Dedicated to Marius Iosifescu
on the occasion of his seventieth birthday

PERTURBED FINITE MARKOV CHAINS

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We consider two types of perturbations of finite Markov chains which are still Markov chains, we call perturbations of the first and of the second type, respectively. In Section 1 we define the class of the [Δ]-simple Markov chains (they were considered in [10] without naming them) which is in fact a subclass of the that of [Δ]-groupable Markov chains. Then we show some [Δ]- and Δ-ergodicity results on [Δ]-groupable Markov chains, [Δ]-simple Markov chains, and their perturbations of the first type. In Section 2 we show some uniform ergodicity results on Markov chains and their perturbations of the second type. In particular, for perturbed finite Markov chains we obtain, with different proofs, all ergodicity and uniform ergodicity results of Fleischer and Joffe [2]. Our tools are ergodicity coefficients and norms. Our methods are the perturbation, the looping through limit Δ-ergodic theory (for short, the looping), and the blocks.

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1. [Δ]- AND Δ-ERGODICITY RESULTS

In this section we define [Δ]-simple Markov chains and we show some [Δ]- and Δ-ergodicity results on [Δ]-groupable Markov chains, [Δ]-simple Markov chains, and their perturbations of the first type.

Consider a finite Markov chain with state space $S = \{1, 2, \ldots, r\}$ and transition matrices $(P_n)_{n\geq 1}$. We shall refer to it as the (finite) Markov chain $(P_n)_{n\geq 1}$. For all integers $m \geq 0$, $n > m$, define

$$P_{m,n} = P_{m+1}P_{m+2}\cdots P_n = ((P_{m,n})_{ij})_{i,j\in S}\,.$$

(The entries of a matrix Z will be denoted Z_{ij}.)

Set

$$\mathrm{Par}\,(E) = \{\Delta \mid \Delta \text{ is a partition of } E\}\,,$$

where E is a nonempty set. Here we shall agree that the partitions do not contain the empty set (an exception, e.g., occurs in [14] (see also [4] and [5]) for the case of bases).

Definition 1.1. Let $\Delta_1, \Delta_2 \in \text{Par}(E)$. We say that Δ_1 *is finer than* Δ_2 if $\forall V \in \Delta_1$, $\exists W \in \Delta_2$ such that $V \subseteq W$.

Write $\Delta_1 \preceq \Delta_2$ when Δ_1 is finer than Δ_2.

First, we give some definition from Δ-ergodic theory (see also [14]).

Definition 1.2 ([6]). Let $i, j \in S$. We say that i and j are in the same *weakly ergodic class* if $\forall m \geq 0, \forall k \in S$ we have

$$\lim_{n\to\infty} [(P_{m,n})_{ik} - (P_{m,n})_{jk}] = 0.$$

Write $i \sim j$ when i and j are in the same weakly ergodic class. Then $\sim$ is an equivalence relation and determines a partition $\Delta = (C_1, C_2, \ldots, C_s)$ of S. The sets $C_1, C_2, \ldots, C_s$ are called *weakly ergodic classes.*

Definition 1.3 ([10]). Let $\Delta = (C_1, C_2, \ldots, C_s)$ be the partition of weakly ergodic classes of a Markov chain. We say that the chain is *weakly* Δ*-ergodic.* In particular, a weakly (S)-ergodic chain is called *weakly ergodic* for short.

Definition 1.4 ([11]). Let $(C_1, C_2, \ldots, C_s)$ be the partition of weakly ergodic classes of a Markov chain with state space S and $\Delta \in \text{Par}(S)$. We say that the chain is *weakly* $[\Delta]$*-ergodic* if $\Delta \preceq (C_1, C_2, \ldots, C_s)$.

Definition 1.5 ([7]). Let C be a weakly ergodic class. We say that C is a *strongly ergodic class* if $\forall m \geq 0, \forall i \in C, \forall j \in S$ the limit

$$\lim_{n\to\infty} (P_{m,n})_{ij} := \pi_{m,j} = \pi_{m,j}(C)$$

exists and does not depend on i.

Definition 1.6 ([14]). Consider a weakly Δ-ergodic Markov chain. We say that the chain is *strongly* Δ*-ergodic* if any $C \in \Delta$ is a strongly ergodic class. In particular, a strongly (S)-ergodic chain is called *strongly ergodic* for short.

Definition 1.7 ([14]). Consider a weakly $[\Delta]$-ergodic Markov chain. We say that the chain is *strongly* $[\Delta]$*-ergodic* if any $C \in \Delta$ is included in a strongly ergodic class.

Further, we give some definitions from limit Δ-ergodic theory (see also [14]). For this, we shall agree that when writing

$$\lim_{u\to\infty} \lim_{v\to\infty} a_{u,v},$$

where $a_{u,v} \in \mathbf{R}, \forall u, v \in \mathbf{N}$ with $u \geq u_1, v \geq v_1(u)$, we assume that $\exists u_0 \geq u_1$ such that

$$\exists \lim_{v\to\infty} a_{u,v}, \forall u \geq u_0.$$

Definition 1.8 ([14]). Let $i, j \in S$. We say that i and j are in the same *limit weakly ergodic class* if $\forall k \in S$ we have

$$\lim_{m\to\infty}\lim_{n\to\infty}\left[(P_{m,n})_{ik} - (P_{m,n})_{jk}\right] = 0.$$

Write $i \overset{l}{\sim} j$ when i and j are in the same limit weakly ergodic class. Then $\overset{l}{\sim}$ is an equivalence relation and determines a partition $\bar{\Delta} = (L_1, L_2, \ldots, L_u)$ of S. The sets $L_1, L_2, \ldots, L_u$ are called *limit weakly ergodic classes.*

Definition 1.9 ([14]). Let $\bar{\Delta} = (L_1, L_2, \ldots, L_u)$ be the partition of limit weakly ergodic classes. We say that the chain is *limit weakly* $\bar{\Delta}$*-ergodic.* In particular, a limit weakly (S)-ergodic chain is called *limit weakly ergodic* for short.

Definition 1.10 ([14]). Let $(L_1, L_2, \ldots, L_u)$ be the partition of limit weakly ergodic classes of a Markov chain with state space S and $\bar{\Delta} \in \operatorname{Par}(S)$. We say that the chain is *limit weakly* $[\bar{\Delta}]$*-ergodic* if $\bar{\Delta} \preceq (L_1, L_2, \ldots, L_u)$.

Definition 1.11 ([14]). Let L be a limit weakly ergodic class. We say that L is a *limit strongly ergodic class* if $\forall i \in L$, $\forall j \in S$ the limit

$$\lim_{m\to\infty}\lim_{n\to\infty}(P_{m,n})_{ij} := \pi_j = \pi_j(L)$$

exists and does not depend on i.

Definition 1.12 ([14]). Consider a limit weakly $\bar{\Delta}$-ergodic Markov chain. We say that the chain is *limit strongly* $\bar{\Delta}$*-ergodic* if any $L \in \bar{\Delta}$ is a limit strongly ergodic class.

Definition 1.13 ([14]). Consider a limit weakly $[\bar{\Delta}]$-ergodic Markov chain. We say that the chain is *limit strongly* $[\bar{\Delta}]$*-ergodic* if any $L \in \bar{\Delta}$ is included in a limit strongly ergodic class.

Let $T = (T_{ij})$ be a real $m \times n$ matrix. Let $\emptyset \neq U \subseteq \{1, 2, \ldots, m\}$ and $\emptyset \neq V \subseteq \{1, 2, \ldots, n\}$. Define

$$T_U = (T_{ij})_{i\in U,\, j\in\{1,2,\ldots,n\}}, \quad T^V = (T_{ij})_{i\in\{1,2,\ldots,m\},\, j\in V},$$

$$T_U^V = (T_{ij})_{i\in U,\, j\in V}, \quad \alpha(T) = \min_{1\le i,j\le m}\sum_{k=1}^{n}\min(T_{ik}, T_{jk})$$

(if T is a stochastic matrix, then $\alpha(T)$ is called *the ergodicity coefficient of Dobrushin of the matrix* T (see, e.g., [1] or [3, p. 56])),

$$\bar{\alpha}(T) = \frac{1}{2}\max_{1\le i,j\le m}\sum_{k=1}^{n}|T_{ik} - T_{jk}|,$$

$$\gamma_\Delta(T) = \min_{K\in\Delta}\alpha(T_K), \quad \bar{\gamma}_\Delta(T) = \max_{K\in\Delta}\bar{\alpha}(T_K),$$

where $\Delta \in \operatorname{Par}(\{1,2,\ldots,m\})$ (see [10] for γ_Δ and $\bar{\gamma}_\Delta$), and

$$|||T|||_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |T_{ij}|$$

(*the* ∞-*norm of* T).

Let

$$S_{m,n} = \{P \mid P \text{ is a stochastic } m\times n \text{ matrix}\},$$

$$S_r = S_{r,r},$$

$$S_{\Delta,\sigma} = \left\{P \mid P \in S_r \text{ and } P_{K_l}^{\mathcal{C}K_{\sigma(l)}} = 0, \forall l \in \{1,2,\ldots,p\}\right\},$$

where $\Delta = (K_1, K_2, \ldots, K_p) \in \operatorname{Par}(\{1,2,\ldots,r\})$, $\sigma \in S(p)$, the collection of all permutations of $\{1,2,\ldots,p\}$, and $\mathcal{C}$ is the complement of a set, and

$$S_\Delta = \bigcup_{\sigma\in S(p)} S_{\Delta,\sigma}$$

(see also [10]).

Definition 1.14. We say that a (finite) Markov chain $(P_n)_{n\ge1}$ is $[\Delta]$-*simple* if $P_n \in S_\Delta$, $\forall n \ge 1$, where $\Delta \in \operatorname{Par}(S)$.

Definition 1.15. We say that a Markov chain $(P_n)_{n\ge1}$ is *diagonal* $[\Delta]$-*simple* if $P_n \in S_{\Delta,\sigma}$, $\forall n \ge 1$, where $\sigma = \mathrm{id}$, i.e., the identity permutation, and $\Delta \in \operatorname{Par}(S)$.

Definition 1.16. We say that a Markov chain $(P_n)_{n\ge1}$ is *cyclic* $[\Delta]$-*simple* if $P_n \in S_{\Delta,\sigma}$, $\forall n \ge 1$, where σ is a cycle and $\Delta \in \operatorname{Par}(S)$.

Also, we consider the following notions:

(a) S_Δ is *the set of* $[\Delta]$-*simple matrices*; $A \in S_\Delta$ is a $[\Delta]$-*simple matrix*;

(b) $S_{\Delta,\sigma}$, where $\sigma = \mathrm{id}$, is *the set of diagonal* $[\Delta]$-*simple matrices*; $A \in S_{\Delta,\sigma}$ is a *diagonal* $[\Delta]$-*simple matrix*;

(c) $\bigcup_\sigma S_{\Delta,\sigma}$, where σ is a cycle, is *the set of cyclic* $[\Delta]$-*simple matrices*; $A \in \bigcup_\sigma S_{\Delta,\sigma}$ is a *cyclic* $[\Delta]$-*simple matrix.* (Also, the cyclic matrices from the homogeneous case are example of cyclic $[\Delta]$-simple matrices, where Δ is, e.g., the partition of cyclic subclasses.)

Remark 1.17. (a) The weak and uniform weak Δ-ergodicity properties of $[\Delta]$-simple Markov chains (without naming them) were studied in [10].

(b) The cyclic $[\Delta]$-simple Markov chains appear, e.g., in Theorems 2.8, 2.11, 2.13, and 2.16 from [13].

(c) If a Markov chain is diagonal $[\Delta]$-simple, this does not mean that it has the nonzero blocks on the main diagonal. But using a permutation matrix we can obtain the diagonal form. Indeed, let, e.g.,

$$P_n = \begin{pmatrix} 1-\frac{1}{n} & 0 & \frac{1}{n} \\ 0 & 1 & 0 \\ \frac{1}{n} & 0 & 1-\frac{1}{n} \end{pmatrix}, \quad \forall n \geq 1.$$

This chain is diagonal $[(\{1,3\},\{2\})]$-simple, but it does not have the nonzero blocks on the main diagonal. Considering the permutation matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

we have

$$Q_n := P'P_nP = \begin{pmatrix} 1-\frac{1}{n} & \frac{1}{n} & 0 \\ \frac{1}{n} & 1-\frac{1}{n} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \forall n \geq 1,$$

in diagonal form, where P' is the transpose of P. A similar thing happens for a cyclic $[\Delta]$-simple Markov chain.

Definition 1.18. Let E be a nonnegative $m \times n$ matrix. We say that E is a *generalized stochastic matrix* if there exist $a \geq 0$ and $F \in S_{m,n}$ such that $E = aF$.

Let

$$G_\Delta = \left\{ P \mid P \in S_r \text{ and } \forall K, L \in \Delta, P_K^L \text{ is a generalized stochastic matrix} \right\},$$

where $\Delta \in \mathrm{Par}(S)$.

Remark 1.19. We have $S_\Delta \subseteq G_\Delta$ and $S_{(S)} = G_{(S)} = G_{(\{i\})_{i \in S}} = S_r$.

Definition 1.20 ([11]). We say that a Markov chain $(P_n)_{n\geq 1}$ is $[\Delta]$-*groupable* if $P_n \in G_\Delta$, $\forall n \geq 1$.

Also, we consider the following notions: G_Δ is *the set of* $[\Delta]$-*groupable matrices*; $A \in G_\Delta$ is a $[\Delta]$-*groupable matrix.*

THEOREM 1.21 ([14]). *Consider a* $[\Delta]$-*groupable Markov chain* $(P_n)_{n\geq 1}$. *Then the chain is weakly* $[\Delta]$-*ergodic if and only if it is limit weakly* $[\Delta]$-*ergodic.*

Proof. See [14]. □

For $[\Delta]$-simple Markov chains we can say more.

THEOREM 1.22. *Consider a* $[\Delta]$-*simple Markov chain* $(P_n)_{n\geq 1}$. *Then the following statements are equivalent.*

(i) *The chain is weakly* $[\Delta]$-*ergodic.*

(ii) *The chain is weakly* Δ*-ergodic.*

(iii) *The chain is limit weakly* $[\Delta]$*-ergodic.*

(iv) *The chain is limit weakly* Δ*-ergodic.*

Proof. (i)$\Leftrightarrow$(iii) See Remark 1.19 and Theorem 1.21.

(i)$\Leftrightarrow$(ii) Obvious.

(ii)$\Rightarrow$(iv) Obvious.

(iv)$\Rightarrow$(iii) Obvious. □

THEOREM 1.23 ([14]). *Consider a Markov chain* $(P_n)_{n\geq 1}$*. If the chain is*

(i) *weakly* $[\Delta]$*-ergodic,*

and

(ii) *limit weakly* Δ*-ergodic,*

then it is weakly Δ*-ergodic.*

Proof. See [14]. □

Using Theorem 1.23 we can generalize (iv)$\Rightarrow$(ii) from Theorem 1.22.

THEOREM 1.24. *Consider a* $[\Delta]$*-groupable Markov chain* $(P_n)_{n\geq 1}$*. If the chain is limit weakly* Δ*-ergodic, then it is weakly* Δ*-ergodic.*

Proof. If the chain is limit weakly Δ-ergodic, then it is limit weakly $[\Delta]$-ergodic. It follows from Theorem 1.21 that it is weakly $[\Delta]$-ergodic. Now, by Theorem 1.23, it is weakly Δ-ergodic. □

Remark 1.25. The converse of Theorem 1.24 is not true. Indeed, let

$$P_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1-\frac{1}{n} & \frac{1}{n} & 0 \end{pmatrix}, \quad \forall n \geq 1.$$

This chain is $[(\{1\},\{2\},\{3\})]$-groupable. We have $P_{m,n} = P_{m+1}, \forall n > m$, therefore the chain is weakly (even strongly) $(\{1\},\{2\},\{3\})$-ergodic. Since

$$P_n \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ as } n \to \infty,$$

it follows that it is limit weakly (even strongly) $(\{1,3\},\{2\})$-ergodic.

In Δ-ergodic theory it is doubtful to find a submultiplicative ergodicity coefficient which generalizes $\bar{\alpha}$ better than $\bar{\gamma}_\Delta$. In this sense see Remark 1.14(3) from [11] and the following basic example. Let

$$P_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{2}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad P_n = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}, \quad \forall n \geq 2.$$

The matrix P_1 destroys the $(\{1,2\}, \{3,4\})$-ergodicity of the chain $(P_n)_{n\geq 2}$. On account of this, we shall invoke the perturbation method (see, e.g., [2], [5], [8], [9], [10], [11], [12], and [14]).

Definition 1.26. Let $(P_n)_{n\geq 1}$ and $(P'_n)_{n\geq 1}$ be two (finite) Markov chains. We say that $(P'_n)_{n\geq 1}$ is a *perturbation of the first type of* $(P_n)_{n\geq 1}$ if

$$\sum_{n\geq 1} |||P_n - P'_n|||_\infty < \infty.$$

Definition 1.27. Let $(P_n)_{n\geq 1}$ and $(P'_n)_{n\geq 1}$ be two Markov chains. We say that $(P'_n)_{n\geq 1}$ is a *perturbation of the second type of* $(P_n)_{n\geq 1}$ if

$$|||P_n - P'_n|||_\infty \to 0 \text{ as } n \to \infty$$

(this is equivalent to $P_n - P'_n \to 0$ as $n \to \infty$).

We note that perturbation of the first type is a good method for the study of the weak and strong $[\Delta]$- and Δ-ergodicity while perturbation of the second type is a good method for the study of the uniform weak and strong $[\Delta]$- and Δ-ergodicity. Also, we note that

$$\exists \lim_{n\to\infty} |||P_{m,n} - P'_{m,n}|||_\infty, \quad \forall m \geq 0,$$

when $(P_n)_{n\geq 1}$ is a Markov chain and $(P'_n)_{n\geq 1}$ is a perturbation of the first type of it. For to prove this, let $m \geq 0$. Let x and y be two limit points of the sequence $\left(|||P_{m,n} - P'_{m,n}|||_\infty\right)_{n>m}$. Let $(n_k)_{k\geq 1}$ and $(t_l)_{l\geq 1}$ be two subsequences of the sequence of natural numbers with $n_1, t_1 > m$ such that

$$\lim_{k\to\infty} |||P_{m,n_k} - P'_{m,n_k}|||_\infty = x \quad \text{and} \quad \lim_{l\to\infty} |||P_{m,t_l} - P'_{m,t_l}|||_\infty = y,$$

respectively. Suppose that $x \neq y$ and, e.g., $x < y$. Using the inequality

$$|||A_1 A_2 \cdots A_p - B_1 B_2 \cdots B_p|||_\infty \leq \sum_{v=1}^{p} |||A_v - B_v|||_\infty,$$

where $p \geq 1$ and $A_1, A_2, \ldots, A_p, B_1, B_2, \ldots, B_p \in S_r$ (see [5]) we obtain

$$\left|\left|\left|P_{m,t_l} - P'_{m,t_l}\right|\right|\right|_\infty \leq \left|\left|\left|P_{m,n_k} - P'_{m,n_k}\right|\right|\right|_\infty + \sum_{v=n_k+1}^{t_l} \left|\left|\left|P_v - P'_v\right|\right|\right|_\infty \leq$$

$$\leq \left|\left|\left|P_{m,n_k} - P'_{m,n_k}\right|\right|\right|_\infty + \sum_{v>n_k} \left|\left|\left|P_v - P'_v\right|\right|\right|_\infty, \quad \forall k, l \geq 1 \text{ with } t_l > n_k.$$

So,

$$y \leq \left|\left|\left|P_{m,n_k} - P'_{m,n_k}\right|\right|\right|_\infty + \sum_{v>n_k} \left|\left|\left|P_v - P'_v\right|\right|\right|_\infty, \quad \forall k \geq 1.$$

Further, it follows that $y \leq x$, so we have reached a contradiction. Therefore $x = y$. Hence $\exists \lim_{n\to\infty} \left|\left|\left|P_{m,n} - P'_{m,n}\right|\right|\right|_\infty$. Moreover, $\left(\left|\left|\left|P_{m,n} - P'_{m,n}\right|\right|\right|_\infty\right)_{n>m}$ is convergent because $\left|\left|\left|P_{m,n} - P'_{m,n}\right|\right|\right|_\infty \in [0, 2]$, $\forall n > m$.

The next theorem is a corrected version of Theorem 3.15 from [14]. It is a basic result for perturbations of the first type. Also, it is one of the results on which is based our method called the looping through limit Δ-ergodic theory. For short, we call it the looping method. The looping method consists in passing from Δ-ergodic theory to limit Δ-ergodic theory, then coming back.

Theorem 1.28. *Let $(P_n)_{n\geq 1}$ be a strongly Δ-ergodic Markov chain and $(P'_n)_{n\geq 1}$ a perturbation of the first type of it.*

(i) *$(P_n)_{n\geq 1}$ is limit weakly $\bar{\Delta}$-ergodic if and only if $(P'_n)_{n\geq 1}$ is limit weakly $\bar{\Delta}$-ergodic.*

(ii) *$\lim_{m\to\infty} \lim_{n\to\infty} P_{m,n} = Q$ if and only if $\lim_{m\to\infty} \lim_{n\to\infty} P'_{m,n} = Q$, where $Q \in S_r$.*

(iii) *$(P_n)_{n\geq 1}$ is limit strongly $\bar{\Delta}$-ergodic if and only if $(P'_n)_{n\geq 1}$ is limit strongly $\bar{\Delta}$-ergodic.*

Proof. See [14]. □

Remark 1.29. Theorem 1.28 cannot be extend to perturbations of the second type. Indeed, if, e.g.,

$$P_n = \begin{pmatrix} 1 - \frac{1}{n} & \frac{1}{n} \\ \frac{1}{n} & 1 - \frac{1}{n} \end{pmatrix}, \quad P'_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \forall n \geq 1,$$

then the chain $(P_n)_{n\geq 1}$ is strongly ergodic while $(P'_n)_{n\geq 1}$ is strongly $(\{1\}, \{2\})$-ergodic. Hence $(P_n)_{n\geq 1}$ is limit weakly and strongly ergodic while $(P'_n)_{n\geq 1}$ is limit weakly and strongly $(\{1\}, \{2\})$-ergodic.

Theorem 1.28(i) cannot be generalized for an arbitrary $(P_n)_{n\geq 1}$. We need a more general equivalence relation than that given in Definition 1.8 to obtain a result similar to Theorem 1.28(i). For this, let $i, j \in S$. We say that i and

j are in the same *limit weakly ergodic class in a generalized sense* if $\forall k \in S$ we have

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \left| (P_{m,n})_{ik} - (P_{m,n})_{jk} \right| = 0.$$

Now, it is easy to verify that a result similar to Theorem 1.28(i) (see its proof) holds in a generalized sense for an arbitrary $(P_n)_{n\geq 1}$. We call it Theorem 1.28(i)$'$. Also, it is easy to verify that Theorems 1.21, 1.22, 1.23, and 1.24 have similar versions in a generalized sense. (Note that in such a theorem and the corresponding one in a generalized sense, $(P_n)_{n\geq 1}$ verifies the same conditions, except for the limit ones.) We call they Theorems 1.21$'$, 1.22$'$, 1.23$'$, and 1.24$'$, respectively.

As concerns weak Δ-ergodicity under perturbation, the following result is the main theorem of this section (we prove it by the looping method).

THEOREM 1.30. *Let $(P_n)_{n\geq 1}$ be a Markov chain and $(P'_n)_{n\geq 1}$ a perturbation of the first type of it. If $(P_n)_{n\geq 1}$ is weakly Δ-ergodic and limit weakly Δ-ergodic in a generalized sense and $(P'_n)_{n\geq 1}$ is weakly Δ'-ergodic, then $\Delta' \preceq \Delta$.*

Proof. It follows from Theorem 1.28(i)$'$ that $(P'_n)_{n\geq 1}$ is limit weakly Δ-ergodic in a generalized sense. Now, it is obvious that $\Delta' \preceq \Delta$ (see also Theorem 2.9 from [14]). $\square$

If $(P_n)_{n\geq 1}$ is $[\Delta]$-simple we can say more.

THEOREM 1.31. *Let $(P_n)_{n\geq 1}$ be a $[\Delta]$-simple Markov chain and $(P'_n)_{n\geq 1}$ a perturbation of the first type of it. If $(P_n)_{n\geq 1}$ is weakly Δ-ergodic and $(P'_n)_{n\geq 1}$ is weakly Δ'-ergodic, then $\Delta' \preceq \Delta$.*

Proof. See Theorems 1.22$'$ and 1.30. $\square$

If $(P_n)_{n\geq 1}$ is $[\Delta]$-simple and $(P'_n)_{n\geq 1}$ is $[\Delta]$-groupable we can say even more.

THEOREM 1.32. *Let $(P_n)_{n\geq 1}$ be a $[\Delta]$-simple Markov chain and $(P'_n)_{n\geq 1}$ a $[\Delta]$-groupable perturbation of the first type of it. Then $(P_n)_{n\geq 1}$ is weakly Δ-ergodic if and only if $(P'_n)_{n\geq 1}$ is weakly Δ-ergodic.*

Proof. "$\Rightarrow$" We use the looping method. If $(P_n)_{n\geq 1}$ is weakly Δ-ergodic then, by Theorem 1.22$'$, it is limit weakly Δ-ergodic in a generalized sense. By Theorem 1.28(i)$'$, the chain $(P'_n)_{n\geq 1}$ is limit weakly Δ-ergodic in a generalized sense. Now, it follows from Theorem 1.24$'$ that $(P'_n)_{n\geq 1}$ is weakly Δ-ergodic.

"$\Leftarrow$" If $(P'_n)_{n\geq 1}$ is weakly Δ-ergodic, then $\exists \Delta' \in \mathrm{Par}\,(S)$ with $\Delta \preceq \Delta'$ such that it is limit weakly Δ'-ergodic in a generalized sense. By Theorem 1.28(i)$'$, $(P_n)_{n\geq 1}$ is limit weakly Δ'-ergodic in a generalized sense. It follows that $\Delta' \preceq \Delta$ because $(P_n)_{n\geq 1}$ is $[\Delta]$-simple. Therefore $\Delta' = \Delta$. Now, by Theorem 1.22′, $(P_n)_{n\geq 1}$ is weakly Δ-ergodic. $\square$

In particular, if $\Delta = (S)$ then we obtain a result from [2].

THEOREM 1.33 ([2]). *Let $(P_n)_{n\geq 1}$ be a Markov chain and $(P'_n)_{n\geq 1}$ a perturbation of the first type of it. Then $(P_n)_{n\geq 1}$ is weakly ergodic if and only if $(P'_n)_{n\geq 1}$ is weakly ergodic.*

Proof. See [2] or Theorem 1.32 (for $\Delta = (S)$). $\square$

Remark 1.34. In general, in Theorems 1.30 and 1.31 we cannot have more than $\Delta' \preceq \Delta$. For this, we give three examples.

(a): an example with a transient state. Let

$$P'_n = \begin{pmatrix} 1-\frac{1}{n^2} & 0 & \frac{1}{n^2} \\ 1-\frac{1}{2n^2} & 0 & \frac{1}{2n^2} \\ 0 & 0 & 1 \end{pmatrix}, \quad \forall n \geq 1.$$

We have

$$\left(P'_n P'_{n+1}\right)^{\{1,2\}} = \begin{pmatrix} \left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{(n+1)^2}\right) & 0 \\ \left(1-\frac{1}{2n^2}\right)\left(1-\frac{1}{(n+1)^2}\right) & 0 \\ 0 & 0 \end{pmatrix}, \quad \forall n \geq 1,$$

$$\left(P'_n P'_{n+1} P'_{n+2}\right)^{\{1,2\}} = \begin{pmatrix} \left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{(n+1)^2}\right)\left(1-\frac{1}{(n+2)^2}\right) & 0 \\ \left(1-\frac{1}{2n^2}\right)\left(1-\frac{1}{(n+1)^2}\right)\left(1-\frac{1}{(n+2)^2}\right) & 0 \\ 0 & 0 \end{pmatrix}, \forall n \geq 1,$$

etc. It follows that

$$\left(P'_{m,n}\right)_{11} = \prod_{k=m+1}^{n}\left(1-\frac{1}{k^2}\right) = \left(1-\frac{1}{(m+1)^2}\right)\prod_{k=m+2}^{n}\left(1-\frac{1}{k^2}\right) \to$$

$$\to \left(1-\frac{1}{(m+1)^2}\right) a_m \text{ as } n\to\infty, \quad \forall m \geq 0,$$

and

$$\left(P'_{m,n}\right)_{21} = \left(1-\frac{1}{2(m+1)^2}\right)\prod_{k=m+2}^{n}\left(1-\frac{1}{k^2}\right) \to$$

$$\to \left(1-\frac{1}{2(m+1)^2}\right) a_m \text{ as } n\to\infty, \quad \forall m \geq 0,$$

where

$$a_m = \lim_{n\to\infty} \prod_{k=m+2}^{n} \left(1 - \frac{1}{k^2}\right).$$

Therefore the chain is weakly (even strongly (see also Theorem 1.43)) $(\{1\}, \{2\}, \{3\})$-ergodic because $0 < a_m < \infty$ (in fact, $a_m < 1$). Instead, if

$$P_n = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \forall n \geq 1,$$

then $(P_n)_{n\geq 1}$ is weakly (even strongly) $(\{1,2\}, \{3\})$-ergodic. Obviously, $(P_n)_{n\geq 1}$ is $[(\{1,2\}, \{3\})]$-simple and $(P'_n)_{n\geq 1}$ is a perturbation of the first type of it.

(b): an example without transient states. Let

$$P_n = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} := A, \quad \forall n \geq 1,$$

and

$$P'_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{2}{4} & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix}, \quad P'_n = A, \quad \forall n \geq 2.$$

Obviously, $(P'_n)_{n\geq 1}$ is a perturbation of the first type of $(P_n)_{n\geq 1}$. The chain $(P_n)_{n\geq 1}$ is weakly (even strongly) $(\{1,2\}, \{3\})$-ergodic while $(P'_n)_{n\geq 1}$ is weakly (even strongly) $(\{1\}, \{2\}, \{3\})$-ergodic. We note that $P'_n = P_n, \forall n \geq 2$, therefore the matrix P'_1 makes $(P'_n)_{n\geq 1}$ not weakly $(\{1,2\}, \{3\})$-ergodic.

(c): an example where Theorem 1.28 is used. Let

$$P_n = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \forall n \geq 1,$$

and

$$P'_n = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} - \frac{1}{4n^2} & \frac{1}{4n^2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \forall n \geq 1.$$

Obviously, $(P_n)_{n\geq 1}$ is weakly (even strongly) $(\{1,2\}, \{3\})$-ergodic and limit weakly (even strongly) $(\{1,2\}, \{3\})$-ergodic. Is $(P'_n)_{n\geq 1}$ weakly (even strongly)

$(\{1,2\},\{3\})$-ergodic? Suppose that $(P'_n)_{n\geq 1}$ is weakly $(\{1,2\},\{3\})$-ergodic. Then

$$\lim_{n\to\infty}\left(\left(P'_{m,n}\right)_{11}-\left(P'_{m,n}\right)_{21}\right)=0,\quad \forall m\geq 0.$$

By Theorem 1.28, $(P'_n)_{n\geq 1}$ is limit weakly $(\{1,2\},\{3\})$-ergodic. Therefore, $2\overset{l}{\nsim}3$. It follows that there exists $u>0$ such that

$$\left(P'_{u,n}\right)_{23}\nrightarrow 1 \text{ as } n\to\infty,$$

because

$$\left(P'_{m,n}\right)_{33}=1,\quad \forall m,n,\ 0\leq m<n.$$

We have

$$\left(P'_{u-1,n}\right)_{11}-\left(P'_{u-1,n}\right)_{21}=\sum_{k=1}^{3}\left(P'_u\right)_{1k}\left(P'_{u,n}\right)_{k1}-\sum_{k=1}^{3}\left(P'_u\right)_{2k}\left(P'_{u,n}\right)_{k1}=$$

$$=\frac{1}{2}\left(P'_{u,n}\right)_{11}+\left(\frac{1}{2}-\frac{1}{4u^2}\right)\left(P'_{u,n}\right)_{21}-\frac{1}{2}\left(P'_{u,n}\right)_{11}-\frac{1}{2}\left(P'_{u,n}\right)_{21}=$$

$$=-\frac{1}{4u^2}\left(P'_{u,n}\right)_{21}\to 0 \text{ as } n\to\infty,$$

because of the hypothesis. Therefore,

$$\left(P'_{u,n}\right)_{21}\to 0 \text{ as } n\to\infty.$$

Since

$$\left(P'_{u,n}\right)_{21}=\sum_{k=1}^{3}\left(P'_{u,n-1}\right)_{2k}\left(P'_n\right)_{k1}=\frac{1}{2}\left(P'_{u,n-1}\right)_{21}+\frac{1}{2}\left(P'_{u,n-1}\right)_{22},$$

it follows that

$$\left(P'_{u,n-1}\right)_{22}\to 0 \text{ as } n\to\infty.$$

Hence

$$\left(P'_{u,n}\right)_{23}\to 1 \text{ as } n\to\infty.$$

Contradiction.

Further, we consider strong Δ-ergodicity.

THEOREM 1.35 ([14]). *Consider a Markov chain $(P_n)_{n\geq 1}$. Then the chain is strongly ergodic with limit Π if and only if it is limit strongly ergodic with limit Π.*

Proof. See [14], Theorem 2.27 (it follows from its proof that Π is a common limit). □

THEOREM 1.36. *Consider a diagonal $[\Delta]$-simple Markov chain $(P_n)_{n\geq 1}$. Then the following statements are equivalent.*

(i) *The chain is strongly Δ-ergodic with limit Π.*

(ii) *The chain is limit strongly Δ-ergodic with limit Π.*

(iii) *The chain* $\left((P_n)_K^K\right)_{n\geq 1}$*is strongly ergodic with limit* Π_K^K, $\forall K \in \Delta$.

(iv) *The chain* $\left((P_n)_K^K\right)_{n\geq 1}$*is limit strongly ergodic with limit* Π_K^K, $\forall K \in \Delta$.

Proof. (i)$\Leftrightarrow$(ii) It follows from Theorem 1.35 applied to the chains $\left((P_n)_K^K\right)_{n\geq 1}, K \in \Delta$.

(i)$\Leftrightarrow$(iii) Obvious.

(iii)$\Leftrightarrow$(iv) See Theorem 1.35. $\square$

THEOREM 1.37. *Consider a* $[\Delta]$*-simple Markov chain* $(P_n)_{n\geq 1}$. *Then it is strongly* Δ*-ergodic if and only if* $\exists n_0 \geq 1$ *such that* $(P_n)_{n\geq n_0}$ *is a diagonal* $[\Delta]$*-simple Markov chain and* $\left((P_n)_K^K\right)_{n\geq n_0}$ *is strongly ergodic,* $\forall K \in \Delta$.

Proof. "$\Rightarrow$" Let $\Delta = (K_1, K_2, \ldots, K_p)$. Obviously, $\forall n \geq 0$, $\exists \sigma_n \in S(p)$ such that $P_n \in S_{\Delta,\sigma_n}$. We can find a permutation matrix P such that $Q_n := P'P_nP$, $n \geq 1$, is a $[\Delta]$-simple Markov chain with the property that the states from K_1 correspond to rows $1, 2, \ldots, |K_1|$, those from K_2 to rows $|K_1|+1$, $|K_1|+2, \ldots, |K_1|+|K_2|$ etc. Let

$$(R_n)_{uv} = \begin{cases} 1, & \text{if } (Q_n)_{K_u}^{K_v} \neq 0 \\ 0, & \text{if } (Q_n)_{K_u}^{K_v} = 0 \end{cases}$$

$\forall u, v \in \{1, 2, \ldots, p\}$, $\forall n \geq 1$. Then R_n is a permutation matrix corresponding to Q_n, $\forall n \geq 1$. Clearly, $\exists \lim_{n\to\infty} R_{m,n}$, $\forall m \geq 0$, if and only if $\exists n_0 \geq 1$ such that $R_n = I_p$, $\forall n \geq n_0$, where $R_{m,n} := R_{m+1}R_{m+2}\cdots R_n$, $\forall m, n$, $0 \leq m < n$. It follows that $\exists n_0 \geq 1$ such that $(P_n)_{n\geq n_0}$ is a diagonal $[\Delta]$-simple Markov chain. Since the chain $(P_n)_{n\geq 1}$ is strongly Δ-ergodic, $(P_n)_{n\geq n_0}$ is strongly Δ-ergodic. Now, it follows from Theorem 1.36 that $\left((P_n)_K^K\right)_{n\geq n_0}$ is strongly ergodic, $\forall K \in \Delta$.

"$\Leftarrow$" Obvious. $\square$

Theorem 1.37 allows us to reduce to the study of diagonal $[\Delta]$-simple Markov chains when we study the strong Δ-ergodicity of $[\Delta]$-simple Markov chains. An example is the following result.

THEOREM 1.38. *Consider a* $[\Delta]$*-simple Markov chain* $(P_n)_{n\geq 1}$. *Then the chain is strongly* Δ*-ergodic if and only if it is limit strongly* Δ*-ergodic.*

Proof. It follows from Theorems 1.36, 1.37, and 1.41. $\square$

Remark 1.39. Obviously, in Theorem 1.38 strong Δ-ergodicity and limit strong Δ-ergodicity in general are with different limits (a excepted case is in Theorem 1.36).

THEOREM 1.40 ([14]). *Consider a Markov chain* $(P_n)_{n\geq 1}$. *If the chain is*

(i) *weakly* $[\Delta]$*-ergodic,*

and

(ii) *limit strongly* Δ*-ergodic,*

then it is strongly Δ*-ergodic.*

Proof. See [14]. $\square$

Using Theorem 1.40 we can generalize an implication from Theorem 1.38 (thus we obtain for strong Δ-ergodicity the similar result to that for weak Δ-ergodicity from Theorem 1.24).

THEOREM 1.41. *Consider a* $[\Delta]$*-groupable Markov chain* $(P_n)_{n\geq 1}$*. If the chain is limit strongly* Δ*-ergodic, then it is strongly* Δ*-ergodic.*

Proof. If the chain is limit strongly Δ-ergodic, then it is limit weakly $[\Delta]$-ergodic. By Theorem 1.21, it is weakly $[\Delta]$-ergodic. Now, by Theorem 1.40, it is strongly Δ-ergodic. $\square$

Remark 1.42. The converse of Theorem 1.41 is not true. For this, see the example from Remark 1.25.

Further, we study strong Δ-ergodicity under perturbation. Some results will be similar those for weak Δ-ergodicity. We begin with a basic result from [5].

THEOREM 1.43 ([5]). *Let* $(P_n)_{n\geq 1}$ *be a Markov chain and* $(P'_n)_{n\geq 1}$ *a perturbation of the first type of it. Then* $\exists \Delta \in \mathrm{Par}(S)$ *such that* $(P_n)_{n\geq 1}$ *is strongly* Δ*-ergodic if and only if* $\exists \Delta' \in \mathrm{Par}(S)$ *such that* $(P'_n)_{n\geq 1}$ *is strongly* Δ'*-ergodic.*

Proof. We give a full proof of this result here since in [5] it was stated without proof and in [14] (see Theorem 3.12) there was given an incorrect proof. By symmetry, it is sufficient to suppose that $(P_n)_{n\geq 1}$ is a strongly Δ-ergodic chain and prove that $\exists \Delta' \in \mathrm{Par}(S)$ such that $(P'_n)_{n\geq 1}$ is strongly Δ'-ergodic.

First, we show that $\left(P'_{m,n}\right)_{n>m}$ is a Cauchy sequence, $\forall m \geq 0$. Let $m \geq 0$. We have

$$\left|\left|\left|P'_{m,n} - P'_{m,n+p}\right|\right|\right|_\infty = \left|\left|\left|P'_{m,t}P'_{t,n} - P'_{m,t}P'_{t,n+p}\right|\right|\right|_\infty \leq$$

$$\leq \left|\left|\left|P'_{m,t}\right|\right|\right|_\infty \left|\left|\left|P'_{t,n} - P'_{t,n+p}\right|\right|\right|_\infty = \left|\left|\left|P'_{t,n} - P'_{t,n+p}\right|\right|\right|_\infty \leq$$

$$\leq \left|\left|\left|P'_{t,n} - P_{t,n}\right|\right|\right|_\infty + |||P_{t,n} - P_{t,n+p}|||_\infty + \left|\left|\left|P_{t,n+p} - P'_{t,n+p}\right|\right|\right|_\infty \leq$$

$$\leq 2 \sum_{k\geq t+1} \left|\left|\left|P_k - P'_k\right|\right|\right|_\infty + |||P_{t,n} - P_{t,n+p}|||_\infty\,, \quad \forall n,t,\ m<t<n,\ \forall p \geq 0$$

(for the last inequality see before Theorem 1.28). Let $\varepsilon > 0$. Then $\exists t_\varepsilon > m$ such that

$$2 \sum_{k \geq t+1} \left|\left|\left| P_k - P'_k \right|\right|\right|_\infty < \frac{\varepsilon}{2}, \quad \forall t \geq t_\varepsilon.$$

Because $(P_{u,v})_{v>u}$ is convergent, $\forall u \geq 0$, it is a Cauchy sequence, $\forall u \geq 0$. Hence $\exists n_\varepsilon > t_\varepsilon$ such that

$$|||P_{t_\varepsilon,n} - P_{t_\varepsilon,n+p}|||_\infty < \frac{\varepsilon}{2}, \quad \forall n \geq n_\varepsilon, \ \forall p \geq 0.$$

Further, it follows that $\exists n_\varepsilon > m$ such that

$$\left|\left|\left| P'_{m,n} - P'_{m,n+p} \right|\right|\right|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall n \geq n_\varepsilon, \ \forall p \geq 0$$

(this is equivalent to $\lim_{n\to\infty} \left(P'_{m,n} - P'_{m,n+p}\right) = 0$ uniformly with respect to $p \geq 0$), i.e., $\left(P'_{m,n}\right)_{n>m}$ is a Cauchy sequence, therefore is convergent.

Now, since $\exists \Delta' \in \mathrm{Par}\,(S)$ such that the chain $(P'_n)_{n\geq 1}$ is weakly Δ'-ergodic and $\left(P'_{m,n}\right)_{n>m}$ is convergent, $\forall m \geq 0$, the chain $(P'_n)_{n\geq 1}$ is strongly Δ'-ergodic. $\square$

Another main result is as follows.

THEOREM 1.44. *Let $(P_n)_{n\geq 1}$ be a Markov chain and $(P'_n)_{n\geq 1}$ a perturbation of the first type of it. If $(P_n)_{n\geq 1}$ is strongly Δ-ergodic and limit strongly Δ-ergodic and $(P'_n)_{n\geq 1}$ is strongly Δ'-ergodic, then $\Delta' \preceq \Delta$.*

Proof. It follows from Theorem 1.28(iii) that $(P'_n)_{n\geq 1}$ is limit strongly Δ-ergodic. Therefore $\Delta' \preceq \Delta$. $\square$

If $(P_n)_{n\geq 1}$ is $[\Delta]$-simple we can say more.

THEOREM 1.45. *Let $(P_n)_{n\geq 1}$ be a $[\Delta]$-simple Markov chain and $(P'_n)_{n\geq 1}$ a perturbation of the first type of it. If $(P_n)_{n\geq 1}$ is strongly Δ-ergodic and $(P'_n)_{n\geq 1}$ is strongly Δ'-ergodic, then $\Delta' \preceq \Delta$.*

Proof. See Theorems 1.38 and 1.44. $\square$

If $(P_n)_{n\geq 1}$ is $[\Delta]$-simple and $(P'_n)_{n\geq 1}$ is $[\Delta]$-groupable we can say even more.

THEOREM 1.46. *Let $(P_n)_{n\geq 1}$ be a $[\Delta]$-simple Markov chain and $(P'_n)_{n\geq 1}$ a $[\Delta]$-groupable perturbation of the first type of it. Then $(P_n)_{n\geq 1}$ is strongly Δ-ergodic if and only if $(P'_n)_{n\geq 1}$ is strongly Δ-ergodic.*

Proof. "$\Rightarrow$" We use the looping method. If $(P_n)_{n\geq 1}$ is strongly Δ-ergodic, then by Theorem 1.38 it is limit strongly Δ-ergodic. By Theorem 1.28(iii), the chain $(P'_n)_{n\geq 1}$ is limit strongly Δ-ergodic. Now, it follows from Theorem 1.41 that $(P'_n)_{n\geq 1}$ is strongly Δ-ergodic.

"$\Leftarrow$" If $(P'_n)_{n\geq 1}$ is strongly Δ-ergodic, then it is weakly Δ-ergodic. By Theorem 1.32, $(P_n)_{n\geq 1}$ is weakly Δ-ergodic. Now, it follows from Theorem 1.43 that $(P_n)_{n\geq 1}$ is strongly Δ-ergodic. $\square$

In particular, if $\Delta = (S)$, then we obtain a result of Fleischer and Joffe [2].

THEOREM 1.47 ([2]). *Let $(P_n)_{n\geq 1}$ be a Markov chain and $(P'_n)_{n\geq 1}$ a perturbation of the first type of it. Then $(P_n)_{n\geq 1}$ is strongly ergodic with limit Π if and only if $(P'_n)_{n\geq 1}$ is strongly ergodic with limit Π.*

Proof. See [2] or the following lines. By Theorem 1.46 for $\Delta = (S)$, $(P_n)_{n\geq 1}$ is strongly ergodic if and only if $(P'_n)_{n\geq 1}$ is strongly ergodic. Now, we show that both chains have the same limit. By symmetry, it is sufficient to prove just an implication. We use the looping method. If $(P_n)_{n\geq 1}$ is strongly ergodic with limit Π, then by Theorem 1.35 it is limit strongly ergodic with limit Π. By Theorem 1.28(ii), this implies that $(P'_n)_{n\geq 1}$ is limit strongly ergodic with limit Π. By Theorem 1.35 again, $(P'_n)_{n\geq 1}$ is strongly ergodic with limit Π. $\square$

Remark 1.48. (a) In general, in Theorems 1.44 and 1.45 we cannot have more than $\Delta' \preceq \Delta$. For this, see the examples from Remark 1.34.

(b) In Theorem 1.46 it is possible that the chains have different limits ($P_{m,n} \to \Pi_m,\ P'_{m,n} \to \Pi'_m$ as $n \to \infty$, $\forall m \geq 0$). Indeed, for

$$P_n = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} := A, \quad \forall n \geq 1.$$

and

$$P'_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad P'_n = A, \quad \forall n \geq 2,$$

both chains are $[(\{1,2\},\{3\})]$-simple and $(P'_n)_{n\geq 1}$ is a perturbation of the first type of $(P_n)_{n\geq 1}$. But $(P_n)_{n\geq 1}$ is strongly $(\{1,2\},\{3\})$-ergodic with (unique) limit

$$\Pi = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

while $(P'_n)_{n\geq 1}$ is strongly $(\{1,2\},\{3\})$-ergodic with limits

$$\Pi'_0 = P'_1\Pi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad \text{and} \quad \Pi'_m = \Pi, \quad \forall m \geq 1.$$

(c) Theorem 1.46 gives a simpler and more general criterion of strong Δ-ergodicity than that from Theorem 3.19 of [14].

(d) If $(P_n)_{n\geq 1}$ is $\left[(\{i\})_{i\in S}\right]$-simple and weakly (respectively, strongly) $(\{i\})_{i\in S}$-ergodic, then any perturbation of the first type of it is weakly (respectively, strongly) $(\{i\})_{i\in S}$-ergodic. An example is $P_n = I_r$, $\forall n \geq 1$, and other, more general, is $P_n = P$, $\forall n \geq 1$, where P is a permutation matrix.

Now, we consider $[\Delta]$-ergodicity under perturbation of the first type. For this, we first give the following result.

THEOREM 1.49. *Let $(P_n)_{n\geq 1}$ be a Markov chain and $(P'_n)_{n\geq 1}$ a perturbation of the first type of it.*

(i) *$(P_n)_{n\geq 1}$ is limit weakly $[\bar{\Delta}]$-ergodic in a generalized sense if and only if $(P'_n)_{n\geq 1}$ is limit weakly $[\bar{\Delta}]$-ergodic in a generalized sense.*

(ii) *$(P_n)_{n\geq 1}$ is limit strongly $[\bar{\Delta}]$-ergodic if and only if $(P'_n)_{n\geq 1}$ is limit strongly $[\bar{\Delta}]$-ergodic.*

Proof. See Theorems 1.28 and 1.28(i)$'$ and Theorem 2.13 from [14]. □

Remark 1.50. Theorems 1.32 and 1.46 cannot be generalized (for Δ-ergodicity) if $(P_n)_{n\geq 1}$ and $(P'_n)_{n\geq 1}$ are $[\Delta]$-groupable Markov chains. Indeed, let

$$P_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad P'_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1-\frac{1}{n^2} & \frac{1}{n^2} & 0 \end{pmatrix}, \quad \forall n \geq 1.$$

Because $P_{m,n} = P_{m+1}$ and $P'_{m,n} = P'_{m+1}$, $\forall m,n,\ 0 \leq m < n$, the chain $(P_n)_{n\geq 1}$ is weakly and strongly $(\{1,3\},\{2\})$-ergodic while $(P'_n)_{n\geq 1}$ is weakly and strongly $(\{1\},\{2\},\{3\})$-ergodic. Obviously, $(P'_n)_{n\geq 1}$ is a perturbation of the first type of $(P_n)_{n\geq 1}$ and both chains are $[(\{1\},\{2\},\{3\})]$-groupable.

Related to Theorems 1.32 and 1.46 and Remark 1.50 we show that weak and strong $[\Delta]$-ergodicity is preserved for $[\Delta]$-groupable Markov chains under perturbations of the first type.

THEOREM 1.51. *Let $(P_n)_{n\geq 1}$ be a $[\Delta]$-groupable Markov chain and $(P'_n)_{n\geq 1}$ a $[\Delta]$-groupable perturbation of the first type of it.*

(i) *$(P_n)_{n\geq 1}$ is weakly $[\Delta]$-ergodic if and only if $(P'_n)_{n\geq 1}$ is weakly $[\Delta]$-ergodic.*

(ii) $(P_n)_{n\geq 1}$ *is strongly* $[\Delta]$*-ergodic if and only if* $(P'_n)_{n\geq 1}$ *is strongly* $[\Delta]$*-ergodic.*

Proof. By symmetry, in both cases it is sufficient to prove just an implication.

(i) Suppose that $(P_n)_{n\geq 1}$ is weakly $[\Delta]$-ergodic. Then, by Theorem 1.21$'$, $(P_n)_{n\geq 1}$ is limit weakly $[\Delta]$-ergodic in a generalized sense. By Theorem 1.49, $(P'_n)_{n\geq 1}$ is limit weakly $[\Delta]$-ergodic in a generalized sense. Now, by Theorem 1.21$'$ again, $(P'_n)_{n\geq 1}$ is weakly $[\Delta]$-ergodic.

(ii) Suppose that $(P_n)_{n\geq 1}$ is strongly $[\Delta]$-ergodic. It follows that it is weakly $[\Delta]$-ergodic. It follows from (i) that $(P'_n)_{n\geq 1}$ is weakly $[\Delta]$-ergodic. But, by Theorem 1.43, $\exists \Delta' \in \operatorname{Par}(S)$ such that $(P'_n)_{n\geq 1}$ is strongly Δ'-ergodic. Now, the result is obvious because weak $[\Delta]$-ergodicity and strong Δ'-ergodicity imply strong $[\Delta]$-ergodicity. $\square$

Remark 1.52. (a) In particular, Theorem 1.51 can be applied to chains with 'transient states'. For this, let $\Delta = (K_1, K_2, \ldots, K_p, \{i\})_{i\in T} \in \operatorname{Par}(S)$, where $p \geq 1$ and $\emptyset \neq T = S - \bigcup_{u=1}^{p} K_u$, and let $(P_n)_{n\geq 1}$ be a Markov chain such that $\left((P_n)_{\mathcal{C}T}^{\mathcal{C}T}\right)_{n\geq 1}$ is $[(K_1, K_2, \ldots, K_p)]$-simple and $\exists n \geq 1$ such that $(P_n)_T^{\mathcal{C}T} \neq 0$ (in this case we say that $(P_n)_{n\geq 1}$ is with 'transient states'). It follows that the chain $(P_n)_{n\geq 1}$ is $[\Delta]$-groupable. Let $(P'_n)_{n\geq 1}$ be a $[\Delta]$-groupable perturbation of the first type of $(P_n)_{n\geq 1}$. By Theorem 1.51, $(P_n)_{n\geq 1}$ is weakly (strongly) $[\Delta]$-ergodic if and only if $(P'_n)_{n\geq 1}$ is weakly (strongly) $[\Delta]$-ergodic.

(b) (a) and Theorem 1.51(ii) give a better criterion of Δ-ergodicity than that from Theorem 3.17 of [14] (in fact,this theorem is due to Mukherjea and Chaudhuri [5]) in the special case when the chain $(P_n)_{n\geq 1}$ from there is $[\Delta]$-groupable, where Δ is as in (a). (See also Remark 2.15(a).)

Remark 1.53. From this section it is easy to see that we can obtain information about limit $[\Delta]$- and Δ-ergodicity. Thus, if $(P_n)_{n\geq 1}$ is a $[\Delta]$-simple and weakly (strongly) Δ-ergodic Markov chain, then any perturbation of the first type of it is limit weakly (strongly) Δ-ergodic in a generalized sense (in the usual sense).

2. UNIFORM ERGODICITY

In this section we give some results on uniform weak or strong ergodicity. Then considering perturbations of the second type and using the blocks method we obtain other results.

Definition 2.1 (see, e.g., [3, p. 221]). We say that a (finite) Markov chain $(P_n)_{n\geq 1}$ is *uniformly weakly ergodic* if $\forall i, j, k \in S$ we have

$$\lim_{n\to\infty}\left[(P_{m,m+n})_{ik} - (P_{m,m+n})_{jk}\right] = 0$$

uniformly with respect to $m \geq 0$.

Definition 2.2 (see, e.g., [3, p. 226]). We say that a Markov chain $(P_n)_{n\geq 1}$ is *uniformly strongly ergodic* if $\forall i, j \in S$ the limit

$$\lim_{n\to\infty}(P_{m,m+n})_{ij} := \pi_j$$

exists uniformly with respect to $m \geq 0$ and does not depend on i (it is easy to prove that this limit (when it exists) does not also depend on m).

The following theorem is on uniform weak ergodicity, but it can be generalized for uniform weak $[\Delta]$-ergodicity using $[\Delta]$-groupable Markov chains and $\bar{\gamma}_\Delta$ instead of $\bar{\alpha}$ (see Theorem 3.14 from [11] and Theorem 1.5 and Remark 1.6 from [12]).

THEOREM 2.3. *Let $(P_n)_{n\geq 1}$ be a Markov chain and k and k' two natural numbers. Then the following statements are equivalent.*

(i) *The chain is uniformly weakly ergodic.*

(ii) $\limsup_{l\to\infty}\limsup_{n\to\infty}\bar{\alpha}\left(P_{n-l,n}\right) = 0.$

(iii) $\limsup_{l\to\infty}\limsup_{n\to\infty}\bar{\alpha}\left(P_{n,n+l}\right) = 0.$

(iv) $\limsup_{l\to\infty}\limsup_{n\to\infty}\bar{\alpha}\left(P_{n-kl,n}\right) = 0,$ *if* $k \geq 1.$

(v) $\limsup_{l\to\infty}\limsup_{n\to\infty}\bar{\alpha}\left(P_{n,n+kl}\right) = 0,$ *if* $k \geq 1.$

(vi) $\limsup_{l\to\infty}\limsup_{n\to\infty}\bar{\alpha}\left(P_{n-k'l,n+kl}\right) = 0,$ *if* $k + k' \geq 1.$

Proof. (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv)$\Leftrightarrow$(v) See Theorem 3.14 from [11] and Theorem 1.5 and Remark 1.6 from [12].

(v)$\Rightarrow$(vi) The case $k' = 0$ is obvious. For $k' \geq 1$, from

$$\bar{\alpha}\left(P_{n-k'l,n+kl}\right) \leq \bar{\alpha}\left(P_{n-k'l,n}\right)\bar{\alpha}\left(P_{n,n+kl}\right) \leq \bar{\alpha}\left(P_{n,n+kl}\right),$$

using (v), we have (vi).

(vi)$\Rightarrow$(v) The case $k' = 0$ is obvious. Now, let $k' \geq 1$. Setting $u = n - k'l$, we have $n + kl = u + (k + k')\,l$, so that

$$0 = \limsup_{l\to\infty}\limsup_{n\to\infty}\bar{\alpha}\left(P_{n-k'l,n+kl}\right) = \limsup_{l\to\infty}\limsup_{u\to\infty}\bar{\alpha}\left(P_{u,u+(k+k')l}\right).$$

Therefore (v) holds with $k + k'$ instead of k. But (v) implies (i), i.e., the chain is uniformly weakly ergodic. Since (i) implies (v), we obtain (v). $\square$

PROPOSITION 2.4. *Let $P, Q, \Pi \in S_r$ and Π be a stable matrix (i.e., it have all rows identically). Then*
(i) $\bar{\alpha}(P) \leq |||P - \Pi|||_\infty$;
(ii) $|||PQ - \Pi|||_\infty \leq |||Q - \Pi|||_\infty$;
(iii) $|||PQ - \Pi|||_\infty \leq |||P - \Pi|||_\infty$, *if* $\Pi Q = \Pi$;
(iv) $\bar{\alpha}(PQ) \leq |||Q - \Pi|||_\infty$;
(v) $\bar{\alpha}(PQ) \leq |||P - \Pi|||_\infty$, *if* $\Pi Q = \Pi$.

Proof. (i)

$$\bar{\alpha}(P) = \frac{1}{2} \max_{1\leq i,j\leq r} \sum_{k=1}^{r} |P_{ik} - P_{jk}| \leq$$

$$\leq \frac{1}{2} \max_{1\leq i,j\leq r} \sum_{k=1}^{r} (|P_{ik} - \Pi_{ik}| + |\Pi_{jk} - P_{jk}|) \leq$$

$$\leq \frac{1}{2} \max_{1\leq i\leq r} \sum_{k=1}^{r} |P_{ik} - \Pi_{ik}| + \frac{1}{2} \max_{1\leq j\leq r} \sum_{k=1}^{r} |\Pi_{jk} - P_{jk}| =$$

$$= \frac{1}{2} |||P - \Pi|||_\infty + \frac{1}{2} |||P - \Pi|||_\infty = |||P - \Pi|||_\infty .$$

(ii)

$$|||PQ - \Pi|||_\infty = |||PQ - P\Pi|||_\infty = |||P(Q - \Pi)|||_\infty \leq$$

$$\leq |||P|||_\infty |||Q - \Pi|||_\infty = |||Q - \Pi|||_\infty .$$

(iii) Similar to (ii).
(iv) Using (i) and (ii), we have

$$\bar{\alpha}(PQ) \leq |||PQ - \Pi|||_\infty \leq |||Q - \Pi|||_\infty .$$

(v) Similar to (iv). □

THEOREM 2.5 ([9]). *Let $(P_n)_{n\geq 1}$ be a Markov chain. Then it is uniformly strongly ergodic if and only if it is uniformly weakly ergodic and strongly ergodic.*

Proof. See [9]. □

For uniform strong ergodicity the result similar to Theorem 2.3 is

THEOREM 2.6. *Let $(P_n)_{n\geq 1}$ be a Markov chain and k and k' two natural numbers. Then the following statements are equivalent.*
(i) *The chain is uniformly strongly ergodic with limit* Π.
(ii) $\limsup_{l\to\infty} \limsup_{n\to\infty} |||P_{n-l,n} - \Pi|||_\infty = 0$.
(iii) $\limsup_{l\to\infty} \limsup_{n\to\infty} |||P_{n,n+l} - \Pi|||_\infty = 0$.
(iv) $\limsup_{l\to\infty} \limsup_{n\to\infty} |||P_{n-kl,n} - \Pi|||_\infty = 0$, *if* $k \geq 1$.

(v) $\limsup_{l\to\infty}\limsup_{n\to\infty} |||P_{n,n+kl} - \Pi|||_\infty = 0,$ *if* $k \geq 1$.

(vi) $\limsup_{l\to\infty}\limsup_{n\to\infty} \left|\left|\left|P_{n-k'l,n+kl} - \Pi\right|\right|\right|_\infty = 0,$ *if* $k + k' \geq 1$.

Proof. We only prove (i)⇔(iii). The others are left to the reader (see the proof of Theorem 2.3 and use Proposition 2.4).

(i)⇒(iii) By Theorem 2.5, the chain $(P_n)_{n\geq 1}$ is strongly ergodic with limit Π, i.e., $P_{m,n} \to \Pi$ as $n \to \infty$, $\forall m \geq 0$. For $0 \leq m < n < n + l$, we have

$$|||P_{n,n+l} - \Pi|||_\infty \leq |||P_{n,n+l} - P_{m,n+l}|||_\infty + |||P_{m,n+l} - \Pi|||_\infty =$$

$$= |||P_{n,n+l} - P_{m,n}P_{n,n+l}|||_\infty + |||P_{m,n+l} - \Pi|||_\infty \leq$$

(we use a well-known inequality, namely $|||RP|||_\infty \leq |||R|||_\infty\ \bar{\alpha}\,(P)$, where R is an $m \times n$ real matrix with $Re' = 0$, e' is the transpose of $e = (1, 1, \ldots, 1) \in \mathbf{R}^n$ and $P \in S_{n,p}$ (see, e.g., [12]))

$$\leq |||I_r - P_{m,n}|||_\infty\ \bar{\alpha}\,(P_{n,n+l}) + |||P_{m,n+l} - \Pi|||_\infty \leq$$

$$\leq 2\ \bar{\alpha}\,(P_{n,n+l}) + |||P_{m,n+l} - \Pi|||_\infty\,,$$

so that

$$\limsup_{n\to\infty} |||P_{n,n+l} - \Pi|||_\infty \leq 2 \limsup_{n\to\infty}\ \bar{\alpha}\,(P_{n,n+l})\,, \forall l \geq 1.$$

Further,

$$\limsup_{l\to\infty}\limsup_{n\to\infty} |||P_{n,n+l} - \Pi|||_\infty \leq 2 \limsup_{l\to\infty}\limsup_{n\to\infty}\ \bar{\alpha}\,(P_{n,n+l}) = 0$$

because the chain is uniformly weakly ergodic (see Theorems 2.3 and 2.5). Therefore (iii) holds.

(iii)⇒(i) By hypothesis and Proposition 2.4(i) we have

$$\limsup_{l\to\infty}\limsup_{n\to\infty}\ \bar{\alpha}\,(P_{n,n+l}) = 0,$$

i.e., $(P_n)_{n\geq 1}$ is uniformly weakly ergodic. Now, we show that $(P_n)_{n\geq 1}$ is strongly ergodic with limit Π. For this, if $0 \leq m < n$ and $l \geq 1$, by Proposition 2.4(ii) we have

$$|||P_{m,n+l} - \Pi|||_\infty = |||P_{m,n}P_{n,n+l} - \Pi|||_\infty \leq |||P_{n,n+l} - \Pi|||_\infty\,.$$

It follows from (iii) that

$$\limsup_{l\to\infty}\limsup_{n\to\infty} |||P_{m,n+l} - \Pi|||_\infty = 0, \quad \forall m \geq 0.$$

Let $m \geq 0$. Then $\forall \varepsilon > 0, \exists l_\varepsilon \geq 1$ such that

$$\limsup_{n\to\infty} |||P_{m,n+l} - \Pi|||_\infty < \varepsilon, \quad \forall l \geq l_\varepsilon.$$

This implies

$$\limsup_{n\to\infty} |||P_{m,n+l_\varepsilon} - \Pi|||_\infty < \varepsilon.$$

By the definition of lim sup we have

$$\lim_{n\to\infty} \sup_{p\geq n} |||P_{m,p+l_\varepsilon} - \Pi|||_\infty < \varepsilon,$$

so that $\exists n_\varepsilon \geq 1$ such that

$$\sup_{p\geq n} |||P_{m,p+l_\varepsilon} - \Pi|||_\infty < \varepsilon, \quad \forall n \geq n_\varepsilon.$$

Thus,

$$|||P_{m,p+l_\varepsilon} - \Pi|||_\infty < \varepsilon, \quad \forall p \geq n_\varepsilon.$$

Setting $u = p + l_\varepsilon$ and $u_\varepsilon = n_\varepsilon + l_\varepsilon$, we have

$$|||P_{m,u} - \Pi|||_\infty < \varepsilon, \quad \forall u \geq u_\varepsilon,$$

i.e., $P_{m,u} \to \Pi$ as $u \to \infty$. Finally, by Theorem 2.5 we obtain (i). □

Now, we consider perturbations of the second type to get other results.

THEOREM 2.7 ([8]). *Let $(P_n)_{n\geq 1}$ be a Markov chain and $(P'_n)_{n\geq 1}$ be a perturbation of the second type of it. Then the chain $(P_n)_{n\geq 1}$ is uniformly weakly ergodic if and only if the chain $(P'_n)_{n\geq 1}$ is uniformly weakly ergodic.*

Proof. See [8]. □

Further, we give a similar result of Theorem 2.7 which is due to Fleischer and Joffe [2].

THEOREM 2.8 ([2]). *Let $(P_n)_{n\geq 1}$ be a Markov chain and $(P'_n)_{n\geq 1}$ be a perturbation of the second type of it. Then the chain $(P_n)_{n\geq 1}$ is uniformly strongly ergodic with limit Π if and only if the chain $(P'_n)_{n\geq 1}$ is uniformly strongly ergodic with limit Π.*

Proof. See [2] or the following lines. By symmetry, it is sufficient to prove just an implication. Suppose that $(P_n)_{n\geq 1}$ is uniformly strongly ergodic with limit Π . Let $Q_n = P'_n - P_n$, $\forall n \geq 1$. Then $P'_n = P_n + Q_n$, $\forall n \geq 1$, and $Q_n \to 0$ as $n \to \infty$. Let $l \geq 1$ and $n \geq 0$. Then

$$\left|\left|\left|P'_{n,n+l} - \Pi\right|\right|\right|_\infty = |||P_{n,n+l} + R(n, n+l) - \Pi|||_\infty \leq$$
$$\leq |||P_{n,n+l} - \Pi|||_\infty + |||R(n, n+l)|||_\infty ,$$

where $R(n, n+l)$ is the sum of the terms left (there are $2^l - 1$ such terms and this number does not depend on n). Further,

$$\limsup_{n\to\infty} \left|\left|\left|P'_{n,n+l} - \Pi\right|\right|\right|_\infty \leq \limsup_{n\to\infty} |||P_{n,n+l} - \Pi|||_\infty , \quad \forall l \geq 1,$$

because $\lim_{n\to\infty} |||R(n,n+l)|||_\infty = 0$. Therefore

$$\limsup_{l\to\infty} \limsup_{n\to\infty} \left|\left|\left|P'_{n,n+l} - \Pi\right|\right|\right|_\infty = 0,$$

i.e., by Theorem 2.6, $(P'_n)_{n\geq 1}$ is uniformly strongly ergodic with limit Π. $\square$

Remark 2.9. (a) Related to Theorem 2.8, we mention that in [9] we obtained the following equivalence: $(P_n)_{n\geq 1}$ is uniformly strongly ergodic if and only if $(P'_n)_{n\geq 1}$ is uniformly strongly ergodic. Theorem 2.8 improves this result since it says, moreover, that both chains have the same limit.

(b) Using Theorem 2.8, it is easy to give another proof of a theorem of J.L. Mott (see, e.g., [3, pp. 226]). For the case $\lim_{n\to\infty} P_{n,n+k} = P$, where $k \geq 2$, see Theorem 2.13.

To get a generalization of Theorem 2.8 we shall use the blocks method. For this, consider two sequences $(k_1(n))_{n\geq 1}$ and $(k_2(n))_{n\geq 0}$ of nonzero natural numbers. With $(P_n)_{n\geq 1}$ we associate $\left(P_{n-k_1(n),n}\right)_{n\geq 1}$, if $n-k_1(n) \geq 0$, $\forall n \geq 1$, and $\left(P_{n,n+k_2(n)}\right)_{n\geq 0}$. These will be called *the sequence of left-hand blocks* and *the sequence of right-hand blocks*, respectively. Moreover, for sequences $(k_1(n))_{n\geq 0}$ and $(k_2(n))_{n\geq 0}$ of natural numbers with $k_1(n)+k_2(n) \geq 1$, $\forall n \geq 0$, with $(P_n)_{n\geq 1}$ we associate $\left(P_{n-k_1(n),n+k_2(n)}\right)_{n\geq 0}$, if $n-k_1(n) \geq 0, \forall n \geq 0$. This will be called *the sequence of bilateral blocks*. We used bilateral blocks in, e.g., Theorems 2.3 and 2.6 from this section and in Theorem 2.16 from [13].

Further, we shall use only left-hand blocks. The others are left to the reader. Consider a sequence of left-hand blocks $\left(P_{n-k(n),n}\right)_{n\geq 1}$. Either this or the sequence $(k(n))_{n\geq 1}$ determines a nondirected graph $(\mathbf{N}, L)$, where the set of natural numbers $\mathbf{N}$ is the set of vertices and $L = \{[n-k(n), n] \,|\, n \geq 1\}$ is the set of edges. Following [12], we call it *the graph of left-hand blocks*.

A *chain* in a nondirected graph (V, E) is a (finite or infinite) sequence of vertices $(n_s)_{s\in I}$ such that $[n_{s-1}, n_s] \in E, \forall s \in I - \{0\}$, where $I = \{0, 1, \ldots, t\}$ in the finite case and $I = \mathbf{N}$ in the infinite case. Set $C = [n_0, n_1, \ldots, n_t]$ for a finite chain and $C = [n_0, n_1, \ldots]$ for an infinite chain. Let

$$\mathcal{L} = \{C = [n_0, n_1, \ldots, n_t] \,|\, [n_s, n_{s+1}] \in L \text{ and } n_s < n_{s+1}, \forall s \in \{0, 1, \ldots, t-1\},$$
$$n_0 = 0, \text{ and } \nexists l \in \mathbf{N} \text{ such that } l - k(l) = n_t\}$$

and

$$\mathcal{M} = \{C = [n_0, n_1, \ldots] \,|\, [n_s, n_{s+1}] \in L \text{ and } n_s < n_{s+1}, \forall s \geq 0, \text{ and } n_0 = 0\}.$$

Definition 2.10 ([12]). Let $n \in \mathbf{N}^*$. We say that n is a *right-hand end* of the graph $(\mathbf{N}, L)$ if $\exists C = [n_0, n_1, \ldots, n_t] \in \mathcal{L}$ such that $n_t = n$.

Let $m \geq 0$. Consider the condition

(CRm) $\forall n \geq m, \exists l > n$ such that $n = l - k(l)$,

that we call *the condition of continuation to the right* for natural numbers (vertices) greater than or equal to m.

Let (here we correct $\mathcal{N}_n$ from [12]; we also mention that it is not used in the theorem we are going to prove)

$$\mathcal{N}_n = \Big\{C = [n_0, n_1, \ldots, n_t] \in \mathcal{L} \,|\, n_t > n \text{ and } \nexists s \in \{0, 1, \ldots, t-1\} \text{ with } n_s > n \text{ for which } \exists\ \widetilde{C} \in \mathcal{M} \text{ such that } n_s \in \widetilde{C}\Big\}, \quad \forall n \geq 0$$

(in words, $\mathcal{N}_n, n \geq 0$, is the set of finite chains from $\mathcal{L}$ with right-hand ends $> n$ which are connected with infinite chains from $\mathcal{M}$ at vertices $\leq n$).

Theorem 2.11 ([12]). *Let $(k(n))_{n\geq 1}$ be a sequence of nonzero natural numbers such that $k(n) \leq M$, $\forall n \geq 1$. Then*
(i) $1 \leq |\mathcal{M}| \leq M$;
(ii) $|\mathcal{N}_n| \leq M - |\mathcal{M}|$, $\forall n \geq 0$.

Proof. See [12]. □

For left-hand blocks, Theorem 2.7 has the following generalization.

Theorem 2.12 ([12]). *Let $(k(n))_{n\geq 1}$ be a bounded sequence of nonzero natural numbers such that* (CRv) *holds for some $v \geq 0$. Let $(P_n)_{n\geq 1}$ and $(P'_n)_{n\geq 1}$ be two Markov chains such that $P_{n-k(n),n} = P'_{n-k(n),n} + Q_n, \forall n \geq 1$, where $\lim\limits_{n\to\infty} Q_n = 0$. Then the chain $(P_n)_{n\geq 1}$ is uniformly weakly ergodic if and only if the chain $(P'_n)_{n\geq 1}$ is uniformly weakly ergodic.*

Proof. See [12]. □

With left-hand blocks, too, we can generalize Theorem 2.8.

Theorem 2.13. *Let $(k(n))_{n\geq 1}$ be a bounded sequence of nonzero natural numbers such that* (CRv) *holds for some $v \geq 0$. Let $(P_n)_{n\geq 1}$ and $(P'_n)_{n\geq 1}$ be two Markov chains such that $P_{n-k(n),n} = P'_{n-k(n),n} + Q_n$, $\forall n \geq 1$, where $\lim\limits_{n\to\infty} Q_n = 0$. Then the chain $(P_n)_{n\geq 1}$ is uniformly strongly ergodic with limit Π if and only if the chain $(P'_n)_{n\geq 1}$ is uniformly strongly ergodic with limit Π.*

Proof. Similar to that of Theorem 2.12. We consider two cases.

Case 1. $k(n) = 1$, $\forall n \geq 1$. See Theorem 2.8.

Case 2. $\exists n \geq 1$ such that $k(n) > 1$. Suppose that $k(n) \leq M$, $\forall n \geq 1$. Since condition (CRv) holds for some $v \geq 0$, then for any $n \geq v$, $n \neq 0$, $\exists C = C(n) = [n_0, n_1, \ldots] \in \mathcal{M}$ for which $\exists t \geq 1$ such that $n = n_t$.

Step 1. We construct two Markov chains. The chain C determines two Markov chains $\left(P_{n_{s-1},n_s}\right)_{s\geq 1}$ and $\left(P'_{n_{s-1},n_s}\right)_{s\geq 1}$ (clearly, these chains depend on n and we write $\left(P_{n_{s-1},n_s}(n)\right)_{s\geq 1}$ and $\left(P'_{n_{s-1},n_s}(n)\right)_{s\geq 1}$, respectively, when confusion can arise). Putting $R_n = P_{n-k(n),n}$ and $R'_n = P'_{n-k(n),n}$, $\forall n \geq 1$, we have $\left(P_{n_{s-1},n_s}\right)_{s\geq 1} = (R_{n_s})_{s\geq 1}$ and $(P'_{n_{s-1},n_s})_{s\geq 1} = (R'_{n_s})_{s\geq 1}$. Clearly, $R_{n_s} = R'_{n_s} + Q_{n_s}$, $\forall s \geq 1$, where $Q_{n_s} \to 0$ as $s \to \infty$.

Step 2. We show that $(R_{n_s})_{s\geq 1}$ is uniformly strongly ergodic with limit Π if and only if $\left(R'_{n_s}\right)_{s\geq 1}$ is uniformly strongly ergodic with limit Π. This is obvious from Theorem 2.8 (or Case 1).

Step 3. We show that $(P_n)_{n\geq 1}$ is uniformly strongly ergodic with limit Π if and only if $(P'_n)_{n\geq 1}$ is uniformly strongly ergodic with limit Π. By symmetry, it is sufficient to prove that $(P_n)_{n\geq 1}$ is uniformly strongly ergodic with limit Π when $(P'_n)_{n\geq 1}$ is uniformly strongly ergodic with limit Π. Let $n \geq l \geq M$. Clearly, $\exists s = s(l)$, $1 \leq s \leq t$, such that $n - l \leq n_{t-s}$ and suppose that it is the greatest number with this property. Setting

$$A\,(l,n) = \begin{cases} I_r, & \text{if } n-l = n_{t-s}, \\ P_{n-l,n_{t-s}}, & \text{if } n-l < n_{t-s}, \end{cases}$$

by Proposition 2.4(ii) we have

$$|||P_{n-l,n} - \Pi|||_\infty = \left|\left|\left|A\,(l,n)\,P_{n_{t-s},n} - \Pi\right|\right|\right|_\infty \leq$$

$$\leq \left|\left|\left|P_{n_{t-s},n} - \Pi\right|\right|\right|_\infty = \left|\left|\left|R_{n_{t-s},n_t} - \Pi\right|\right|\right|_\infty,$$

where $R_{n_v,n_w} := R_{n_{v+1}} R_{n_{v+2}} \dots R_{n_w}$, $\forall v, w,\ 0 \leq v < w$. Since $(P'_n)_{n\geq 1}$ is uniformly strongly ergodic with limit Π, $\left(R'_{n_s}\right)_{s\geq 1}$ is uniformly strongly ergodic with limit Π for any chain $\left(R'_{n_s}\right)_{s\geq 1}$. Further, by Step 2 and Theorem 2.6((i)$\Rightarrow$(ii)) we obtain

$$\limsup_{s\to\infty} \limsup_{w\to\infty} \left|\left|\left|R_{n_{w-s},n_w} - \Pi\right|\right|\right|_\infty = 0$$

for any chain $(R_{n_s})_{s\geq 1}$. By Theorem 2.11(i) and the inequality above we have

$$\limsup_{l\to\infty} \limsup_{n\to\infty} |||P_{n-l,n} - \Pi|||_\infty \leq \limsup_{s\to\infty} \limsup_{t\to\infty} \left|\left|\left|R_{n_{t-s},n_t}\,(n) - \Pi\right|\right|\right|_\infty = 0$$

($n = n_t$ and we write $R_{n_{t-s},n_t}\,(n)$ because confusion can arise). Hence the chain $(P_n)_{n\geq 1}$ is uniformly strongly ergodic with limit Π. To complete the proof we shall show that

$$\limsup_{s\to\infty} \limsup_{t\to\infty} \left|\left|\left|R_{n_{t-s},n_t}\,(n) - \Pi\right|\right|\right|_\infty = 0.$$

By Theorem 2.11(i) we can suppose that $\mathcal{M} = \{C_1, C_2, \ldots, C_N\}$, where $N \geq 1$. By the definition of lim sup we have

$$\limsup_{t\to\infty} \left|\left|\left|R_{n_{t-s},n_t}(n) - \Pi\right|\right|\right|_\infty = \lim_{t\to\infty} \sup_{k\geq t} \left|\left|\left|R_{n_{k-s},n_k}(n) - \Pi\right|\right|\right|_\infty =$$

$(n = n_k \in C_i = C_{i(n)}$, where $i \in \{1, 2, \ldots, N\})$

$$= \lim_{t\to\infty} \sup \bigcup_{i=1}^{N} \left\{ \left|\left|\left|R_{n_{k-s},n_k}(n) - \Pi\right|\right|\right|_\infty | n_k \in C_i, k \geq t \right\} =$$

$$= \lim_{t\to\infty} \max \left(\sup_{\substack{k\geq t,\\ n_k\in C_1}} \left|\left|\left|R_{n_{k-s},n_k} - \Pi\right|\right|\right|_\infty, \ldots, \sup_{\substack{k\geq t,\\ n_k\in C_N}} \left|\left|\left|R_{n_{k-s},n_k} - \Pi\right|\right|\right|_\infty \right) =$$

(by continuity of max)

$$= \max \left(\lim_{t\to\infty} \sup_{\substack{k\geq t,\\ n_k\in C_1}} \left|\left|\left|R_{n_{k-s},n_k} - \Pi\right|\right|\right|_\infty, \ldots, \lim_{t\to\infty} \sup_{\substack{k\geq t,\\ n_k\in C_N}} \left|\left|\left|R_{n_{k-s},n_k} - \Pi\right|\right|\right|_\infty \right) =$$

$$= \max \left(\limsup_{\substack{t\to\infty,\\ n_t\in C_1}} \left|\left|\left|R_{n_{t-s},n_t} - \Pi\right|\right|\right|_\infty, \ldots, \limsup_{\substack{t\to\infty,\\ n_t\in C_N}} \left|\left|\left|R_{n_{t-s},n_t} - \Pi\right|\right|\right|_\infty \right).$$

Further,

$$\lim_{s\to\infty} \max \left(\limsup_{\substack{t\to\infty,\\ n_t\in C_1}} \left|\left|\left|R_{n_{t-s},n_t} - \Pi\right|\right|\right|_\infty, \ldots, \limsup_{\substack{t\to\infty,\\ n_t\in C_N}} \left|\left|\left|R_{n_{t-s},n_t} - \Pi\right|\right|\right|_\infty \right) =$$

$$= \max \left(\lim_{s\to\infty} \limsup_{\substack{t\to\infty,\\ n_t\in C_1}} \left|\left|\left|R_{n_{t-s},n_t} - \Pi\right|\right|\right|_\infty, \ldots, \lim_{s\to\infty} \limsup_{\substack{t\to\infty,\\ n_t\in C_N}} \left|\left|\left|R_{n_{t-s},n_t} - \Pi\right|\right|\right|_\infty \right) =$$

$$= \max(0, \ldots, 0) = 0.$$

Therefore,

$$\limsup_{s\to\infty} \limsup_{t\to\infty} \left|\left|\left|R_{n_{t-s},n_t}(n) - \Pi\right|\right|\right|_\infty = 0. \quad \square$$

Example 2.14. Let

$$P_{2n-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_{2n} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

$$P'_{2n-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P'_{2n} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2n} & \frac{1}{2} - \frac{1}{2n} \\ \frac{1}{2} + \frac{1}{3n} & \frac{1}{2} - \frac{1}{3n} \end{pmatrix}, \quad \forall n \geq 1$$

(this was used in [12] for an example related to Theorem 2.12). Theorem 2.8 cannot be used because

$$P_{2n-1} = P'_{2n-1} + \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \forall n \geq 1.$$

On the other hand, Theorem 2.13 can be used for $k(1) = 1$, $k(n) = 2$, $\forall n \geq 2$. Indeed, we have

$$P_{2n-2,2n} = P_{2n-1}P_{2n} = P_{2n}, \quad P'_{2n-2,2n} = P'_{2n-1}P'_{2n} = P'_{2n},$$

$$P_{2n-1,2n+1} = P_{2n}P_{2n+1} = P_{2n}, \quad P'_{2n-1,2n+1} = P'_{2n}P'_{2n+1} = P'_{2n}, \quad \forall n \geq 1,$$

so that

$$P_{n-2,n} = P'_{n-2,n} + \begin{pmatrix} -\frac{1}{2\left[\frac{n}{2}\right]} & \frac{1}{2\left[\frac{n}{2}\right]} \\ -\frac{1}{3\left[\frac{n}{2}\right]} & \frac{1}{3\left[\frac{n}{2}\right]} \end{pmatrix}, \quad \forall n \geq 2.$$

The chain $(P_n)_{n\geq 1}$ is uniformly strongly ergodic with limit

$$\Pi = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

because

$$P_{m,n} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad \forall m \geq 0, \ \forall n \geq m+2.$$

Now, by Theorem 2.13, the chain $(P'_n)_{n\geq 1}$ is uniformly strongly ergodic with limit Π as above.

Finally, from this paper we also draw among other things the conclusions below.

Remark 2.15. (a) Weak and strong ergodicity (see Section 1) are preserved under perturbations of the first type. Also, weak and strong $[\Delta]$- and Δ-ergodicity in some cases. Moreover, for strong ergodicity, the limit is preserved. It follows that when we decide whether a chain $(P_n)_{n\geq 1}$ is weakly or/and strongly ergodic (also weakly or/and strongly $[\Delta]$- and Δ-ergodic in some cases), for simplification we can assign value 0 to each entry (i,j) for which $\sum\limits_{n\geq 1} (P_n)_{ij} < \infty$, adding then $(P_n)_{ij}$, $\forall n \geq 1$, to a entry (i,k) for which $\sum\limits_{n\geq 1} (P_n)_{ik} = \infty$.

(b) Uniform weak and strong ergodicity (see this section) are preserved under perturbations of the second type. Moreover, for uniform strong ergodicity, the limit is preserved. It follows that when we decide whether a chain $(P_n)_{n\geq 1}$ is uniformly weakly or/and strongly ergodic, for simplification we can assign value 0 to each entry (i,j) for which $(P_n)_{ij} \to 0$ as $n \to \infty$, adding then $(P_n)_{ij}, \forall n \geq 1$, to a entry (i,k) for which $(P_n)_{ik} \nrightarrow 0$ as $n \to \infty$.

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