Dedicated to Dr. TOADER MOROZAN
on the occasion of his 70th birthday

"LOST" CASES IN THE THEORY OF STABILITY FOR LINEAR TIME DELAY SYSTEMS

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Some basic process control systems with standard feedback controllers are described by second order linear equations with time lag whose characteristic equation may be reduced to

$$P(z) \equiv (a_2z^2 + a_1z + a_0)\cosh z + (b_2z^2 + b_1z + b_0)\sinh z = 0$$

The Routh-Hurwitz type conditions for this polynomial are given in a famous classical memoir of Čebotarev and Meiman. Due to a regrettable omission, some cases which are most probably to describe practical situations were not considered. The present paper aims to explain this paradoxical situation and to complete the existing results in a rigorous way.

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1. INTRODUCTION. STATE OF THE ART

A. One of the simplest and standard control problems in engineering is the PID control, i.e., the use of a standard industry controller with the transfer function $H_c(s) = K_R(1 + (T_is)^{-1} + T_ds)$ on some standard equivalent description of a process by $H(s) = K(1 + Ts)^{-1}e^{-\tau s}$. This problem leads to a linear system with the following characteristic equation whose RHS is a second order quasi-polynomial

$$KK_R\left(\frac{1}{T_i} + s + T_ds^2\right)e^{-\tau s} + s(1 + Ts) = 0. \tag{1}$$

The problem is known since the pioneering period of the automatic control – the 30ies of the XXth century. It is still an actual research field: we may cite the books of Górecki [3], Stépán [14], Górecki *et al* [4]. A group of quite recent engineering papers [11, 12, 13] offer an algorithmic way of solving parameter choice for the stability of these systems; these results are enclosed in an even more recent book.

B. In order to explain the aim of this paper let us consider the characteristic equation (1) for which the Routh–Hurwitz problem (i.e., localization of its roots in the half plane $\Re e(s) < 0$) is analyzed. The roots of (1) coincide with the roots of

$$(2)\quad KK_R\left(\frac{1}{T_i}+s+T_ds^2\right)\exp(-\tau s/2)+s(1+Ts)\exp(\tau s/2)=$$
$$=\left[\frac{KK_R}{T_i}+(1+KK_R)s+(KK_RT_d+T)s^2\right]\cosh(\tau s/2)+$$
$$+\left[-\frac{KK_R}{T_i}+(1-KK_R)s+(T-KK_RT_d)s^2\right]\sinh(\tau s/2)=0.$$

Introducing the new variable $z = \tau s/2$, the characteristic equation becomes

$$(3)\quad \left[\frac{KK_R}{T_i}+\frac{2}{\tau}(1+KK_R)z+\frac{4}{\tau^2}(KK_RT_d+T)z^2\right]\cosh z+$$
$$+\left[-\frac{KK_R}{T_i}+\frac{2}{\tau}(1-KK_R)z+\frac{4}{\tau^2}(T-KK_RT_d)z^2\right]\sinh z=0.$$

With the notation

$$(4)\qquad \gamma_p=KK_R,\quad \gamma_i=\frac{T}{T_i},\quad \gamma_d=\frac{T_d}{T},\quad \delta=\frac{2T}{\tau}$$

the characteristic equation becomes

$$(5)\qquad [\gamma_p\gamma_i+\delta(1+\gamma_p)z+\delta^2(1+\gamma_p\gamma_d)z^2]\cosh z+$$
$$+[-\gamma_p\gamma_i+\delta(1-\gamma_p)z+\delta^2(1-\gamma_p\gamma_d)z^2]\sinh z=0,$$

with the left hand side belonging to the class of quasi-polynomials

$$(6)\qquad p(z)=(a_2z^2+a_1z+a_0)\cosh z+(b_2z^2+b_1z+b_0)\sinh z$$

which were considered in the memoir of Čebotarev and Meiman [2] from the point of view of the Routh-Hurwitz problem.

It is a well known fact that for polynomials, the Routh-Hurwitz conditions are expressed through a finite set of inequalities and this was shown to be true for quasi-polynomials, too, in the sense that the solution is obtained after a procedure with a finite number of steps. To fix the ideas, let $a_0 > 0$, as in [2]. Then the inequalities

$$(7)\qquad a_1+b_0>0,\quad a_2+\frac{a_0}{2}+b_1>0,\quad a_2>0,\quad b_2>0,$$

called of Stodola type, are necessary for localization of the roots of (6) in the half plane $\Re e(s) < 0$.

Further, necessary and sufficient conditions were obtained for solving the Routh-Hurwitz problem. The approach in [2] is based on the Sturm procedure.

This approach which is valid for polynomials cannot be applied to all entire functions to which the quasi-polynomials belong; the necessary condition is the validity of the Hermite-Biehler theorem; this theorem is not valid for all entire functions, but *it is* for quasi-polynomials; therefore the approach of Sturm can be used in this case. We first have

PROPOSITION 1 (Theorem 5a in [2]). *If all the zeros of*

$$V(z) = a_2(\cos z)z^2 + b_1(\cos z)z - a_0 \sin z$$

are real, then a_0 *and* a_2 *have the same sign.*

PROPOSITION 2 (Theorem 5b in [2]). *If all the zeros of*

$$V_1(z) = -b_2(\sin z)z^2 + a_1(\cos z)z + b_0 \sin z$$

are real, then b_0 *and* b_2 *have the same sign.*

These results are used to eliminate some sign combinations of the coefficients of (6); since we fixed $a_0 > 0$, we have $2^5 = 32$ sign combinations but, after taking into account the two propositions above, only 4 of them are left as able to give the required results. These are the so-called Cases I–IV of [2], namely,

$$\begin{array}{llll} \mathrm{I}: & b_0 > 0;\ a_1 > 0,\ b_1 > 0; & \mathrm{II}: & b_0 > 0;\ a_1 < 0,\ b_1 < 0; \\ \mathrm{III}: & b_0 > 0;\ a_1 > 0,\ b_1 < 0; & \mathrm{IV}: & b_0 > 0;\ a_1 < 0,\ b_1 > 0. \end{array} \tag{8}$$

On the other hand the quasi-polynomial (5) does not fit these cases since a_0 and b_0 have opposite signs. The natural question would be: is always the feedback system composed of a first order “plant” with time delay and a PID controller unstable? The answer is negative since other methods of analyzing stability say so, and we may send the reader to various references including papers [11, 12, 13]. What then about the classical memoir [2]? An answer will be given below.

2. EXPLANATION FOR THE LOST CASES

In [2] is mentioned that the analysis of the stability inequalities for the quasi-polynomial (6) follows from the direct analysis performed in a Ph.D. Thesis. The main results of the thesis concerning (5) were published in [15]; this reference is cited several times in the years afterwards without any mention of possible mistakes. In the book of Górecki [3] it has been noticed (for the first time, prior to our knowledge) that the generally accepted stability cases I–IV do not cover some practical situations, in particular that described above – where $b_0 < 0$ while $b_2 > 0$. We were unable to discover any proof for the inequalities of the additional cases supplied there; a recent private communication [5], due to one of the co-authors of [3], mentions that the approach was rather formal: a series development of $\exp\{-\tau s\}$ in (5) and some determinant conditions imposed there. While it appears rather questionable as approach, it is as well questionable how some algebraic conditions may give such inequalities as some of those given below. This situation was stimulating to elucidate the problem. A preliminary account of the results is given in [8]. Our first step will be to explain the reasons of omitting some cases – corresponding to $b_0 < 0$ – in the analysis of [15] and, consequently, in that of [2]. The key lies in the apparent symmetry of Propositions 1 and 2 (Theorems 5a and 5b in [2]); this "symmetry" explains why only the proof of Proposition 1 (Theorem 5a) is given at some extent while the proof of Proposition 2 (Theorem 5b) is omitted. Our statement is that *Proposition* 2 *is false,* and we shall prove this by contradiction. Our main tool will be as in most studies on quasi-polynomials a result due to Pontryagin [7], see also the book of Bellman and Cooke [1]

Theorem 1. *Let*

$$f(z,u,v) = \sum_{m=0}^{r}\sum_{n=0}^{s} z^m \varphi_m^{(n)}(u,v) \tag{9}$$

be a polynomial in z, u, v with real coefficients, where $\varphi_m^{(n)}(u,v)$ are homogeneous polynomials of degree n in u and v, with $z^r\varphi_r^{(s)}(u,v)$ the principal term, and let

$$\Phi^{(s)}(z) = \sum_{n=0}^{s} \varphi_r^{(n)}(\cos z, \sin z). \tag{10}$$

If ε is such $\Phi^{(s)}(\varepsilon + \imath\omega) \neq 0$, $\forall\omega \in \mathbb{R}$, then $f(z, \cos z, \sin z)$ has only real roots iff for sufficiently large integers k it has exactly $4sk + r$ zeros within the band $-2k\pi + \varepsilon \leq \Re(z) \leq 2k\pi + \varepsilon$.

Proof of the falsity of Proposition 2. Consider the polynomial in Proposition 2, namely,

$$g(z,u,v) = -b_2vz^2 + a_1uz + b_0v,$$

hence $r = 2$, $s = 1$. Assume that b_0 and b_2 have opposite signs, i.e., $b_0 < 0$ since we know from the necessary conditions that $b_2 > 0$. Were Proposition 2 true we should find at least one combination of the coefficients of $V_1(z)$ such that this quasi-polynomial had non-real roots. We write $V_1(z) = 0$ as

$$(b_0 - b_2z^2)\sin z + (a_1\cos z)z = 0. \tag{11}$$

In turn, this equation can be written as

$$\tan z - \frac{a_1 z}{|b_0| + b_2z^2} = 0, \tag{12}$$

without loosing roots. Indeed, we might have lost the imaginary roots $\pm\imath\sqrt{|b_0|/b_2}$ but these are not roots since $\cos(\pm\imath\sqrt{|b_0|/b_2}) = -\cosh(\sqrt{|b_0|/b_2}) \neq 0$. We might have also lost the real roots $\nu\pi + \pi/2$ of $\cos z = 0$ but these also are not roots of (11) since $\sin(\nu\pi+\pi/2) = (-1)^\nu \neq 0$. It follows that the roots of (11) and (12) coincide.

Let $a_1 > 0$ to fix the ideas. The LHS (left hand side) of (12) is odd hence the analysis for $z > 0$ is sufficient; if $a_1 < 0$ then we may consider the case $z < 0$ and use the change of variable $a_1 z = \zeta > 0$. Denoting by $\psi(z)$ the rational function $a_1 z/(|b_0| + b_2 z^2)$, we find that

$$\psi(0) = 0; \quad \lim_{z\to\infty} \psi(z) = 0; \quad \psi'(z) = \frac{a_1(|b_0| - b_2 z^2)}{(|b_0| + b_2 z^2)^2},$$

hence $\psi(z)$ has a maximum corresponding to $z_M = \sqrt{|b_0|/b_2}$, namely

$$\psi(z_M) = \frac{a_1}{\sqrt{b_2|b_0|} + b_2} > 0.$$

Consider now some interval $(\nu\pi, (\nu + 1)\pi)$, $\nu \geq 1$. Within such an interval one may find a single root of (12) located between $\nu\pi$ and $\nu\pi + \pi/2$, the subinterval where $\tan z > 0$. Since $\tan z$ is monotonically increasing and $\psi(z)$ is monotonically decreasing for $z > z_M$, the root will be given by $\nu\pi + \delta_\nu$ where $\{\delta_\nu\}_\nu$ is a positive bounded sequence tending monotonically to 0. We deduce that there are always $2k - 1$ roots of (12) within the interval $[\pi, 2k\pi]$ hence there are other $2k - 1$ ones within the symmetric interval $[-2k\pi, -\pi]$. Within the central interval $[-\pi, \pi]$ one may find the root $z = 0$ and possibly 2 other ones, located between $(0, \pi/2]$ and $[-\pi/2, 0)$, respectively. For the existence of these roots we need to show that $-\psi(z) > 0$ in the neighborhood of 0, $z > 0$ This will follow from $-\psi'(0) > 0$; but $-\psi'(0) = a_1/|b_0| - 1 > 0$ provided $a_1 + b_0 > 0$. This last inequality has been assumed since it is a necessary condition for the location of the roots of (6) in $\mathbb{C}^-$ – see (7).

It follows that if the necessary conditions hold then there are exactly $4k + 1$ roots within the interval $[-2k\pi, 2k\pi]$ whatever $k > 0$. Let us consider now the shifted interval $[-2k\pi + \varepsilon, 2k\pi + \varepsilon]$, with $\varepsilon > 0$. Obviously, if $\varepsilon > 0$ is small enough, all $4k + 1$ roots still lie within this interval, too. Let now $k > 0$ be large enough, in order that $\delta_k < \varepsilon$; in this way the root of (12) from the interval $(2k\pi, (2k + 1)\pi)$ will be "caught" within the shifted interval $[-2k\pi + \varepsilon, 2k\pi + \varepsilon]$ for sufficiently large $k > 0$. Applying the result of Pontryagin [7], i.e., Theorem 1 here, we deduce that (12) hence (11) has only real roots in spite of our assumption that b_0 and b_2 have opposite signs. The assertion on *falsity* of Proposition 2 is proved. $\square$

We deduce now that the cases with $b_0 < 0$ cannot be eliminated from the stability analysis. If we take into account the sign combinations for a_1 and b_1, we obtain four additional cases. But the cases corresponding to $b_0 < 0$, $a_1 < 0$ have to be eliminated according to the necessary condition $a_1 + b_0 > 0$ which does not hold in these cases. We deduce that we have to consider additionally the two cases

$$\text{V}: \quad b_0 < 0,\ a_1 > 0;\ b_1 > 0; \qquad \text{VI}: \quad b_0 < 0,\ a_1 > 0;\ b_1 < 0, \tag{13}$$

which are exactly the cases mentioned in [3].

3. MAIN RESULT

The two additional cases (called conventionally as in [2, 3] V and VI will be analyzed separately, following the basic line of [2], i.e., by counting the sign changes in the Sturm sequence. Their number has to be, according to the results of [7], $4k+2$ (in the case of (6) – single delay and second degree of the principal term).

A. First of all we count those sign changes that are independent of the analyzed case hence independent of the fact that now $b_0 < 0$. Recall here the Sturm sequence of the problem:

$$\begin{aligned} (14)\quad & V(z) = a_2(\cos z)z^2 + b_1(\cos z)z - a_0 \sin z, \\ & V_1(z) = -b_2(\sin z)z^2 + a_1(\cos z)z + b_0 \sin z, \\ & V_2(z) = -b_2 \sin z(b_1 b_2 \sin^2 z + a_1 a_2 \cos^2 z)z + b_2(a_0 b_2 - a_2 b_0)\sin^2 z \cos z, \\ & V_3(z) = -b_2 \sin^3 z\, \Omega(z), \end{aligned}$$

where

$$\Omega(z) = A\cos^4 z + B\sin^2 z\cos^2 z + C\sin^4 z, \tag{15}$$

$$A = a_0 a_1^2 a_2 > 0,\ B = a_1 b_1(a_0 b_2 + a_2 b_0) - (a_0 b_2 - a_2 b_0)^2\ ,\ C = b_0 b_1^2 b_2 < 0.$$

Substituting $z = \pm 2\nu\pi + \varepsilon$ and neglecting the higher order terms with respect to ε we obtain, as in [2],

$$\begin{aligned} V(\pm 2\nu\pi + \varepsilon) &\approx a_2(2\nu\pi)^2 > 0, & V_1(\pm 2\nu\pi + \varepsilon) &\approx -b_2(2\nu\pi)^2\varepsilon > 0, \\ V_2(\pm 2\nu\pi + \varepsilon) &\approx \mp a_1 a_2 b_2(2\nu\pi)\varepsilon, & V_3(\pm 2\nu\pi + \varepsilon) &\approx -b_2 a_1^2 a_2 a_0 \varepsilon^3 < 0. \end{aligned}$$

We deduce that the number of the sign losses on $[-2k\pi + \varepsilon, 2k\pi + \varepsilon]$, where $k > 0$ is large and $k\varepsilon > 0$ also large, will be

$$P(-2k\pi + \varepsilon) - P(2k\pi + \varepsilon) = 2\operatorname{sgn} a_1. \tag{16}$$

We compute now the sign losses $\ell_{i\nu}$ when crossing the zeros $\nu\pi$ of $\sin z$, the multiplier of $V_3(z)$, where $i = 1, 2$ according to the type of the root: $i = 1$ when the root is of the first type and introduces a sign gain ($\ell_{i\nu} = 1$), and

$i = 2$ when the root is of the second type and introduces a sign loss ($\ell_{i\nu} = -1$). This analysis is also independent of b_0 hence we keep the result of [2], namely,

$$P(-2k\pi + \varepsilon) - P(2k\pi + \varepsilon) - \sum_i \sum_\nu \ell_{i\nu} = 4k + 2\operatorname{sgn} a_1. \tag{17}$$

The next multiplier in the Sturm sequence is given by $\Omega(z)$ and its zeros count in the sign losses provided they are real. But the zeros of the multiplier are real provided the zeros of $A\lambda^2 + B\lambda + C$ are real. Since we discuss the case $b_0 < 0$ and $C < 0$, this polynomial has always two real roots of opposite sign. Since only the positive root counts, we deduce that $\Omega(z) = 0$ has two real roots (mod π). Following [2], we use instead the equation

$$C \tan^4 z + B \tan^2 z + A = 0 \tag{18}$$

with $C < 0$, $A > 0$. The biquadratic equation

$$C\lambda^4 + B\lambda^2 + A = 0$$

has two real roots

$$\lambda_{1,2} = \pm\sqrt{\frac{B}{2|C|} + \sqrt{\left(\frac{B}{2|C|}\right)^2 + \frac{A}{|C|}}}$$

corresponding to the positive real root of the associated second degree equation, to which there correspond the roots of (18), namely

$$z_{1,\nu} = \nu\pi + \tau_1, \quad z_{2,\nu} = \nu\pi - \tau_1, \quad \nu = 0, \pm 1, \pm 2, \ldots \tag{19}$$

and

$$\tau_1 = \arctan\sqrt{\frac{B}{2|C|} + \sqrt{\left(\frac{B}{2|C|}\right)^2 + \frac{A}{|C|}}}, \quad 0 < \tau_1 < \pi/2. \tag{20}$$

Denoting $\tau_2 = \pi - \tau_1$, it follows that in each interval $(\nu\pi, (\nu + 1)\pi)$ we find 2 roots of (18) – or of $\Omega(z) = 0$ – namely, $z_{1,\nu} = \nu\pi + \tau_1$ such that $\nu\pi < z_{1,\nu} < \nu\pi + \pi/2$ and $z_{2,\nu} = (\nu+1)\pi - \tau_1 = \nu\pi + \tau_2$ such that $\nu\pi + \pi/2 < z_{2,\nu} < (\nu+1)\pi$.

Generally speaking, these values are not zeros of the quasi-polynomials V, V_1, V_2 of the Sturm sequence constructed according to [2]; this only happens if the coefficients of (6) are subject to some very special equalities – which clearly are "non-robust" and called "limit cases".

In the general cases, the sign losses $\ell_{i\nu}$ are determined by the behavior of the ratio V_2/V_3 given by

$$\frac{V_2(z)}{V_3(z)} = \frac{(a_1a_2 + b_1b_2 \tan^2 z)z - (a_0b_2 - a_2b_0) \tan z}{\cos^2 z \sin^2 z(C \tan^4 z + B \tan^2 z + A)} \tag{21}$$

when $z_{i\nu} = \nu\pi - \tau_i$, $i = 1, 2$, $\nu = 0, \pm 1, \dots$ If this ratio changes from $-$ to $+$ then $\ell_{i\nu} = +1$ and $z_{i\nu}$ is called a root of V_3 of 1st type; if the ratio changes from $+$ to $-$ then $\ell_{i\nu} = -1$ and $z_{i\nu}$ is called a root of 2nd type of V_3.

Consider first the sign changes of the ratio's denominator. Usual continuity arguments show that when crossing $z_{1\nu}$ the sign changes from $+$ to $-$ and when crossing $z_{2\nu}$ the change is from $-$ to $+$.

B. As known from [2], the behavior of the numerator $V_2(z)$ depends on each analyzed case.

Case V ($b_0 < 0,\ a_1 > 0;\ b_1 > 0$). This case is somehow alike *Case* I already analyzed in [2]: the coefficient of z in the numerator is positive for all z and the free term of the numerator, namely, $-(a_0b_2 - a_2b_0)\tan z = -(a_0b_2 + a_2|b_0|)\tan z$ is negative for $z_{1\nu} = \nu\pi + \tau_1$ and positive for $z_{2\nu} = \nu\pi + \tau_2$. We deduce that

a) for the roots $z_{2\nu} = \nu\pi + \tau_2$ the numerator is positive for all $\nu \geq 0$ and $\nu < 0$ of modulus sufficiently small; if $|\nu|$ for $\nu < 0$ increases, the term in z decreases and the numerator becomes negative; a $k_2 < 0$ may be defined from the change of the sign as satisfying the inequalities

$$(a_1a_2 + b_1b_2\tan^2(k_2\pi + \tau_2))(k_2\pi + \tau_2) - (a_0b_2 - a_2b_0)\tan(k_2\pi + \tau_2) > 0,$$

$$(a_1a_2 + b_1b_2\tan^2((k_2-1)\pi + \tau_2))((k_2-1)\pi + \tau_2) - (a_0b_2 - a_2b_0)\tan((k_2-1)\pi + \tau_2) < 0,$$

which lead after some simple manipulation to

$$k_2 - \frac{\tau_1}{\pi} + \frac{1}{\pi}\cdot\frac{(a_0b_2 - a_2b_0)\tan\tau_1}{a_1a_2 + b_1b_2\tan^2\tau_1} < 0 < k_2 + 1 - \frac{\tau_1}{\pi} + \frac{1}{\pi}\cdot\frac{(a_0b_2 - a_2b_0)\tan\tau_1}{a_1a_2 + b_1b_2\tan^2\tau_1},$$

hence

$$k_2 = \left[\frac{\tau_1}{\pi} - \frac{1}{\pi}\cdot\frac{(a_0b_2 - a_2b_0)\tan\tau_1}{a_1a_2 + b_1b_2\tan^2\tau_1}\right]_e; \tag{22}$$

b) for the roots $z_{1\nu} = \nu\pi + \tau_1$, the numerator is positive for $\nu > 0$ sufficiently large and negative for $\nu < 0$ and $\nu \geq 0$ sufficiently small; a $k_1 > 0$ may be defined from the change of the sign, finally given by

$$k_1 = \left[-\frac{\tau_1}{\pi} + \frac{1}{\pi}\cdot\frac{(a_0b_2 - a_2b_0)\tan\tau_1}{a_1a_2 + b_1b_2\tan^2\tau_1}\right]_e + 1. \tag{23}$$

In the following we shall count $\sum_i\sum_\nu \ell_{i\nu}$ as follows. We consider an interval $[-2k\pi + \varepsilon, 2k\pi + \varepsilon]$ with $k > 0$ sufficiently large, i.e., larger than $\max\{k_1, -k_2\}$ and also than that k for which we showed that Proposition 2 was false; $\varepsilon > 0$ is such that $k\varepsilon$ is still very large e.g. $\varepsilon = k^{-1/7}$.

Now, for the intervals $(\nu\pi, (\nu+1)\pi)$ with $\nu \leq k_2 - 1$ we find easily that $\ell_{1\nu} = +1$, $\ell_{2\nu} = -1$, hence the sum is 0. For the intervals with $k_2 \leq \nu \leq k_1 - 1$, we deduce $\ell_{i\nu} = 1$, $\nu = 1, 2$, hence $\sum_{k_2}^{k_1-1}\sum_i \ell_{i\nu} = 2(k_1 - k_2)$. For $\nu \geq k_1$ we deduce again that the sum is zero. Therefore, the real roots of (21) introduce now $2(k_1 - k_2)$ sign changes and, since $a_1 > 0$, the overall number of the sign changes will be

$$N_1 - N_2 = 4k + 2 - 2(k_1 - k_2)$$

while the Pontryagin type result requires $N_1 - N_2 \geq 4k + 2$. Therefore the necessary and sufficient condition will be $k_1 - k_2 = 0$, i.e., $k_1 = k_2$. Using (22) and (23) we deduce the necessary and sufficient condition

$$\left[-\frac{\tau_1}{\pi} + \frac{1}{\pi} \cdot \frac{(a_0b_2 - a_2b_0)\tan\tau_1}{a_1a_2 + b_1b_2\tan^2\tau_1}\right]_e + 1 = \left[\frac{\tau_1}{\pi} - \frac{1}{\pi} \cdot \frac{(a_0b_2 - a_2b_0)\tan\tau_1}{a_1a_2 + b_1b_2\tan^2\tau_1}\right]_e, \tag{24}$$

or

$$\left[-\frac{\tau_1}{\pi} + \frac{1}{\pi} \cdot \frac{(a_0b_2 - a_2b_0)\tan\tau_1}{a_1a_2 + b_1b_2\tan^2\tau_1}\right]_e = \left[-\frac{\tau_2}{\pi} + \frac{1}{\pi} \cdot \frac{(a_0b_2 - a_2b_0)\tan\tau_2}{a_1a_2 + b_1b_2\tan^2\tau_2}\right]_e. \tag{25}$$

Equation (25) is exactly the one given without proof by Górecki [3].

Case VI ($b_0 < 0,\ a_1 > 0;\ b_1 < 0$). In this case the coefficient of z in the numerator is positive for small $z > 0$ and decreases on $(0, \pi/2)$ since $b_1 < 0$. To see the sign for $z = \tau_1$ we compare $\tan^2\tau_1$ which corresponds to the positive root of the second degree equation associated with (22) and $-(a_1a_2)/(b_1b_2)$ which makes the coefficient 0. We deduce easily that

$$|C|\left(-\frac{a_1a_2}{b_1b_2}\right)^2 + B\frac{a_1a_2}{b_1b_2} - A = -\frac{a_1a_2}{b_1b_2}(a_0b_2 - a_2b_0)^2 > 0,$$

hence $-(a_1a_2)/(b_1b_2) > \tan^2\tau_1$. The coefficient of z in the numerator is thus positive in some neighborhood of the root $z_{i\nu}$ where the sign change is counted. The free term of the numerator is as previously. We deduce that the analysis coincides with the previous one, k_1 and k_2 are determined as previously and the stability conditions are (24) and (25) as previously.

It follows that in this case the formulae of Górecki [3], given without proof, *are not correct* ; one may suppose that they have been obtained from a supposed analogy of Case VI and Case III.

To conclude this section we shall consider (24) in some detail. From the well known equality

$$[x]_e + [-x]_e = -1,$$

valid for non-entire x, for (24) we obtain

$$\left[\frac{\tau_1}{\pi} - \frac{1}{\pi} \cdot \frac{(a_0b_2 - a_2b_0)\tan\tau_1}{a_1a_2 + b_1b_2\tan^2\tau_1}\right]_e = 0,$$

hence

$$-1 < -\frac{\tau_1}{\pi} + \frac{1}{\pi} \cdot \frac{(a_0b_2 - a_2b_0)\tan\tau_1}{a_1a_2 + b_1b_2\tan^2\tau_1} < 0,$$

or

$$-\pi + \tau_1 < \frac{(a_0b_2 - a_2b_0)\tan\tau_1}{a_1a_2 + b_1b_2\tan^2\tau_1} < \tau_1.$$

But we already showed that the denominator of the ratio above is positive, as well as the numerator, while $-\pi + \tau_1 < 0$ since $0 < \tau_1 < \pi/2$. It follows that (24) is fulfilled provided

$$\frac{(a_0b_2 - a_2b_0)\tan\tau_1}{a_1a_2 + b_1b_2\tan^2\tau_1} < \tau_1. \tag{26}$$

4. APPLICATIONS

We have already applied the above results to the motivational application of the first order object with PID controller [8], to the vibration control for a mechanical rotor [9], too. Here we shall consider another application – the delayed resonator [6]. The characteristic equation of the controlled system is

$$p(s) \equiv m_a s^2 + c_a s + k_a + g\,\mathrm{e}^{-s\tau} = 0 \tag{27}$$

with all coefficients being strictly positive. Its equivalent equation will be

$$\begin{aligned}\tilde{p}(z) \equiv \mathrm{e}^z p(2z/\tau) \equiv \left(\frac{4m_a}{\tau^2}z^2 + \frac{2c_a}{\tau}z + k_a + g\right)\cosh\ z+ \\ + \left(\frac{4m_a}{\tau^2}z^2 + \frac{2c_a}{\tau}z + k_a - g\right)\sinh\ z = 0.\end{aligned} \tag{28}$$

The necessary (but not sufficient) stability conditions of Stodola type read here as (see (7))

$$\frac{4m_a}{\tau^2} + \frac{k_a + g}{2} + \frac{2c_a}{\tau} > 0, \quad g < k_a + \frac{2c_a}{\tau}. \tag{29}$$

The first one is automatically satisfied since the coefficients are positive, while the second one gives an upper limit of the control gain g which diminishes with the increase of the delay τ. Letting $\tau \to \infty$ one gets $g < k_a$ – a "small gain condition" what suggests that small gain could ensure *delay independent stability*. For this reason we first discuss a standard case to which we are directed by the small gain condition. In this case all coefficients of (28) are positive (Case I) and we have to check the inequality

$$|\sqrt{a_0b_2} - \sqrt{a_2b_0}\,| - \sqrt{a_1b_1} < 0$$

which (after some manipulation) reads

$$g < \frac{c_a}{2m_a}\sqrt{4m_ak_a - c_a^2}.$$

This may hold provided $c_a < 2\sqrt{m_ak_a}$, i.e., for a limited damping. Otherwise, we shall have a delay-dependent condition. We do not insist on this case any longer since our development is concerned with $g > k_a$ – the "high gain". Condition (26) will yield

$$\frac{\tau g}{c_a} < \tau_1(\tan\ \tau_1 + \cot\ \tau_1),$$

$\tan\ \tau_1$ being the positive root of the auxiliary equation (18) for the corresponding expressions of the coefficients a_i, b_i, $i = 1, 2$. Note that these expressions do not depend on the delay τ, but they *do depend* on the gain $g > 0$. Therefore, if we fix the gain we obtain the upper limit of the delay while the upper limit of the gain could have a quite complicated dependence on the coefficients m_a, c_a, k_a and the delay $\tau > 0$.

5. CONCLUSIONS

We have presented here the rigorous proof (in fact a sketch) of some stability cases for time lag systems of second order that have been omitted ("lost") by a mistake in treating an auxiliary equation. The practitioners have not been in fact very affected by this omission since they were using complementary approaches. From the mathematical point of view, this is a completion of the classical results.

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