)DA} \left[PB\left(\mathbf{u} - \left(\mathbf{p}_k + \mathbf{q}_{k-1}\right), c_{q,k-1}\right)\right](s)\,\mathrm{d}s \right|,
$$

and, finally,

$$
\begin{gathered}
\left|\left(c_p - c_{p,k}\right)(t)\right| \leq \\
\leq C \max_{0\leq t\leq T} \left| \int_0^t \mathrm{e}^{-(t-s)DA} \left[ PB\left(\mathbf{u}-\left(\mathbf{p}_k + \mathbf{q}_{k-1}\right), c_{p,k}\right) + PB\left(\mathbf{u}, c_q - c_{q,k-1}\right)\right](s)\,\mathrm{d}s \right. \\
\left. + \int_0^t \mathrm{e}^{-(t-s)DA} \left[ PB\left(\mathbf{u}-\left(\mathbf{p}_k + \mathbf{q}_{k-1}\right), c_{q,k-1}\right)\right](s)\,\mathrm{d}s \right|.
\end{gathered}
$$

Then, by using the induction hypotheses, we obtain

$$
\begin{aligned}
\left| PB\left(\mathbf{u}-\left(\widehat{\mathbf{p}_k + \mathbf{q}_{k-1}}\right), c_{p,k}\right)^i_{j,k}\right| &\leq \left|\mathbf{u}-\left(\mathbf{p}_k + \mathbf{q}_{k-1}\right)\right| \|c_{p,k}\| \leq C\eta_1' \delta^{(2+k)/2}, \\
\left| PB\left(\mathbf{p}, \widehat{c_q - c}_{q,k-1}\right)^i_{j,k}\right| &\leq C\delta^{(2+k)/2}\left(\frac{1}{\lambda_{m-j+1}} + \frac{1}{\lambda_{m-k+1}}\right), \\
\left| PB\left(\mathbf{q}, \widehat{c_q - c}_{q,k-1}\right)^i_{j,k}\right| &\leq |\mathbf{q}| \|c_q - c_{q,k-1}\| \leq C\delta\delta^{(1+k)/2}, \\
\left| PB\left(\mathbf{u}-\left(\widehat{\mathbf{p}_k + \mathbf{q}_{k-1}}\right), c_{q,k-1}\right)^i_{j,k}\right| &\leq \left|\mathbf{u}-\left(\mathbf{p}_k + \mathbf{q}_{k-1}\right)\right| \|c_{q,k-1}\| \leq C\delta^{(2+k)/2}\delta^{1/2}.
\end{aligned}
$$

From here, with the use of the Lemma at the beginning of the Section,

$$
\left|\left(c_p - c_{p,k}\right)(t)\right| \leq C\delta^{5/4+k/2}.
$$

We have still to find estimates for $|c_q - c_{q,k}|$. As above we subtract the relation

$$
\begin{aligned}
D\Delta c_{q,k} &= QB\left(\mathbf{p}_k, c_{p,k}\right) + QB\left(\mathbf{p}_k, c_{q,k-1}\right) + QB\left(\mathbf{q}_{k-1}, c_{p,k}\right) + \\
&\quad + QB(\mathbf{q}_{k-2}, c_{q,k-2}) + \frac{\mathrm{d}c_{q,k-2}}{\mathrm{d}t} - Qh,
\end{aligned}
$$

that is equivalent with the definition relation of $c_{q,k}$, from (56) and get

$$
\begin{aligned}
D\Delta\left(c_q - c_{q,k}\right) &= QB\left(\mathbf{u}, c\right) - \left[QB\left(\mathbf{p}_k, c_{p,k}\right) + QB\left(\mathbf{p}_k, c_{q,k-1}\right) + \right. \\
&\quad \left. + QB\left(\mathbf{q}_{k-1}, c_{p,k}\right) + QB(\mathbf{q}_{k-2}, c_{q,k-2})\right] + \frac{dc_q}{\mathrm{d}t} - \frac{\mathrm{d}c_{q,k-2}}{\mathrm{d}t}
\end{aligned}
$$

By taking the norm and grouping the terms in a convenient way we obtain
$|D\Delta\left(c_q - c_{q,k}\right)| \leq |QB\left(\mathbf{p}, c_p\right) - QB\left(\mathbf{p}_k, c_{p,k}\right)| + |QB\left(\mathbf{p}, c_q\right) - QB\left(\mathbf{p}_k, c_{q,k-1}\right)| +$
$+ |QB\left(\mathbf{q}, c_p\right) - QB\left(\mathbf{q}_{k-1}, c_{p,k}\right)| + |QB\left(\mathbf{q}, c_q\right) - QB(\mathbf{q}_{k-2}, c_{q,k-2})| +$
$+ \left|\frac{\mathrm{d}c_q}{\mathrm{d}t} - \frac{\mathrm{d}c_{q,k-2}}{\mathrm{d}t}\right|.$

We can treat the first three terms of the rhs as the similar terms from the step **k=1** and we find that their sum is of the order of $\delta^{(k+1)/2}$. For the fourth term we have

$|QB\left(\mathbf{q}, c_q\right) - QB(\mathbf{q}_{k-2}, c_{q,k-2})| \leq |QB\left(\mathbf{q}, c_q - c_{q,k-2}\right)| + \left|QB(\mathbf{q} - \mathbf{q}_{k-2}, c_{q,k-2})\right|$
$\leq c_1 |\mathbf{q}|^{\frac{1}{2}} |\mathbf{\Delta q}|^{\frac{1}{2}} \|c_q - c_{q,k-2}\| + c_1 \left|\mathbf{q} - \mathbf{q}_{k-2}\right|^{\frac{1}{2}} \left|\mathbf{\Delta}\left(\mathbf{q} - \mathbf{q}_{k-2}\right)\right|^{\frac{1}{2}} \|c_{q,k-2}\|$

$$\leq C\delta^{1/2}\delta^{k/2} + C\delta^{(k+1)/2}\delta^{(k-1)/2}\delta^{1/2} \leq C\delta^{(k+1)/2},$$

while for the fifth,

$$\left|\frac{\mathrm{d}c_q}{\mathrm{d}t} - \frac{\mathrm{d}c_{q,k-2}}{\mathrm{d}t}\right| \leq C\delta^{(k+1)/2}.$$

We thus obtain

$$\begin{aligned} |\Delta\,(c_q - c_{q,k})| &\leq C\delta^{(k+1)/2}, \\ \|c_q - c_{q,k}\| &\leq C\delta^{(k+2)/2}, \\ |c_q - c_{q,k}| &\leq C\delta^{(k+3)/2}. \end{aligned}$$

The argument used at the level **k=0** can be used also here to prove

$$\left|\frac{\mathrm{d}c_q}{\mathrm{d}t} - \frac{\mathrm{d}c_{q,k}}{\mathrm{d}t}\right| \leq C\delta^{(k+3)/2}.$$

We see that the induction hypothesis is confirmed and this concludes the proof.
□

## 9. Comments on the R-APP Galerkin method

When compared to the Galerkin classical method, the repeatedly adjusted and post-processed (R-APP) Galerkin method proposed by us presents the advantage of using small dimension projection spaces, since accuracy may be increased by using several levels of the method.

When compared to the NL Galerkin method, that shares the above property, the R-APP Galerkin method presents some advantages coming from the fact that we use induced trajectories instead of approximate inertial manifolds as basis of our approximations. As was asserted in the literature (e.g. [21]), the use of some high accuracy a.i.m. in the NL Galerkin method leads to equations whose numerical schemes are difficult to construct (these are equations of the type (7) and the definition of a.i.m.s is recursive, a high order a.i.m. $\mathcal{M}_k$ appealing, in its definition, the definition of all $\mathcal{M}_j$, with $0 \leq j \leq k-1$). In the R-APP method, in the equation for $\mathbf{p}_k$ (or $c_{p,k}$), $\mathbf{q}_{k-1}$ (respectively $c_{q,k-1}$) are known from the previous level and this makes the equations much easier to program (they have qualitatively the same level of difficulty, in what concerns the programming, as the classical Galerkin equations).

We must however say that, since the R-APP method requires some successive numerical integrations (at each level a numerical integration in time), a special attention must be payed to the accuracy of these, in order to not affect the good accuracy predicted by the theoretical computations.

To have an idea concerning how low dimensional the projection space can be, let us remark, as in [12] that, if we choose $m = 6$, after having passed through five levels of the method we obtain an approximate solution that bears an error of the order of $10^{-5}$ since $\delta^{13/4} = \frac{1}{7^{13/2}} \simeq 0.0000032$.

For $m = 10$, we need only four levels of the method for an error of the order of $10^{-5}$ ($\delta^{11/4} = \frac{1}{11^{11/2}} \simeq 0.00000187$).

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