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Algebraic Full Multigrid in Image Reconstruction

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In this paper we propose an algebraic full multigrid algorithm for efficient and robust numerical solution of arbitrary linear systems of equations arising in image reconstruction from projections in computerized tomography. Numerical experiments and comparisons with the classical Kaczmarz algebraic reconstruction technique are presented.

1. Introduction

Tomographic reconstruction is the process of reconstructing an object or its cross section from several images of its projections. In the 2D case the object is illuminated by a fan-beam of X-rays, where the signal is attenuated by the object. Within ART the object is represented as a linear combination of basis functions, typically pixels, with some unknown coefficients. This leads to a linear system of equations with a sparse system matrix, because each observation is influenced only by the pixels on the corresponding beam path. The drawbacks of all ART techniques are the computational costs of the iterative formula applied to huge data sets (in practice the reconstruction of a 256^3 or 512^3 volume and 150 X-ray images of size 1024^2 is a common situation). Our idea in the present paper was to think of these iterative methods as smoothers within an Algebraic Full MultiGrid (AFMG) solver (cf. [5], [6]). From this view point we first tried to adapt the basic steps of a multigrid procedure – smoothing and correction (see [2]) – to general least squares problems. We shall essentially refer to Computerized Tomography (CT) image reconstruction from projections for medical investigations, in which the $m \times n$ matrix problem A,

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obtained after the scanning procedure (see Figure 1) is very big, sparse, rank-deficient and ill-conditioned . As an example, in a 2D experimental situation, if we use e.g. 150 sources and 1024 detectors and a 256×256 pixel resolution we get $m = 150 \times 1024 = 153600$ and $n = 256 \times 256 = 65656$. In the 3D case, these values are amplified to $m = 150 \times 1024^2 = 157286400$ and $n = 256^3 = 16777216$. In such cases, the matrix A cannot be any more stored in the computer memory (not even in a compressed form!), thus it has to be re-generated (row by row) during each iteration of a solver as, e.g. Kaczmarz projection algorithm. This bad aspect, together with the slow convergence of Kaczmarz's iterations (due to the ill-conditioning of A) can slow very much the reconstruction procedure, such that it becomes useless from a practical view point. For overcoming this difficulty in section 2 of the present paper we propose our AFMG algorithm. It uses in an efficient way the structure of pixels discretization of the image, together with the row-by-row generation of A. In section 3 we present numerical experiments and comparisons of our algorithm with the classical Kaczmarz one on a 2D Shepp-Logan phantom (see [3]). Former results on 3D real medical image can be found in [4].

2. The algebraic full multigrid algorithm

Let A be the $m \times n$ (sparse) matrix and $b \in \mathbb{R}^m$ the measurements vector obtained after the CT (medical) scanning procedure (see Figure 1). For simplifying the exposure, we shall suppose that the algebraic reconstruction system of equations is consistent and can be written in a classical formulation

$$Ax = b. \tag{1}$$

Because the matrix A is rank-deficient, the system (1) has infinitely many solutions, among which we are usually looking for the (unique) minimal norm one, denoted in what follows by x_{LS}. In our algebraic reconstruction problem, the dimension n of A represents the number of pixels (voxels in 3D) used for the discretization of the (square) scanned area. We shall suppose without restricting the generality that

$$n = 2^q \cdot 2^q = 2^{2q}, \tag{2}$$

for some $q \geq 1$, i.e. each edge of the square area containing the image is divided into 2^q equal parts. This will be the "finest grid" (Ω_h) of our multigrid type algorithm. We then consider a "coarser grid" (denoted by Ω_H) formed with 4 times bigger pixels (see Figure 2.(A) for $q = 2$).

Then, a schematic presentation of our AFMG algorithm with 2 grids is given in Figure 3. There, the meaning of the signs and arrows used is the following:

1. $\square$: exact solution on the coarse grid Ω_H
2. $\nearrow$: prolongation of the coarse grid solution from Ω_H to Ω_h
3. $\bullet$: q relaxation (Kaczmarz) sweeps on Ω_h applied to the prolongated solution

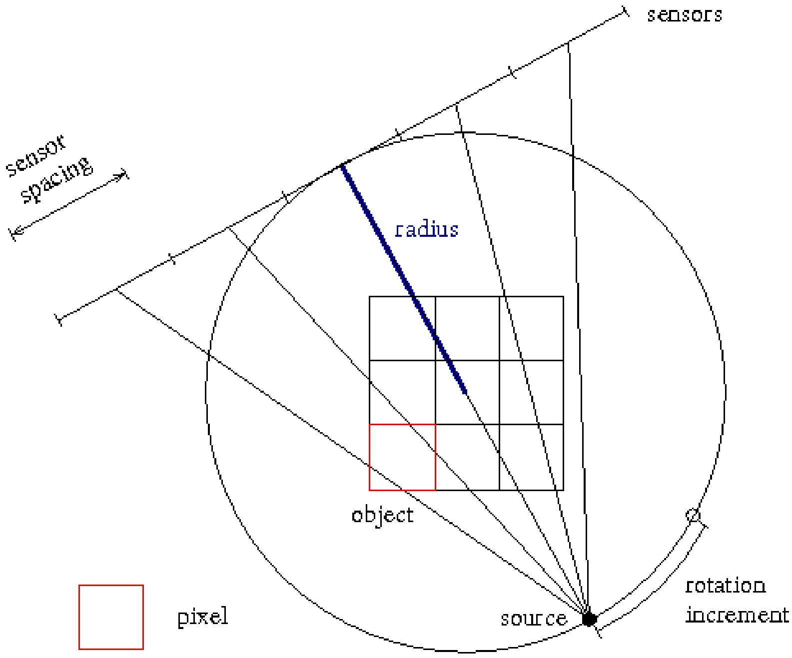

Fig. 1. Setup and construction of projection matrix.

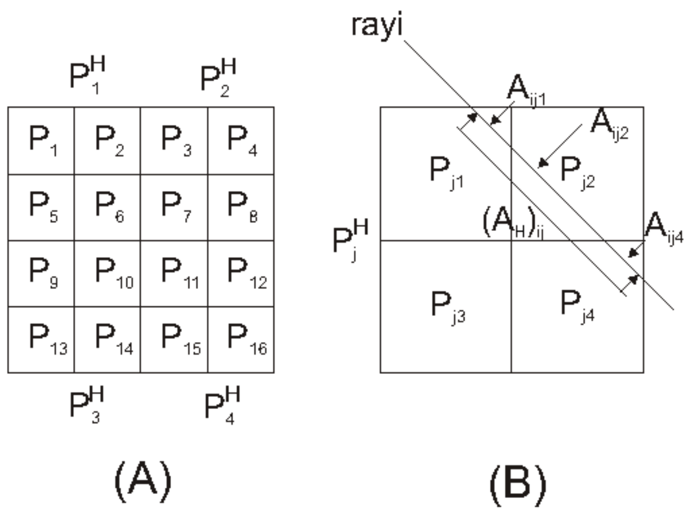

Fig. 2. Relation between fine and coarse grid matrix entries.

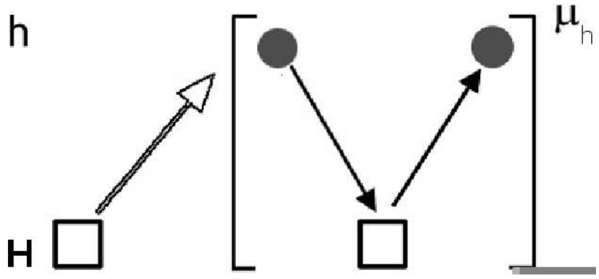

Fig. 3. AFMG algorithm (with 2 grids).

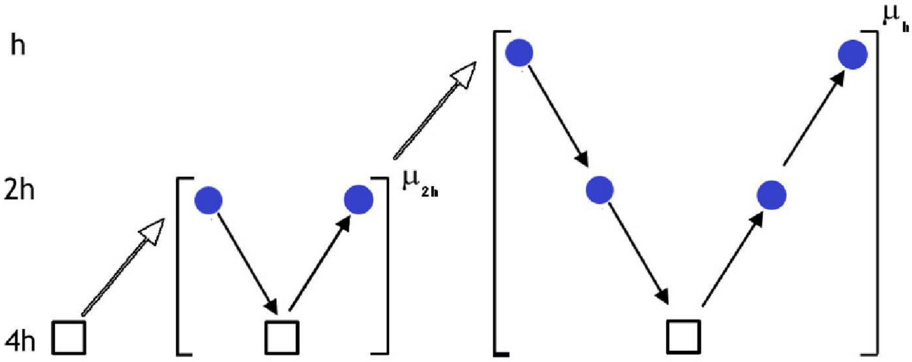

Fig. 4. AFMG algorithm (with 3 grids).

4. : restriction of the Ω_h, - residual to Ω_H
5. : prolongation of the Ω_H, - correction to Ω_h and addition to the corresponding previous Ω_h-approximation
6. $[\ \ ]^{\mu_h}$: the corresponding procedure (2 grid, V-cycle) is applied μ_h times

The construction of the prolongation and restriction operators, I_H^h, R_h^H, respectively and the coarse grid matrix A_H are presented below, with respect to the notations from Figure 2. For $j = 1, \ldots n/4,\ i = 1, \ldots, n,\ S(j) = \{j_1, j_2, j_3, j_4\}$ we have

$$(I_H^h)_{ij} = \begin{cases} 1 & , i \in S(j) \\ 0 & , i \notin S(j) \end{cases} \quad , \quad R_h^H = \text{ identity} : \mathbb{R}^m \longrightarrow \mathbb{R}^m, \tag{3}$$

$$A_H = A \cdot I_H^h,\ (A_H)_{ij} = \sum_{k \in S(j)} A_{ik},\ i = 1, \ldots, M,\ j = 1, \ldots, n/4. \tag{4}$$

With these elements our 2-grid AFMG algorithm, from Figure 3 can be written as follows.

Step 1. Compute exactly the minimal norm solution of the (possible inconsistent!) coarse grid problem

$$\| A_H \ x_{LS,H} - b \| = \min! \tag{5}$$

as

$$x_{LS,H} = A_H^+ b, \tag{6}$$

where A_H^+ is the Moore-Penrose pseudoinverse of A_H.

Step 2. Interpolate $x_{LS,H}$ to Ω_h as

$$x_h = I_H^h x_{LS,H}. \tag{7}$$

Step 3. Relax x_h, $\mu_h \geq 1$ times on Ω_h

$$x_h = \text{Kaczmarz}^{(\mu_h)}(x_h). \tag{8}$$

Step 4. Solve on Ω_H the least squares problem with the right hand side given by the residual $d_h = b_h - A_h x_h$ (see (3)),

$$\| A_H \ v_H - d_h \| = \min! \tag{9}$$

for v_H, (formally) given by

$$v_H = A_H^+ d_h. \tag{10}$$

Step 5. Interpolate v_H to Ω_h and correct x_h by

$$x_h = x_h + I_H^h v_h. \tag{11}$$

Step 6. Relax x_h, μ_h times on Ω_h

$$x_h = \text{Kaczmarz}^{(\mu_h)}(x_h). \tag{12}$$

Remark 1. *If we use more than* 2 *consecutive (pixels) discretization levels, the above algorithm can be easily extended (see Figure* 4 *for* 3 *consecutive grids:* $\Omega_h, \Omega_{2h}, \Omega_{4h}$*).*

The fact that, for the 2-grid AFMG algorithm in Figure 3, the exact coarse grid solution on Ω_H interpolated to Ω_h can be a better approximation than the (usual) null one, as starting point for the μ_h sweeps of the 2-grid V cycle on Ω_H and Ω_h, is described in the following result.

Proposition 1. *Let* $x_h^1 = 0, x_h^2 = I_H^h \ x_{LS,H}$ *and suppose that*

$$(I_H^h)^{\mathrm{T}} A^{\mathrm{T}} b \neq 0. \tag{13}$$

Then, the corresponding residuals on Ω_h

$$r_h^i = A x_h^i - b, \ i = 1, 2 \tag{14}$$

satisfy

$$\| r_h^2 \| \ < \ \| r_h^1 \| . \tag{15}$$

Proof. Because $x_h^1 = 0$ we get

$$r_h^1 = Ax_h^1 - b = -b. \tag{16}$$

For r_h^2, using (5) and the definition of A_H in (3) we successively get

$$r_h^2 = Ax_h^2 - b = AI_H^h A_H^+ b - b = A_H A_H^+ b - b =$$

$$P_{R(A_H)} b - b = -(I - P_{R(A_H)})b = -P_{N(A_H^{\mathrm{T}})} b,$$

where $P_S x$ denotes the orthogonal projection of x onto a vector subspace $S \subset \mathbb{R}^m$. Then, because of the orthogonal decomposition $\mathbb{R}^m = N(A_H^{\mathrm{T}}) \oplus R(A_H)$ (see [1]) we obtain

$$\| r_h^1 \|^2 \;=\; \| b \|^2 \;=\; \| P_{N(A_H^{\mathrm{T}})} b + P_{R(A_H)} b \|^2 \;=$$

$$\| P_{N(A_H^{\mathrm{T}})} b \|^2 + \| P_{R(A_H)} b \|^2 \;>\; \| P_{N(A_H^{\mathrm{T}})} b \|^2 \;=\; \| r_h^2 \|^2,$$

in which the last inequality is strict because of the assumption (15). This completes the proof.

Remark 2. *If* (15) *doesn't hold, then the normal equation associated to the coarse grid problem* (4) *becomes*

$$(I_H^h)^{\mathrm{T}} A^{\mathrm{T}} A \; I_H^h \; x_{LS,H} = (I_H^h)^{\mathrm{T}} A^{\mathrm{T}} b = 0$$

giving us $x_{LS,H} = 0$*, which is not the case in real practical applications.*

3. Numerical experiments

For the numerical results in 2D we have implemented the setup of the projection matrix and the solution of the least squares problem in Matlab. In the 3D case we used a C++ implementation and some results were communicated in [4]. Figure 5 shows the original image (size 256^2), a Shepp-Logan phantom available in Matlab and the corresponding sinogram (see [3]), computed by the Matlab routine *fanbeam*. For our experiments we resized the image to $n = 24^2 = 576$ and used $m = 39 \times 72 = 2808$ rays, i.e. 39 rays for each of the 72 positions of the source (see Figure 1). The structure of the projection matrix $A \in \mathbb{R}^{2808 \times 576}$ (that has full column rank for this setup) is shown in Figure 6. The results in Figure 7 show that the full algebraic multigrid method is able to reduce both the error (the euclidean norm of the difference between original image x^{ex} and the reconstructed one x^k) and the residual (the euclidean norm of $b - Ax^k$), during 10 iterations of the classical Kaczmarz ART method and our AFMG(2,0) algorithm (i.e. $\mu_h = 10$ in Figure 3). The right hand side b in (1) was defined as $b = Ax^{ex}$, where x^{ex} is the resized Shepp-Logan phantom (24×24). However, we can not expect the usual multigrid convergence rates for elliptic PDEs. In the medical application it is sufficient to use only a few Kaczmarz steps to get acceptable results. Due to the huge size of the real problems even only one Kaczmarz

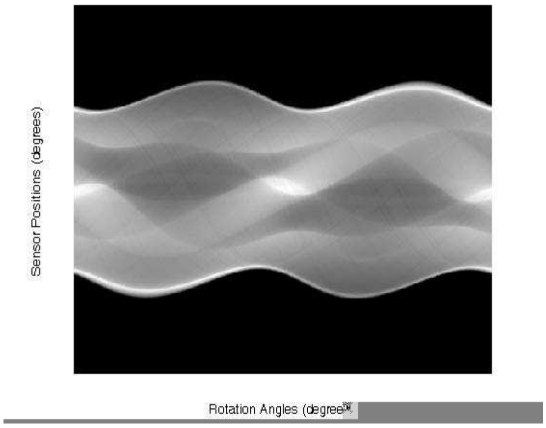

Fig. 5. Modified Shepp-Logan phantom (left) and sinogram (right).

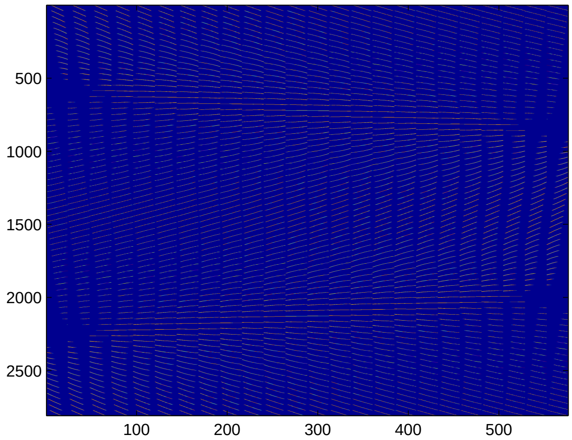

Fig. 6. Structure of projection matrix.

step on the finest level can take several minutes and therefore the computational time can be reduced drastically by saving one or more sweeps on the finest level.

Conclusions. We presented an algebraic full multigrid algorithm for efficient and robust numerical solution of arbitrary linear systems of equations arising in image reconstruction from projections in computerized tomography. Numerical experiments and comparisons with the classical Kaczmarz ART method show that our AFMG algorithm method can reduce the computational effort substantially.

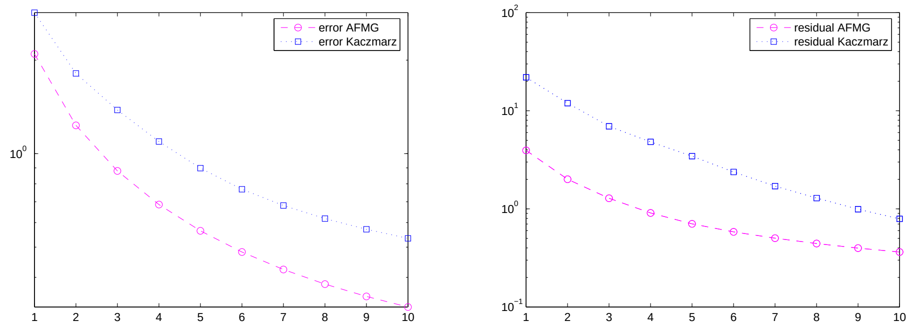

Fig. 7. Comparison of errors (top) and residuals (bottom) for AFMG(2,0)-cycles using only 1 level (Kaczmarz) and using 2 levels with a direct solver on the coarse grid.

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