Dedicated to Dr. TOADER MOROZAN
on the occasion of his 70th birthday

STOCHASTIC HOPFIELD NETWORKS WITH MULTIPLICATIVE NOISE AND JUMP MARKOV PARAMETERS

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A non symmetric version of Hopfield networks subject to state-multiplicative noise, pure time delay and Markov jumps is considered. Such networks seem to arise in the context of visuo-motor control loops and may, therefore, be used to mimic their complex behaviour. In this paper, we adopt the Lur'e-Postnikov systems approach to analyze the stochastic stability and the L_2 gain of generalized Hopfield networks including these effects.

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1. INTRODUCTION

Hopfield networks are symmetric recurrent neural networks which exhibit motions in the state space which converge to minima of energy. Symmetric Hopfield networks can be used to solve practical complex problems such as implement associative memory, linear programming solvers and optimal guidance problems. Recurrent networks which are non symmetric versions of Hopfield networks seem to play an important role in understanding human motor tasks involving visual feedback (see [2]–[3] and the references therein). Such networks seem to be subject to effects of state-multiplicative noise, pure time delay (see [10], [11] and [16]) and even multiple attractors which can be caused by Markov jumps. In this paper, we adopt the Lur'e-Postnikov systems approach to analyze the stochastic stability and L_2 gain of generalized Hopfield networks including the effects of state-multiplicative noise and Markov jumps. The paper is organized as follows. In Section 2, the problem is formulated while in Section 3 Linear Matrix Inequalities (LMIs) based conditions are derived for L_2 gain analysis. The case of a fixed and known pure time delay which is encountered in [2]–[3] is considered, using Padé approximations, in Section 4 to derive stability and disturbance attenuation properties of the stick balancing dynamics. Finally, Section 5 includes concluding remarks. Throughout the paper $\mathcal{R}^n$ denotes the n dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, (respectively, $P \geq 0$) for $P \in \mathcal{R}^{n\times n}$ means that P is symmetric and positive definite (respectively, semi-definite). Throughout the paper $(\Omega, \mathcal{F}, P)$ is a given probability space; the argument $\theta \in \Omega$ will be suppressed. Expectation is denoted by $E\{\cdot\}$ and conditional expectation of x on the event $\theta(t) = i$ is denoted by $E[x|\theta(t) = i]$.

2. PROBLEM FORMULATION

The neural network proposed by Hopfield, can be described by an ordinary differential equation of the form

$$\dot{v}_i(t) = a_i v_i(t) + \sum_{j=1}^{n} b_{ij} g_j(v_j(t)) + \bar{c}_i = \kappa_i(v), \quad 1 \leq i \leq n, \tag{2.1}$$

where v_i represents the voltage on the input of the ith neuron, $a_i < 0$, $1 \leq i \leq n$, $b_{ij} = b_{ji}$ and the activations $g_i(\cdot)$, $i = 1, \ldots, n$ are C^1-bounded and strictly increasing functions. This network is usually analyzed by defining the network energy functional

$$E(v) = -\sum_{i=1}^{n} a_i \int_0^{v_i} u \frac{\mathrm{d}g_i(u)}{\mathrm{d}u} \mathrm{d}u - \frac{1}{2} \sum_{i,j=1}^{n} b_{ij} g_i(v_i) g_j(v_j) - \sum_{i=1}^{n} \bar{c}_i g_i(v_i), \tag{2.2}$$

where it can be seen that $\frac{\mathrm{d}E}{\mathrm{d}t} = -\sum \frac{\mathrm{d}g_i(v_i)}{\mathrm{d}v_i} \kappa_i(v)^2 \leq 0$, where the zero rate of the energy is obtained only at the equilibrium points, also referred to as attractors, where

$$\kappa_i(v^0) = 0, \quad 1 \leq i \leq n. \tag{2.3}$$

However, the neural network may be subject to environmental noise and to connection matrix perturbations which can be modelled as $\sum_{j=1}^{n} d_{ij} w_j(t)$ added to the right hand side of (2.1). The network subject to the combination of these two effects can be then described in matrix form as

$$\dot{v}(t) = Av(t) + Bg(v) + Dw(t) + \bar{C}, \quad 1 \leq i \leq n, \tag{2.4}$$

where

$$A := \mathrm{diag}(a_1, \ldots, a_n), \quad B := [b_{ij}]_{i,j=1,\ldots,n}, \quad \bar{C} := \begin{bmatrix} \bar{c}_1 & \bar{c}_2 & \cdots & \bar{c}_n \end{bmatrix}^{\mathrm{T}},$$
$$v := \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^{\mathrm{T}}$$

and $g(v) := \begin{bmatrix} g_1(v_1) & g_2(v_2) & \cdots & g_n(v_n) \end{bmatrix}^{\mathrm{T}}$ The stochastic version of this network driven by white noise, has been considered in [10] where the stochastic stability of (2.1) has been analyzed and where it has been shown that the network is almost surely stable when the condition $\frac{\mathrm{d}E}{\mathrm{d}t} \leq 0$ is replaced by $\mathcal{L}E \leq 0$, where $\mathcal{L}$ is the infinitesimal generator associated with the Itô type stochastic differential equation (2.4). This condition has been shown in [10] to be only satisfied in cases where the driving noise in (2.1) is not persistent. This non persistent white noise can be interpreted as a white-noise type uncertainty in A and B, but it does not infer any stability results for the practical case of real uncertainties. Hopfield [15] considered networks with Markov jump parameters. Encouraged by the insight gained in [2] and [3] regarding the role of state-multiplicative noise and time delay (see also [13]) in visuo-motor control loops, we generalize the results of [15] to include this effect. The Lur'e-Postnikov systems approach ([14], [1]) is invoked to analyze stability and disturbance attenuation (in the H_∞ norm sense) and the results are given in terms of Linear Matrix Inequalities (LMI).

To analyze the effect of $w(t)$ we first define the error of the Hopfield network output with respect to its equilibrium points by

$$x(t) = v(t) - v^0. \tag{2.5}$$

and assume that the errors vector $x(t)$ satisfy

$$\begin{aligned} \mathrm{d}x = {} & (A_0(\theta(t))\, x + B_0(\theta(t))\, f(x) + Dw)\mathrm{d}t + \\ & + A_1(\theta(t))\, x\mathrm{d}\eta + B_1(\theta(t))\, f(x)\mathrm{d}\xi, \end{aligned} \tag{2.6}$$

where the system output is

$$z = L(\theta(t))\, x \tag{2.7}$$

and $\eta(t)$ and $\xi(t)$ are mutually independent standard Wiener processes. Note that (2.6) was obtained from (2.4) by replacing $A\mathrm{d}t$ by $A_0\mathrm{d}t + A_1\mathrm{d}\xi$, $B\mathrm{d}t$ by $B_0\mathrm{d}t + B_1 g\mathrm{d}\xi$ and $f(x) = g(x + v_0) - g(v_0)$. Note that $A_0(\theta(t)), A_1(\theta(t))$, $B_0(\theta(t)), B_1, D(\theta(t))$ and $L(\theta(t))$ are piecewise constant matrices of appropriate dimensions whose entries are dependent upon the mode $\theta(t) \in \{1, \ldots, r\}$, where r is a positive integer denoting the number of possible models between which the Hopfield network parameters can jump. Namely, $A_0(\theta(t))$ attains the values of $A_{0,1}, A_{0,2}, \ldots, A_{0,r}$, etc. It is assumed that $\theta(t), t \geq 0$ is a right continuous–time homogeneous Markov chain on $\mathcal{D} = \{1, \ldots, r\}$ with a probability transition matrix

$$P(t) = \mathrm{e}^{Qt}; \quad Q = [q_{ij}]; \quad q_{ii} < 0; \quad \sum_{j=1}^{r} q_{ij} = 0; \quad i = 1, 2, \ldots, r. \tag{2.8}$$

Given the initial condition $\theta(0) = i$, at each time instant t, the mode may maintain its current state or jump to another mode $i \neq j$. The transitions between the r possible states $i \in \mathcal{D}$ may be the result of random fluctuations of the actual network components (i.e., resistors, capacitors) characteristics or can used to artificially model deliberate jumps which are the result of parameter changes in an optimization problem the network is used to solve. In visuo-motor tasks one may conjecture that proportional and derivative feedbacks are applied on the basis of time sharing, where transition probabilities define the statistics of switching between the two modes. Although there is no evidence for this conjecture, the model analyzed in the present paper can be used to check its stability and L_2 gain. The model (2.6), (2.7) will be analyzed in Section 3. To include the delay effect needed to derive stability and disturbance attenuation properties for the stick balancing problem of [2]–[3] we will employ Padé (see e.g. [19]) approximations of the time-delay. Namely, the system

$$\begin{aligned} \mathrm{d}x(t) &= \left[A_0\left(\theta(t)\right)x(t) + B_0\left(\theta\left(t\right)\right)f\left(x\left(t-\tau\right)\right) + D\left(\theta(t)\right)w(t)\right]\mathrm{d}t \\ &\quad + A_1\left(\theta(t)\right)x(t)\mathrm{d}\eta(t) + B_1\left(\theta\left(t\right)\right)f\left(x\left(t-\tau\right)\right)\mathrm{d}\xi\left(t-\tau\right), \end{aligned} \tag{2.9}$$

$$z(t) = L\left(\theta\left(t\right)\right)x\left(t\right), \tag{2.10}$$

where $\xi(t)$ is another standard Wiener process independent of η and τ is the pure time delay, will be replaced by

$$\begin{aligned} \mathrm{d}x(t) &= \left[A_0\left(\theta(t)\right)x(t) + B_0\left(\theta(t)\right)f\left(\delta(t)\right) + D\left(\theta(t)\right)w(t)\right]\mathrm{d}t \\ &\quad + A_1\left(\theta(t)\right)x(t)\mathrm{d}\eta(t) + B_1\left(\theta(t)\right)f\left(\delta\left(t\right)\right)\mathrm{d}\xi(t), \end{aligned} \tag{2.11}$$

$$z(t) = L\left(\theta(t)\right)x\left(t\right), \tag{2.12}$$

where $\delta(t)$ is the Nth order Padé approximation to $x(t-\tau)$. Choosing the so called diagonal Padé approximation (see [19]) the i element $x_i(t-\tau)$ is approximated by the output $\delta_i(t)$ of the system feeded by $x_i(t)$ and where the transfer-function of which is

$$h_i(s) = \frac{\sum_{k=0}^{N}\mathcal{A}_n s^n}{\sum_{k=0}^{N}\mathcal{B}_n s^n}.$$

The coefficients in this equation where $\mathcal{B}_0 = 1$ can be calculated using [19]:

$$\Phi\begin{bmatrix}\mathcal{B}_1\\ \mathcal{B}_2\\ \vdots\\ \mathcal{B}_N\end{bmatrix} = -\begin{bmatrix}a_1\\ a_2\\ \vdots\\ a_N\end{bmatrix},$$

where $\Phi_{ij} = a_{N+i-j}$ and $\sum_{k=0}^{n} a_k s^k$ is the nth order power series of $\mathrm{e}^{-s\tau}$. The coefficients $\mathcal{A}_0, \ldots \mathcal{A}_N$ are then obtained by $\mathcal{A}_k = \sum_{j=0}^{k} a_{k-j}\mathcal{B}_j$. Taking $N = 1$, the first order Padé approximation to $\mathrm{e}^{-s\tau}$ is just $h_i(s) = \frac{1-s\tau/2}{1+s\tau/2}$. Namely, $\delta_i = 2\zeta_i - x_i$, where $\dot{\zeta}_i = -\frac{2}{\tau}\zeta_i + \frac{2}{\tau}x_i$. The Padé approximation allows to express the time-delay equation (2.11) in the form (2.6) without delay but rewritten in the extended state space $[x^{\mathrm{T}}\ \ \zeta^{\mathrm{T}}]^{\mathrm{T}}$ where $\zeta = [\zeta_1, \ldots, \zeta_n]^{\mathrm{T}}$.

However, this representation of the delay requires dependence of $f(\cdot)$ on linear combinations of the state-vector components, rather than the components. We, therefore, reformulate (2.6), (2.7) as

$$\begin{aligned} \mathrm{d}x = &(A_0(\theta(t))x + B_0(\theta(t))f(y) + Dw)\mathrm{d}t + \\ &+ A_1(\theta(t))x\mathrm{d}\eta + B_1(\theta(t))f(y)\mathrm{d}\xi, \end{aligned} \tag{2.13}$$

where the system output is

$$z = L(\theta(t))x \tag{2.14}$$

and

$$y = C(\theta(t))x. \tag{2.15}$$

Remind that in the time–delay case the state matrices are expanded with the corresponding dynamics of the Padé approximation. Only for the sake of simplicity we kept the above equations an identic notation with the case without delay. In the forthcoming analysis, we will assume that the components f_i, $i = 1, \ldots, n$ of $f(y)$ satisfy the sector conditions

$$0 \le y_i f_i(y_i) \le \sigma_i y_i^2 \tag{2.16}$$

which are equivalent to

$$-F_i(y_i, f_i) := f_i(y_i)(f_i(y_i) - \sigma_i y_i) \le 0. \tag{2.17}$$

We shall further assume that

$$\frac{\partial f_i}{\partial y_i} \le \sigma_i, \quad i = 1, \ldots, n. \tag{2.18}$$

Although the latter assumption of (2.18) further restricts the sector-type one class of (2.17), the applicability of our results remains since it is fulfilled by the usual nonlinearities as saturation, sigmoid, etc., used in the neural networks. Some additional notation is now in place. Define

$$S = \mathrm{diag}\begin{pmatrix} \sigma_1, & \sigma_2, & \cdots, & \sigma_n \end{pmatrix},$$

where σ_i are the nonlinearity gains of (2.17).

As mentioned above, we shall analyze stability (in stochastic sense) conditions for systems (2.6), (2.7) and (2.11), (2.10), respectively, together with the L_2 gain boundness condition expressed as

$$J = E\left\{\int_0^\infty \left(z^{\mathrm{T}}(t)z(t) - \gamma^2 w^{\mathrm{T}}(t)w(t)\right)\mathrm{d}t\right\} < 0, \quad x(0) = 0. \tag{2.19}$$

3. L_2 GAIN ANALYSIS

Introduce the Lyapunov-type function

$$V\left(x(t),\theta(t)\right)=x^{\mathrm{T}}(t)P\left(\theta(t)\right)x(t)+2\sum_{k=1}^{n}\lambda_k\int_0^{C_kx}f_k\left(s\right)\mathrm{d}s \tag{3.20}$$

depending on the nonlinearities $f_i(y_i)=f_i(C_ix)$ via the constants λ_i, where C_i is the ith row in C. As was mentioned in [1], V defines a parameter-dependent Lyapunov function. To see this, consider the simple case of $f_i(x_i)=x_i\sigma_i$ and get $V(x,\sigma_1,\sigma_2,\ldots,\sigma_n)=x^{\mathrm{T}}(P+S^{\frac{1}{2}}\Lambda S^{\frac{1}{2}})x$ which depends on the parameters σ_i, $i=1,2,\ldots,n$ and on the constants λ_i, $i=1,2,\ldots,n$ via (3.23) in the sequel. Applying the Itô-type formula (see [5], [8]) for $V\left(x,\theta\right)$, it follows that

$$E\left\{V\left(x,\theta(t)|\theta\left(0\right)\right)\right\}-E\left\{V\left(0,\theta\left(0\right)|\theta\left(0\right)\right)\right\}=E\left\{\int_0^t\mathcal{L}V\left(x,\theta\left(s\right)\right)\mathrm{d}s\right\},$$

where

$$\begin{aligned}\mathcal{L}V\left(x,\theta\right):=&\left(A_0\left(\theta\right)x+B_0\left(\theta\right)f\left(x\right)+D\left(\theta\right)w\right)^{\mathrm{T}}\frac{\partial V}{\partial x}\\&+x^{\mathrm{T}}A_1^{\mathrm{T}}\left(\theta\right)\left(P\left(\theta\right)+\mathrm{diag}\left(\lambda_1\frac{\partial f_1}{\partial x_1},\ldots,\lambda_n\frac{\partial f_n}{\partial x_n}\right)\right)A_1\left(\theta\right)x\\&+f^{\mathrm{T}}B_1^{\mathrm{T}}\left(\theta\right)\left(P\left(\theta\right)+\mathrm{diag}\left(\lambda_1\frac{\partial f_1}{\partial x_1},\ldots,\lambda_n\frac{\partial f_n}{\partial x_n}\right)\right)B_1\left(\theta\right)f+\sum_{j=1}^{r}q_{ij}x^{\mathrm{T}}P_jx\end{aligned}$$

and where for simplicity we have used the notation $f:=f(y(t))$. Then condition (2.19) is satisfied if

$$\mathcal{L}V\leq\gamma^2w^{\mathrm{T}}w-z^{\mathrm{T}}z, \tag{3.21}$$

which becomes

$$\begin{aligned}&\left(x^{\mathrm{T}}A_{0i}^{\mathrm{T}}+f^{\mathrm{T}}B_{0i}^{\mathrm{T}}+w^{\mathrm{T}}D_i^{\mathrm{T}}\right)\left(P_ix+C^{\mathrm{T}}\Lambda f\right)+\\&+\left(x^{\mathrm{T}}P_i+f^{\mathrm{T}}\Lambda C\right)\left(A_{0i}x+B_{0i}f+D_iw\right)+\\&+x^{\mathrm{T}}A_{1i}^{\mathrm{T}}\left(P_i+\mathrm{diag}\left(\lambda_1\frac{\partial f_1}{\partial x_1},\ldots,\lambda_n\frac{\partial f_n}{\partial x_n}\right)\right)A_{1i}x+\\&+f^{\mathrm{T}}B_{1i}^{\mathrm{T}}\left(P_i+\mathrm{diag}\left(\lambda_1\frac{\partial f_1}{\partial x_1},\ldots,\lambda_n\frac{\partial f_n}{\partial x_n}\right)\right)B_{1i}f+\\&+\sum_{j=1}^{r}q_{ij}x^{\mathrm{T}}P_jx+x^{\mathrm{T}}L_i^{\mathrm{T}}L_ix-\gamma^2w^{\mathrm{T}}w\leq0,\quad i=1,\ldots,r,\end{aligned} \tag{3.22}$$

where

$$\Lambda:=\mathrm{diag}\begin{pmatrix}\lambda_1, & \lambda_2, & \cdots, & \lambda_n\end{pmatrix}. \tag{3.23}$$

In order to explicitly express (3.22), assumption (2.18) will be used. Indeed, using inequalities (2.18) it follows that conditions (3.22) are satisfied if the following inequalities are satisfied:

(3.24)

$$
\begin{aligned}
&-F_{i0}(x,f) := x^{\mathrm{T}}\Big[A_{0i}^{\mathrm{T}}P_i + P_iA_{0i} + A_{1i}^{\mathrm{T}}\left(P_i + C^{\mathrm{T}}S^{\frac{1}{2}}\Lambda S^{\frac{1}{2}}C\right)A_{1i}+\\
&+L_i^{\mathrm{T}}L_i + \sum_{j=1}^{r} q_{ij}P_j + \gamma^{-2}P_iD_iD_i^{\mathrm{T}}P_i\Big]x+\\
&+f^{\mathrm{T}}\left(B_{0i}^{\mathrm{T}}C^{\mathrm{T}}\Lambda+\Lambda CB_{0i}+\gamma^{-2}\Lambda CD_iD_i^{\mathrm{T}}C^{\mathrm{T}}\Lambda+B_{1i}^{\mathrm{T}}\left(P_i+C^{\mathrm{T}}S^{\frac{1}{2}}\Lambda S^{\frac{1}{2}}C\right)B_{1i}\right)f\\
&+f^{\mathrm{T}}\big(B_{0i}^{\mathrm{T}}P_i+\Lambda CA_{0i}+\gamma^{-2}\Lambda D_iD_i^{\mathrm{T}}P_i\big)\,x+x^{\mathrm{T}}\big(P_iB_{0i}+A_{0i}^{\mathrm{T}}C^{\mathrm{T}}\Lambda+\gamma^{-2}P_iD_iD_i^{\mathrm{T}}\Lambda\big)f\\
&-\left(\gamma w^{\mathrm{T}}-\gamma^{-1}x^{\mathrm{T}}P_iD_i-\gamma^{-1}f^{\mathrm{T}}\Lambda CD_i\right)\left(\gamma w-\gamma^{-1}D_i^{\mathrm{T}}P_ix-\gamma^{-1}D_i^{\mathrm{T}}C^{\mathrm{T}}\Lambda f\right)\leq 0.
\end{aligned}
$$

Using the $\mathcal{S}$-procedure ([1]) one, therefore, obtains that (3.21) subject to (2.17) is satisfied if there exist $\tau_i \geq 0$, $i = 1, 2, \ldots, n$, such that

$$F_{i0}(x,f) - \sum_{k=1}^{n} \tau_k F_k(x,f) \geq 0. \tag{3.25}$$

Denoting

$$T := \operatorname{diag}\left(\ \tau_1,\quad \tau_2,\quad \cdots,\quad \tau_n\ \right) \tag{3.26}$$

and noticing that

$$-\sum_{k=1}^{n}\tau_kF_k(x,f)=\sum_{k=1}^{n}\tau_kf_k^2-\tau_k\sigma_kf_ky_k=f^{\mathrm{T}}Tf-\frac{1}{2}f^{\mathrm{T}}TCSx-\frac{1}{2}x^{\mathrm{T}}SC^{\mathrm{T}}Tf,$$

(3.25) yields

$$
\begin{aligned}
&x^{\mathrm{T}}\Big[A_{0i}^{\mathrm{T}}P_i + P_iA_{0i} + A_{1i}^{\mathrm{T}}\left(P_i + C^{\mathrm{T}}S^{\frac{1}{2}}\Lambda S^{\frac{1}{2}}C\right)A_{1i} + L_i^{\mathrm{T}}L_i + \sum_{j=1}^{r}q_{ij}P_j+\\
&+\gamma^{-2}P_iD_iD_i^{\mathrm{T}}P_i\Big]x + f^{\mathrm{T}}\left(B_{0i}^{\mathrm{T}}P_i + \Lambda CA_{0i} + \gamma^{-2}\Lambda CD_iD_i^{\mathrm{T}}P_i + \frac{1}{2}TCS\right)x+\\
&+x^{\mathrm{T}}\left(P_iB_{0i} + A_{0i}^{\mathrm{T}}C^{\mathrm{T}}\Lambda + \gamma^{-2}P_iD_iD_i^{\mathrm{T}}C^{\mathrm{T}}\Lambda + \frac{1}{2}SC^{\mathrm{T}}T\right)f+\\
&+f^{\mathrm{T}}\Big[B_{0i}^{\mathrm{T}}C^{\mathrm{T}}\Lambda + \Lambda CB_{0i} + \gamma^{-2}\Lambda CD_iD_i^{\mathrm{T}}C^{\mathrm{T}}\Lambda+\\
&+B_{1i}^{\mathrm{T}}\left(P_i + C^{\mathrm{T}}S^{\frac{1}{2}}\Lambda S^{\frac{1}{2}}C\right)B_{1i} - T\Big]f \leq 0,
\end{aligned}
$$

$i = 1, \dots, r$. These conditions are satisfied if

$$\begin{bmatrix} \mathcal{Z}_{i11} & \mathcal{Z}_{i12} & P_i D_i \\ \mathcal{Z}_{i12}^{\mathrm{T}} & \mathcal{Z}_{i22} & \Lambda C D_i \\ D_i^{\mathrm{T}} P & D_i^{\mathrm{T}} C^{\mathrm{T}} \Lambda & -\gamma^2 I \end{bmatrix} \leq 0, \quad i = 1, 2, \dots, r, \tag{3.27}$$

where

$$\mathcal{Z}_{i11} := A_{0i}^{\mathrm{T}} P_i + P_i A_{0i} + A_{1i}^{\mathrm{T}} \big(P_i + C^{\mathrm{T}} S^{\frac{1}{2}} \Lambda S^{\frac{1}{2}} C\big) A_{1i} + L_i^{\mathrm{T}} L_i + \sum_{j=1}^{r} q_{ij} P_j, \tag{3.28}$$

$$\mathcal{Z}_{i12} := P_i B_{0i} + A_{0i}^{\mathrm{T}} C^{\mathrm{T}} \Lambda + \frac{1}{2} S C^{\mathrm{T}} T,$$

$$\mathcal{Z}_{i22} := B_{0i}^{\mathrm{T}} C^{\mathrm{T}} \Lambda + \Lambda C B_{0i} + B_{1i}^{\mathrm{T}} \big(P_i + C^{\mathrm{T}} S^{\frac{1}{2}} \Lambda S^{\frac{1}{2}} C\big) B_{1i} - T, \; i = 1, \dots, r,$$

with the unknown variables P_i, Λ and T. The above developments are concluded in the following result:

THEOREM 1. *System* (2.6), (2.7) *is stochastically stable and its L_2 gain is less than $\gamma > 0$ if there exist symmetric matrices $P_i > 0$, $i = 1, \dots, r, \Lambda > 0$ and $T > 0$ of the form* (3.23) *and* (3.26)*, respectively, satisfying the system of* LMI*s* (3.27) *with notation* (3.28).

4. EXAMPLE – STICK BALANCING

In [4] the problem of human stick balancing was considered. Actual test data showed that the task of stick balancing which is essentialy an inverted pendulum control problem is performed by humans using the combination of delay and state-multiplicative noise. The model

$$\dddot{\phi} = -\Gamma \dot{\phi} + q \sin(\phi) + f(-k(\theta)(R_0 + R_1 \nu)\phi(t-1)) \tag{4.29}$$

was used in [4] to explain the test results, where $\Gamma = \frac{3\bar{\gamma}}{m}$, $q = \frac{3g\tau_r^2}{\ell}$, $c = \frac{3\tau_r^2}{m\ell}$ and m and ℓ are stick mass and length respectively, τ_r is the delay and ν is the state-multiplicative noise. The time in model (4.29) is normalized with respect to the delay τ_r. For the stability analysis in the sequel we take $\sin(\phi) \approx \phi$. We also take $f(x) = 1000 \tanh(x/1000)$. Defining the state-vector of (4.29) by $x = \begin{bmatrix} \phi & \dot{\phi} \end{bmatrix}^{\mathrm{T}}$ we obtain the state-space description of (2.10), where $A_0 = \begin{bmatrix} 0 & 1 \\ q & -\Gamma \end{bmatrix}$, $B_0 = \begin{bmatrix} 0 & 0 \\ k(\theta) R_0 & 0 \end{bmatrix}$ and $B_1 = \begin{bmatrix} 0 & 0 \\ k(\theta) R_1 & 0 \end{bmatrix}$, assuming that $|\phi| \ll 1$. We also take $A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The nonlinearities are $f_1(x_1) = 1000 \tanh(x_1/1000)$ and $f_2(x_2) = 0$. We also assume for this model that $k(\theta) = 1$, namely this model does not involve Markov jumps. We also note that we took a nonzero D just for the sake of illustrating computation of the disturbance attenuation factor γ but for simulations we assumed $w = 0$. It is pointed out in [4] that corrective movements of the inverted pendulum tip (i.e., the tip of the stick) occur also at time scales shorter than the delay. This phenomenon is referred to in [4] as on-off intermittency and seems to have a stabilizing effect. Namely, the noise statistically drives ϕ to zero, and since the noise is state-multiplicative, its effect near origin reduces and the pendulum is somewhat noisily stabilized. In [4] the stability of the inverted pendulum model in the presence of delay and state-multiplicative noise was studied using simulations. Our results may provide an analytic tool to predict this stability including the state-multiplicative noise using the Padé approximation for delay. To illustrate our results, we take $\tau_r = 0.07, R_1 = 10$, $\bar{\gamma} = 100$, $m = 0.035$ and $\ell = 0.62$. Since the above model is written in terms of time normalized with respect to τ_r, we take $\tau = 1$ for our analysis. To include the delay effect, we choose the first order Padé approximation. We, therefore, redefine the state-vector of (4.29) to be $x = \begin{bmatrix} \phi & \dot{\phi} & \zeta & \dot{\zeta} \end{bmatrix}^{\mathrm{T}}$ and obtain the augmented state-space description of (2.12), where $A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ q & -\Gamma & 0 & 0 \\ 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \end{bmatrix}$, $B_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & k(\theta)R_0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & k(\theta)R_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{bmatrix}$, assuming that $|\phi| \ll 1$. We also take $A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. The nonlinearities are $f_1(y_1) = 0$, $f_2(y_2) = 0$, $f_3(y_3) = 1000 \tanh(y_3/1000)$ and $f_4(y_4) = 0$. We take $R_0 = 0.2266981$, $R_1 = 10$ and assume that the control is on ($\theta(t) = 1$) and off ($\theta(t) = 2$) according to the Markov chain on $\mathcal{D} = \{1, 2\}$ with a probability transition matrix from (2.8) with $Q = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}$, where $\alpha = 0.25, \beta = 0.125$. It follows that $\mathcal{P}(t) = \mathrm{e}^{Qt} = (\alpha t + \beta t)^{-1} \begin{bmatrix} \beta t + \mu(t)\alpha t & \alpha t - \mu(t)\alpha t \\ \beta t - \mu(t)\beta t & \alpha t + \mu(t)\beta t \end{bmatrix}$, where $\mu(t) = \mathrm{e}^{-(\alpha+\beta)t}$. We take $k(\theta) = 3$ for $\theta(t) = 1$ and $k(\theta) = 0$ for $\theta(t) = 2$. In the steady state, $\mathcal{P}(t) = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$, namely, control is off most of the time. Substituting the above parameter values in Theorem 1, we find that $\gamma_{\min} = 0.025$. Therefore, adding Markov jumps to the system still results in a stable dynamics. In Fig. 1a-d ($\phi, \dot{\phi}, \Delta z/\ell = \cos(\phi)$, and $u = -(R_0 + R_1\nu)k(\theta(t))f(\phi(t-1))$ simulation results are depicted. The Markov jumps are best seen in Fig. 2, where the times between 100 and 150 are only shown. Both the LMI based analysis and simulation results indicate the stability of the stick balancing scheme also in the presence of Markov jumps between on-and-off states of the control. Although some of the numerical values need, probably, to be tuned to realistically model human stick balancing, the results may encourage further research about the possible effect of Markov jumps on balancing tasks.

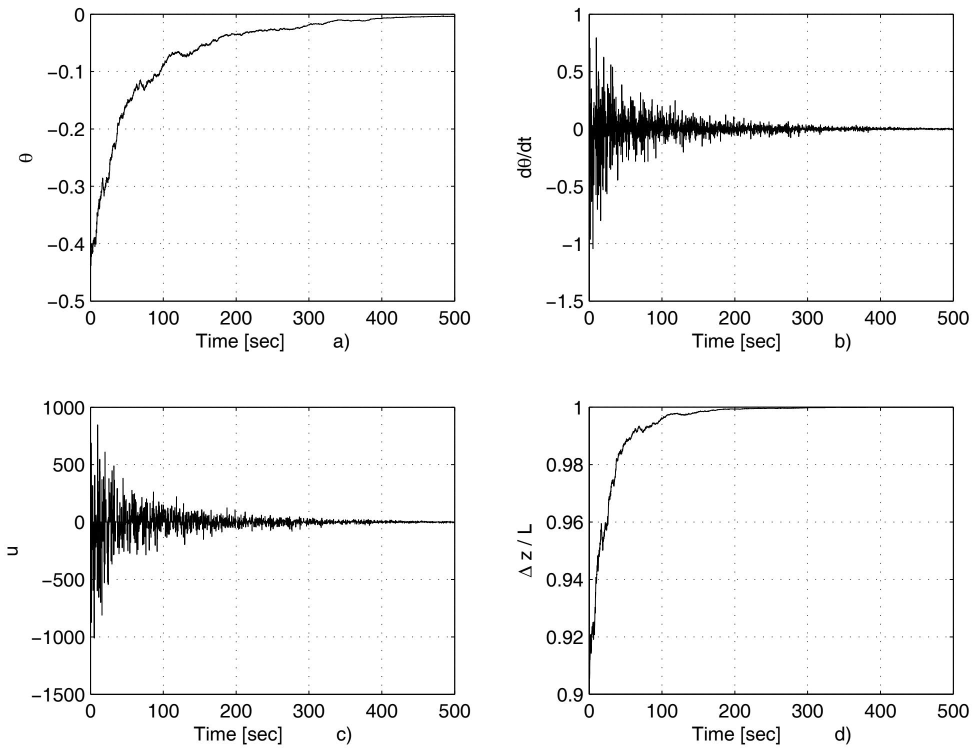

Fig. 1

5. CONCLUSIONS

A class of stochastic Hopfield networks subject to state-multiplicative noise, where the network weights jump according a Markov process have been considered. Stochastic stability and disturbance attenuation analysis in an H_∞ setup have been derived in terms of Linear Matrix Inequalities. The results have been illustrated by a stick balancing related example, where the delay in the control has been replaced by its Padé approximation.

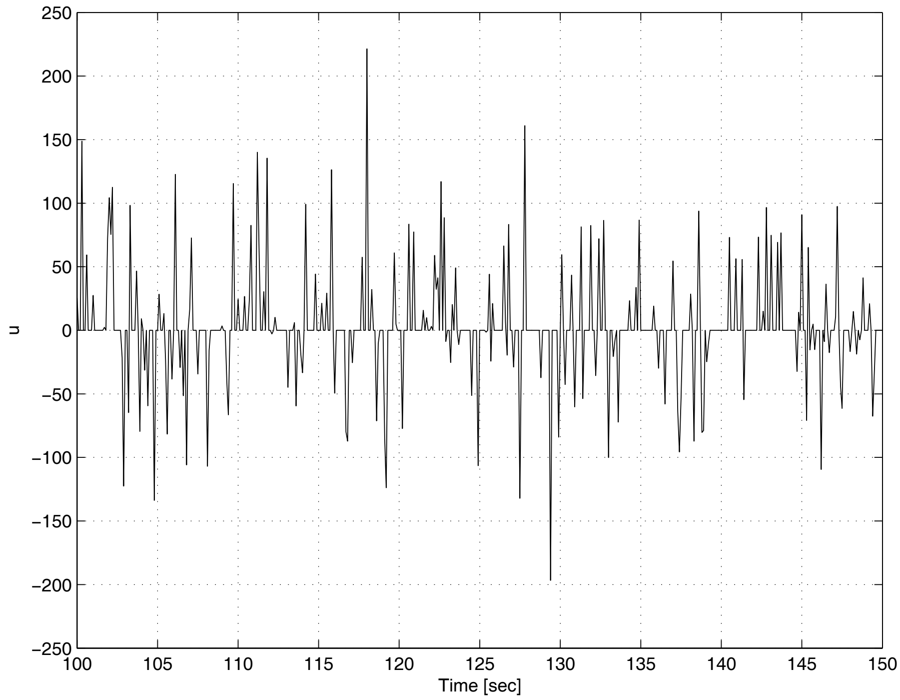

Fig. 2

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