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# A weighted Kaczmarz algorithm in image reconstruction

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The reconstruction of images in borehole electromagnetic geotomography gives rise to large sparse rank-deficient linear least squares problems. From technical reasons, the underground area scanning process is limited by the positions of electromagnetic wave sources and receptors in the boreholes made for this. As a consequence, the reconstructed images by using classical Algebraic Reconstruction Techniques are not always satisfactory because of "shadows" created by the null space of the system matrix. For overcoming these difficulties, we propose in this paper a weighted Kaczmarz algorithm. The weights are predefined for each hyperplane of the system of equations. A convergence analysis of the new method together with some numerical experiments are also presented.

## 1. Introduction

For numerical solution of problems arising in image reconstruction from projections in computerized tomography, a special class of iterative algorithms, called Algebraic Reconstruction Techniques (ART, for short) has been designed in early 80's. They are essentially based on the original Kaczmarz's projection algorithm (see [4]) and use in an efficient manner the "row-by-row" generation of the reconstruction matrix. The development of ART has been done in parallel in different directions (see for an almost complete overview the well known monograph [2]). One of them is related to the introduction of some parameters in the relaxation process, and although this idea can be already found in the original paper by Kaczmarz [4], we may consider that the first essential step has been made by Herman, Lent and Lutz in [3].

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In what follows we shall briefly describe their result. Let in this sense $A$ be an $m \times n$ matrix and $b \in \mathbb{R}^m$ such that the system

$$Ax = b \tag{1}$$

is consistent. We shall denote by $A^T$, $A_i \neq 0$, $N(A)$, $R(A)$, $S(A;b)$ the transpose, $i$-th row, null space, range of $A$ and the set of solutions of (1); also $\langle\cdot\rangle$, $\|\cdot\|$ will denote the euclidean scalar product and norm on some space $\mathbb{R}^q$. Usually in CT image reconstruction problems the matrix $A$ is also rank-deficient ($N(A) \neq 0$), thus $S(A;b)$ contains an infinity of elements among which the (unique) minimal norm one will be denoted by $x_{LS}$. Then, Herman, Lent and Lutz (HLL) version of Kaczmarz's algorithm can be written as follows:
**Algorithm HLL (simplified version)**: Let $x^0 \in \mathbb{R}^n$; for $k = 0, 1, \ldots$ do

$$\begin{array}{l} x = x^k \\ \text{for } i = 1 : m \\ x = x - \omega_i \frac{\langle x^k, A_i\rangle - b_i}{\|A_i\|^2} A_i \qquad (*) \\ \text{end for} \\ x^{k+1} = x \end{array} \tag{2}$$

where $\omega_i > 0$ are some fixed weights.

**Theorem 1.** ([3]) *For any $x^0 \in \mathbb{R}^n$, if the weights $\omega_i$ satisfy*

$$\max_{1\leq i\leq m} |1 - \omega_i| < 1 \Leftrightarrow \omega_i \in (0, 2), \forall i = 1, \ldots, m, \tag{3}$$

*then the sequence $(x^k)_{k\geq 0}$ generated by the HLL algorithm converges to a solution of* (1)*. If $x^0 = 0$, then $\lim_{k\to\infty} x^k = x_{LS}$.*

**Remark 1.** *For $\omega_i = 1$, $\forall i = 1, \ldots, m$ we get in* (2) *the original Kaczmarz's projections algorithm. In this case, the step* (*) *in* (2) *corresponds to the orthogonal projections of $x$ onto the hyperplane $H_i$ defined by the i-th equation in* (1)*, whereas the cases $\omega_i < 1$ and $\omega_i > 1$ are showed in Figure* 1*.*

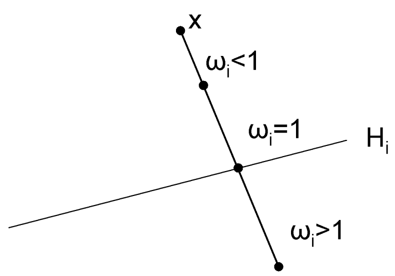


Fig. 1. Geometric interpretation of step (*) in (2).

## 2. The new Weighted Kaczmarz algorithm

If we want to create a new set of weights $\omega_i$, $i = 1, \ldots, m$ for (2), we have to follow two steps: first, to find a clear scope of our construction and second to prove that these weights verify the convergence assumption (3). According to the first aspect, let $i \in \{1, \ldots, m\}$ be fixed and $x$ the current approximation before the projection onto $H_i$ hyperplane. Let us also suppose that

$$S_i = \sum_{j=1}^{m} a_{ij} \neq 0. \tag{4}$$

If $r = Ax - b \in \mathbb{R}^m$ is the residual, we construct the new approximation $\bar{x} = (\bar{x}_j)_{j=1,\ldots,n} \in \mathbb{R}^n$ by modifying the components $x_j$ of $x$ as

$$\bar{x}_j = x_j - \frac{\omega_{ij}}{S_i} r_i, \; j = 1, \ldots, n, \tag{5}$$

where $\omega_{ij} > 0$ and $r_i$ is the $i$-th component of $r$. In this way, if $\bar{r} = A\bar{x} - b$ is the new residual we get

$$\begin{aligned} \bar{r}_i &= (A\bar{x} - b)_i = \sum_{j=1}^{n} a_{ij}\bar{x}_j - b_i \\ &= \sum_{j=1}^{n} a_{ij} x_j - b_i - \sum_{j=1}^{n} a_{ij} \frac{\omega_{ij}}{S_i} r_i \\ &= r_i \left( 1 - \frac{1}{S_i} \sum_{j=1}^{n} a_{ij}\ \omega_{ij} \right) \end{aligned} \tag{6}$$

The idea of the construction of new weights $\bar{\omega}_i$ in (2) can then be related to the reduction of the absolute value of $r_i$, which means

$$\left| 1 - \frac{1}{S_i} \sum_{j=1}^{n} a_{ij}\ \omega_{ij} \right| < 1 \tag{7}$$

or

$$0 < \sum_{j=1}^{n} a_{ij}\ \omega_{ij} < 2S_i, \; \forall i = 1, \ldots, m. \tag{8}$$

**Remark 2.** *The smallest value in* (7) *is obtained when*

$$\sum_{j=1}^{n} a_{ij}\ \omega_{ij} = S_i, \tag{9}$$

*for which* $\bar{r}_i = 0$.

Now, let us observe that, if the weights $\omega_{ij}$ are of the form

$$\omega_{ij} = c_i a_{ij}, \ \forall j = 1, \ldots, n, \tag{10}$$

for a $c_i \neq 0$, then the transformations (5) can be written as

$$\bar{x} = x - \frac{c_i}{S_i} A_i (b_i - \langle x, A_i \rangle) = x - \frac{c_i \|A_i\|^2}{S_i} \cdot \frac{\langle x, A_i \rangle - b_i}{\|A_i\|^2} A_i \tag{11}$$

According to the above considerations, the New Weighted Kaczmarz algorithm (NWK) that we propose in this paper is the following.
**Algorithm NWK**: Let $x^0 \in \mathbb{R}^n$; for $k = 0, 1, \ldots$ do

$$\begin{array}{l} x = x^k \\ \text{for } i = 1 : m \\ x = x - \tilde{\omega}_i \frac{\langle x, A_i \rangle - b_i}{\|A_i\|^2} A_i \ (*) \\ \text{end for} \\ x^{k+1} = x \end{array} \tag{12}$$

with

$$\tilde{\omega}_i = \frac{c_i \|A_i\|^2}{S_i}, \ i = 1, \ldots, m \tag{13}$$

and $c_i \neq 0$ from (10).

**Remark 3.** *If*

$$c_i = \frac{S_i}{\|A_i\|^2}, \ i = 1, \ldots, m \tag{14}$$

*we get $\tilde{\omega}_i = 1$, i.e. the classical Kaczmarz's algorithm (see [4]). This corresponds to the choice $\omega_{ij} = \frac{a_{ij} S_i}{\|A_i\|^2}$, $j = 1, \ldots, n$ for which the "ideal" case (9) holds (which is well known for the Kaczmarz's iteration in which, by projecting onto the $i$-th hyperplane $H_i$ the $i$-th equation in (1) is satisfied, thus the residual is reduced to zero).*

For the convergence analysis of the algorithm NWK we shall restrict ourselves to the case of electromagnetic geotomography (EG, for short; see [8]). In this context the matrix coefficients satisfy

$$0 \leq a_{ij} \leq \sqrt{2}, \tag{15}$$

(see Figure 2; the pixels are considered squares with unitary edges) and for any $i \in \{1, \ldots, m\}$ the $i$-th ray will intersect at least one pixel $P_j$, i.e. $a_{ij} \neq 0$ which gives us $S_i \neq 0$, that is the weights $\tilde{\omega}_i$ in (13) are always well defined. Moreover, for the sum $S_i$ we have (for $n = q^2$)

$$q = S_{\min} \leq S_i \leq S_{\max} = \sqrt{q^2 + (q-1)^2} < q\sqrt{2} \tag{16}$$

Now we are able to prove the convergence result for the algorithm NWK.

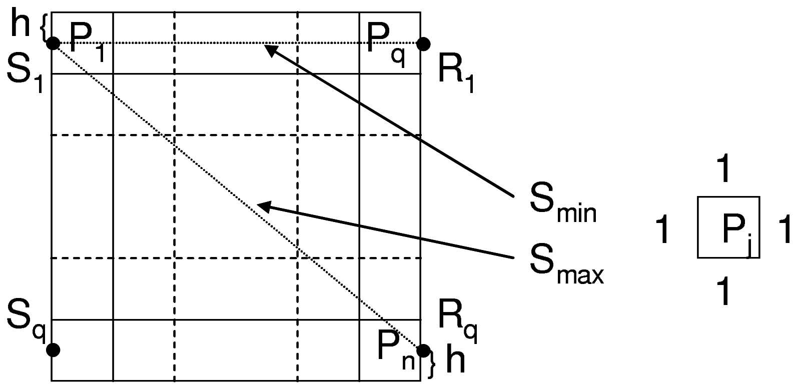


Fig. 2. Scanning process in EG.

**Theorem 2.** *Let $x^0 \in \mathbb{R}^n$ be the initial approximation. There exists two values $\epsilon_1, \epsilon_2 \in (0,1)$ such that, if the constants $c_i$ in (13) satisfy*

$$1 - \epsilon_2 \frac{S_i}{\|A_i\|^2} < c_i < 1 + \epsilon_1 \frac{S_i}{\|A_i\|^2},\ i = 1, \ldots, m, \tag{17}$$

*then the sequence $(x^k)_{k\geq 0}$ generated by the NWK algorithm converges to a solutions of (1). Moreover, if (17) holds and $x^0 = 0$, then $\lim_{k\to\infty} x^k = x_{LS}$.*

*Proof.* **Step 1.** Let $i \in \{1, \ldots, m\}$ be arbitrary fixed and $B_{-(i)}, B_{+(i)} \subset \{1, \ldots, n\}$, the sets of indices defined by (see (15))

$$B_{-(i)} = \{j, 0 \leq a_{ij} < 1\}\,, \quad B_{+(i)} = \{j, 1 \leq a_{ij} \leq \sqrt{2}\}. \tag{18}$$

Then, we get

$$q \leq \|A_i\|^2 = \sum_{j=1}^{n} a_{ij}^2 \leq M, \tag{19}$$

where

$$q < M = \text{card}(B_{-(i)}) + 2\ \text{card}(B_{+(i)}) < 2q. \tag{20}$$

From (16) and (19)–(20) we then get

$$-1 + \epsilon_1 \leq 1 - \frac{\|A_i\|}{S_i} \leq 1 - \epsilon_2, \tag{21}$$

where

$$\epsilon_2 = \frac{q}{\sqrt{q^2 + (q-1)^2}} \in \left(\frac{1}{2}, 1\right);\ \epsilon_1 = 2\frac{M}{q} \in (0,1)\,. \tag{22}$$

**Step 2.** According to Theorem 1, the convergence condition for the NWK algorithm will be

$$|1 - \tilde{\omega}_i| < 1 \Leftrightarrow -1 < 1 - \frac{c_i \|A_i\|^2}{S_i} < 1,\ \forall i = 1, \ldots, m. \tag{23}$$

But using (21) we obtain

$$-1+\epsilon_1+\frac{\|A_i\|^2}{S_i}(1-c_i)\leq 1-\frac{c_i\|A_i\|^2}{S_i}\leq 1-\epsilon_2+\frac{\|A_i\|^2}{S_i}(1-c_i). \tag{24}$$

Then, according to (23) we may ask that

$$1-\epsilon_2+\frac{\|A_i\|^2}{S_i}(1-c_i)<1 \tag{25}$$

and

$$-1<-1+\epsilon_1+\frac{\|A_i\|^2}{S_i}(1-c_i). \tag{26}$$

But, if $c_i$ $(i=1,\dots,m)$ are as (17), the previous two inequalities hold and the proof is complete.

## 3. The constrained version

Although the components of the original image $x^{ex}$ satisfy

$$0\leq x_i^{ex}\leq 1,\ \forall i=1,\dots,n, \tag{27}$$

during the computations with NWK algorithm (12) and because of the structure of $N(A)$ (see [6]), it may happen that components outside $[0,1]$ can appear in the solution $x_{LS}$. For eliminating this unpleasant aspect, which can generate “shadows” in the reconstructed image, in [7] was proposed a constraining strategy. According to this, after each iteration of (12) we must “force” the components of the current approximation $x^k$ to remain in $[0,1]$. By adapting these ideas to our NWK algorithm we get the following “constrained” version of if.

**Algorithm CNWK**: Let $x^0\in\mathbb{R}^n$; for $k=0,1,\dots$ do

$$\begin{array}{l} x=x^k \\ \text{for } i=1:m \\ x=x-\tilde{\omega}_i\frac{\langle x,A_i\rangle-b_i}{\|A_i\|^2}A_i \\ \text{end for} \\ \text{for } i=1:m \\ x_i^{k+1}=\min(1,\max(0,x_i)) \\ \text{end for}\end{array} \tag{28}$$

The formulation of the CNWK algorithm (28) tells us that Theorem 3 in [7] directly applies in the consistent case of (1) and we get the following convergence result.

**Theorem 3.** *If $\tilde{\omega}_i$ are as in* (13) *and $c_i$ satisfy* (17), *then the sequence $(x^k)_{k\geq 0}$ generated with the algorithm* (28) *converges to a solution of* (1). *Moreover, for $x^0=0$ we have $\lim_{k\to\infty}x^k=x_{LS}$.*

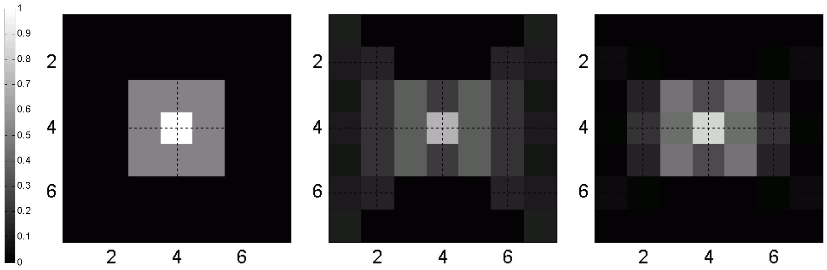

Fig. 3. Original image 1 and NWK and CNWK reconstructions.

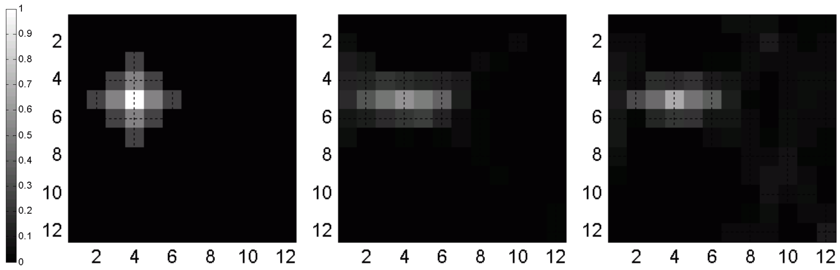

Fig. 4. Original image 2 and NWK and CNWK reconstructions.

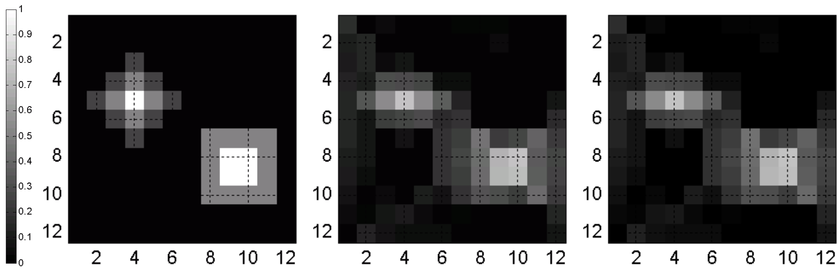

Fig. 5. Original image 3 and NWK and CNWK reconstructions.

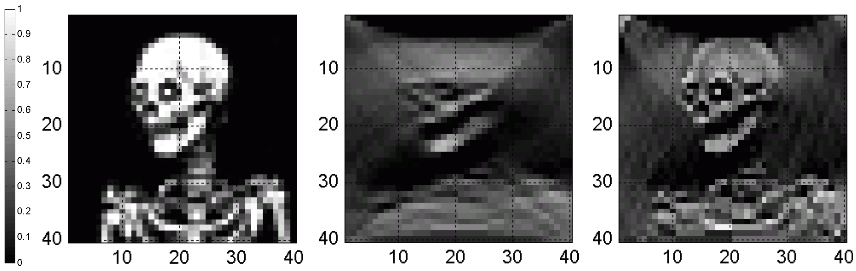

Fig. 6. Original image 4 and NWK and CNWK reconstructions.

## 4. Numerical experiments

In order to test NWK and CNWK algorithms, we selected a set of four images with various resolution and complexity (as in the numerical experiments from [7]).

The first three images have few unique colors, but their complexity increases because the resolution and/or the content complexity grows ($8 \times 8$, $12 \times 12$, and $12 \times 12$ pixels). The complexity of the fourth image is much larger because of a greater resolution ($40 \times 40$ pixels), color depth (64 unique colors), and complexity. The numbers of iterations used in these experiments for NWK and CNWK algorithms where respectively 50, 100, 100, and 200. All the experiments from below show the good behaviour of our NWK and CNWK algorithms.

## References

[1] Björck, A., *Numerical methods for least squares problems*, SIAM Philadelphia, 1996.

[2] Censor Y., Stavros A. Z., *Parallel optimization: theory, algorithms and applications*, Numer. Math. and Sci. Comp. Series, Oxford Univ. Press, New York, 1997.

[3] Herman, G.T., Lent, A., Lutz, P.H., *Relaxation methods for image reconstruction*, Communications of ACM, **21** (1978), 152–158.

[4] Kaczmarz S., *Angenaherte Auflosung von Systemen linearer Gleichungen*, Bull. Acad. Polonaise Sci. et Lettres **A** (1937), pp. 355–357.

[5] Koltracht I., Lancaster P. and Smith D., *The structure of some matrices arising in tomography*, Linear Algebra and its Applic., **130** (1990), 193–218.

[6] Koltracht I. and Lancaster P., *Constraining strategies for linear iterative processes*, IMA Journal of Numerical Analysis, **10** (1990), 555–567.

[7] Popa C. and Zdunek R., *Kaczmarz extended algorithm for tomographic image reconstruction from limited-data*, Math. and Computers in Simulation, **65** (2004), 579–598.