ON THE POSSIBILISTIC APPROACH
TO A PORTFOLIO SELECTION PROBLEM

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Predictions about investor portfolio holdings can provide powerful tests of asset pricing theories. In the context of Markowitz portfolio selection problem, this paper develops a possibilistic mean VaR model with multi assets. Furthermore, through the introduction of a set of investor-specific characteristics, the methodology accommodates either homogeneous or heterogeneous anticipated rates of return models. Thus, we consider a mathematical programming model with probabilistic constraints and solve it by transforming this problem into a multiple objective linear programming problem. Also, we obtain our results in the framework of weighted possibilistic mean value approach.

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1. INTRODUCTION

The process of selecting a portfolio can be divided into two stages. The first stage starts with observation and experience and ends with beliefs about the future performance of available securities. The second stage starts with the relevant beliefs about future performances and ends with the choice of portfolio [9]. This paper is concerned with the second stage.

The problem of standard portfolio selection is as follows. Assume (a) n securities, (b) an initial sum of money to be invested, (c) the beginning of a holding period, (d) the end of the holding period.

The portfolio selection model of Markowitz [8, 9] consists of two interrelated modules:

- a nonlinear programming problem where risk-averse investors solve a utility maximization problem involving the risk and the expected rate of return of any portfolio, subject to the constraint of an efficiency frontier. The latter is defined pointwise, as a sequence of solutions to a quadratic programming problem which minimizes the risk associated with each possible portfolio’s
expected rate of return subject to the constraint that the elements of the portfolio be non-negative and sum to unity, and

– a parametric stochastic returns-generating process by which, in each period, the investors determine the requisite vector of expectations and the variance-covariance matrix of the investors’ anticipated rates of return on all risky assets.

The managers are constantly faced with the dilemma of guessing the direction of market moves in order to meet the return target for assets under management. Given the uncertainty inherent in financial markets, the managers are very cautious in expressing their market views. The information content in such cautious views can be best described as being “fuzzy” or vague, in terms of both the direction and size of market moves.

The rest of the paper is organized in the following manner. Section 2 proposes a formulation of the mean VaR portfolio selection model with multi assets problem. Section 3 considers an overview of the possibility theory and proposes a possibilistic mean VaR portfolio selection model. Some results relatively to efficient portfolios are stated. Also, in Section 4 we use results obtained for efficient portfolios in the frame of the weighted possibilistic mean value approach. Thus, we are able to extend some recent results in this field [4, 6, 7].

2. MEAN VaR PORTFOLIO SELECTION MULTIOBJECTIVE MODEL WITH TRANSACTION COSTS

2.1. MEAN DOWNSIDE-RISK FRAMEWORK

In this section we extend [7, 8, 11] to the case of $i \geq 1$ assets. In practice, investors are concerned about the risk that their portfolio value falls below a certain level. That is the reason why different measures of downside-risk are considered in the multi asset allocation problem. Let $v_i$, $i = 1, \ldots, k$, be the future portfolio value, i.e., the value of the portfolio by the end of the planning period. Then the probability

$$P(v_i < (VaR)_i), \quad i = 1, \ldots, k,$$

that the portfolio value $v_i$ falls below the $(VaR)_i$ level, is called the **shortfall probability**. The conditional mean value of the portfolio given that the portfolio value has fallen below $(VaR)_i$, called the **expected shortfall**, is defined as

$$E(v_i | v_i < (VaR)_i).$$

Other risk measures used in practice are the mean absolute deviation

$$E \left\{ |v_i - E(v_i)| | v_i < E(v_i) \right\},$$
and the semi-variance

$$E((v_i - E(v_i))^2 | v_i < E(v_i)),$$

where we only consider the negative deviations from the mean.

Let $x_j, j = 1, \ldots, n$, be the proportion of the total amount of money devoted to security $j$, and $M_{1j}$ and $M_{2j}$ the minimum and maximum proportions of the total amount of money devoted to security $j$. For $j = 1, \ldots, n$, $i = 1, \ldots, k$ let $r_{ji}$ be a random variable which is the rate of the $i$ return of security $j$. We then have $v_i = \sum_{j=1}^{n} r_{ji} x_j$.

Assume that an investor wants to allocate his/her wealth among $n$ risky securities. If the risk profile of the investor is determined in terms of $(VaR)_i$, $i = 1, \ldots, k$, a mean-$VaR$ efficient portfolio will be a solution of the multiobjective optimization problem

$$\begin{align*}
\text{max} \ & \ [E(v_1), \ldots, E(v_k)] \\
\text{subject to} \ & \ P\{v_i \leq (VaR)_i\} \leq \beta_i, \ i = 1, \ldots, k, \\
& \sum_{j=1}^{n} x_j = 1, \\
& M_{1j} \leq x_j \leq M_{2j}, \ j = 1, \ldots, n.
\end{align*}$$

In this model, the investor tries to maximize the future value of his/her portfolio, assuming the probability that the future value of his portfolio falls below $(VaR)_i$ is not greater than $\beta_i$, $i = 1, \ldots, k$.

2.2. THE PROPORTIONAL TRANSACTION COST MODEL

The introduction of transaction costs adds considerable complexity to the optimal portfolio selection problem. The problem is simplified if one assumes that the transaction costs are proportional to the amount of the risky asset traded, and there are no transaction costs on trades in the riskless asset. Transaction cost is one of the main sources of concern to managers see [1, 16].

Assume the rate of transaction cost of security $j$, $j = 1, \ldots, n$, and allocation of $i, i = 1, \ldots, k$, assets is $c_{ji}$, thus the transaction cost of security $j$ and allocation of $i$ assets is $c_{ji} x_j$. The transaction cost of portfolio $x = (x_1, \ldots, x_n)$ is $\sum_{j=1}^{n} c_{ji} x_j, \ i = 1, \ldots, k$. Considering the proportional transaction cost and
the shortfall probability constraint, we propose the mean VaR portfolio selection model with transaction costs formulated as

\[
\max_{x \in \mathbb{R}^n} \left[ E(v_1) - \sum_{j=1}^{n} c_{j1}x_j, \ldots, E(v_k) - \sum_{j=1}^{n} c_{jk}x_j \right]
\]

subject to

\[
P\{v_i \leq (VaR)_i\} \leq \beta_i, \quad i = 1, \ldots, k,
\]

\[
\sum_{j=1}^{n} x_j = 1,
\]

\[
M_{1j} \leq x_j \leq M_{2j}, \quad j = 1, \ldots, n.
\]

3. POSSIBILISTIC MEAN VaR PORTFOLIO SELECTION MODEL

3.1. POSSIBILISTIC THEORY

We consider the possibilistic theory proposed by Zadeh [17]. Let \( \tilde{a} \) and \( \tilde{b} \) be two fuzzy numbers with membership functions \( \mu_{\tilde{a}} \) and \( \mu_{\tilde{b}} \), respectively. The possibility operator (Pos) is defined (see [5]) as follows:

\[
\begin{align*}
\text{Pos}(\tilde{a} \leq \tilde{b}) &= \sup\{\min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)) \mid x, y \in \mathbb{R}, \ x \leq y\} \\
\text{Pos}(\tilde{a} < \tilde{b}) &= \sup\{\min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)) \mid x, y \in \mathbb{R}, \ x < y\} \\
\text{Pos}(\tilde{a} = \tilde{b}) &= \mu_{\tilde{a}}(\tilde{b})
\end{align*}
\]

In particular, when \( \tilde{b} \) is a crisp number \( b \), we have

\[
\begin{align*}
\text{Pos}(\tilde{a} \leq b) &= \sup\{\mu_{\tilde{a}}(x) \mid x \in \mathbb{R}, \ x \leq b\} \\
\text{Pos}(\tilde{a} < b) &= \sup\{\mu_{\tilde{a}}(x) \mid x \in \mathbb{R}, \ x < b\} \\
\text{Pos}(\tilde{a} = b) &= \mu_{\tilde{a}}(b).
\end{align*}
\]

Let \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) be a binary operation over real numbers. Then it can be extended to the set of fuzzy numbers. If for fuzzy numbers \( \tilde{a}, \tilde{b} \) we define \( \tilde{c} = f(\tilde{a}, \tilde{b}) \), then the membership function \( \mu_{\tilde{c}} \) is obtained from the membership functions \( \mu_{\tilde{a}} \) and \( \mu_{\tilde{b}} \) by the equation

\[
\mu_{\tilde{c}}(z) = \sup\{\min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)) \mid x, y \in \mathbb{R}, \ z = f(x, y)\}
\]

for \( z \in \mathbb{R} \). That is, the possibility that the fuzzy number \( \tilde{c} = f(\tilde{a}, \tilde{b}) \) achieves value \( z \in \mathbb{R} \) is as great as the most possibility combination of the real numbers \( x, y \) such that \( z = f(x, y) \), where the value of \( \tilde{a} \) and \( \tilde{b} \) are \( x \) and \( y \), respectively.
3.2. TRIANGULAR AND TRAPEZOIDAL FUZZY NUMBERS

Let the rate of return on security given by a trapezoidal fuzzy number be 
\[ \tilde{r} = (r_1, r_2, r_3, r_4), \] where \( r_1 < r_2 \leq r_3 < r_4 \). Then the membership function of 
the fuzzy number \( \tilde{r} \) is

\[
\mu(x) = \begin{cases} 
\frac{x - r_1}{r_2 - r_1} & r_1 \leq x \leq r_2, \\
1 & r_2 \leq x \leq r_3, \\
\frac{x - r_4}{r_3 - r_4} & r_3 \leq x \leq r_4, \\
0 & \text{otherwise}.
\end{cases}
\] (3.4)

We recall that a trapezoidal fuzzy number is said to be triangular if 
\( r_2 = r_3 \). Let us consider two trapezoidal fuzzy numbers \( \tilde{r} = (r_1, r_2, r_3, r_4) \) and 
\( \tilde{b} = (b_1, b_2, b_3, b_4) \), as shown in Figure 3.1.

If \( r_2 \geq b_3 \) then we have

\[
\text{Pos}(\tilde{r} \leq \tilde{b}) = \sup\{\min(\mu_{\tilde{r}}(x), \mu_{\tilde{b}}(y)) \mid x \leq y\} \geq \\
\geq \min\{\mu_{\tilde{r}}(r_2), \mu_{\tilde{b}}(b_3)\} = \min\{1, 1\} = 1,
\]

which implies that \( \text{Pos}(\tilde{r} \leq \tilde{b}) = 1 \). If \( r_2 \geq b_3 \) and \( r_1 \leq b_4 \), then the supremum 
occurs at the abscissa \( \delta_x \) of the intersection of the graphs of the functions 
\( \mu_{\tilde{r}}(x) \) and \( \mu_{\tilde{b}}(x) \). A simple computation shows that

\[
\text{Pos}(\tilde{r} \leq \tilde{b}) = \delta = \frac{b_4 - r_1}{(b_4 - b_3) + (r_2 - r_1)}
\]

and

\[
\delta_x = r_1 + (r_2 - r_1) \delta.
\]

If \( r_1 > b_4 \) then for any \( x < y \) at least one of the equations

\[
\mu_{\tilde{r}}(x) = 0, \quad \mu_{\tilde{b}}(y) = 0
\]
holds. Thus we have \( \text{Pos}\{\tilde{r} \leq \tilde{b}\} = 0 \). Now, we summarize the above results as

\[
\text{Pos}\{\tilde{r} \leq \tilde{b}\} = \begin{cases} 
1 & r_2 \leq b_3, \\
\delta & r_2 \geq b_3, r_1 \leq b_4, \\
0 & r_1 \geq b_4.
\end{cases}
\]

Especially, when \( \tilde{b} \) is the crisp number 0 we have

\[
\text{Pos}\{\tilde{r} \leq 0\} = \begin{cases} 
1 & r_2 \leq 0, \\
\delta & r_1 \leq 0 \leq r_2, \\
0 & r_1 \geq 0,
\end{cases}
\]

where \( \delta = \frac{r_1}{r_1 - r_2} \).

We now state

**Lemma 3.1 ([5]).** Consider the trapezoidal fuzzy number \( \tilde{r} = (r_1, r_2, r_3, r_4) \). Then for any given confidence level \( \alpha, 0 \leq \alpha \leq 1 \), we have \( \text{Pos}(\tilde{r} \leq 0) \geq \alpha \) if and only if \( (1 - \alpha)r_1 + \alpha r_2 \leq 0 \).

The \( \lambda \) level set of a fuzzy number \( \tilde{r} = (r_1, r_2, r_3, r_4) \) is a crisp subset of \( R \) denoted by \( [\tilde{r}]^\lambda = \{x | \mu(x) \geq \lambda, \ x \in R\} \). According to Carlsson et al. [4] we have

\[
[\tilde{r}]^\lambda = \{x | \mu(x) \geq \lambda, \ x \in R\} = [r_1 + \lambda(r_2 - r_1), r_4 - \lambda(r_4 - r_3)].
\]

Given \( [\tilde{r}]^\lambda = \{a_1(\lambda), a_2(\lambda)\} \), the crisp possibilistic mean value of \( \tilde{r} = (r_1, r_2, r_3, r_4) \) is

\[
\tilde{E}(\tilde{r}) = \int_0^1 \lambda(a_1(\lambda) + a_2(\lambda))d\lambda,
\]

where \( \tilde{E} \) stands for the fuzzy mean operator. We can see that if \( \tilde{r} = (r_1, r_2, r_3, r_4) \) is a trapezoidal fuzzy number, then

\[
\tilde{E}(\tilde{r}) = \int_0^1 \lambda(r_1 + \lambda(r_2 - r_1) + r_4 - \lambda(r_4 - r_3))d\lambda = \frac{r_2 + r_3}{3} + \frac{r_1 + r_4}{6}.
\]

### 3.3. Efficient Portfolios

Let \( x_j \) the the proportion of the total amount of money devoted to security \( j \), \( M_1 \) and \( M_2 \), the minimum and maximum proportions of the total amount of money devoted to security \( j \). The trapezoidal fuzzy number of \( r_{ji} \) is \( \tilde{r}_{ji} = (r_{(ji)1}, r_{(ji)2}, r_{(ji)3}, r_{(ji)4}) \), where \( r_{(ji)1} < r_{(ji)2} \leq r_{(ji)3} < r_{(ji)4} \). In addition, let us represent the \((\text{VaR})_i\) level by the trapezoidal fuzzy number \( \tilde{b}_i = (b_{i1}, b_{i2}, b_{i3}, b_{i4}), i = 1, \ldots, k \). So, we see that the model given by (2.5)–(2.8) reduces to the form from the next result.
**Theorem 3.1.** The possibilistic mean VaR portfolio selection model for the vector mean VaR efficient portfolio model (2.5)–(2.8) is

\[
\max_{x \in \mathbb{R}^n} \left\{ \tilde{E} \left( \sum_{i=1}^{n} \tilde{r}_{ji} x_j \right) - \sum_{i=1}^{n} c_{j1} x_j - \cdots - \tilde{E} \left( \sum_{i=1}^{n} \tilde{r}_{jk} x_j \right) - \sum_{i=1}^{n} c_{jk} x_j \right\}
\]

subject to

\[
\text{Pos} \left( \sum_{i=1}^{n} \tilde{r}_{jk} x_j < \tilde{b}_i \right) \leq \beta_i, \quad i = 1, \ldots, k,
\]

\[
\sum_{i=1}^{n} x_j = 1,
\]

\[
M_{1j} \leq x_j \leq M_{2j}, \quad j = 1, \ldots, n.
\]

In what follows, to obtain the efficient portfolios given by Theorem 3.1 we use White [15], namely,

**Theorem 3.2.** If \( \lambda_i > 0, \ i = 1, \ldots, k \), then an efficient portfolio for the possibilistic model is an optimal solution of the problem

\[
\max_{x \in \mathbb{R}^n} \sum_{i=1}^{n} \lambda_i \left[ \tilde{E} \left( \sum_{i=1}^{n} \tilde{r}_{ji} x_j \right) - \sum_{i=1}^{n} c_{ji} x_j \right]
\]

subject to

\[
\text{Pos} \left( \sum_{i=1}^{n} \tilde{r}_{jk} x_j < \tilde{b}_i \right) \leq \beta_i, \quad i = 1, \ldots, k,
\]

\[
\sum_{i=1}^{n} x_j = 1,
\]

\[
M_{1j} \leq x_j \leq M_{2j}, \quad j = 1, \ldots, n.
\]

We can now state

**Theorem 3.3.** Let the rate of return on security \( j, \ j = 1, \ldots, n \), be represented by the trapezoidal fuzzy number \( \tilde{r}_{ji} = (r_{(ji)1}, r_{(ji)2}, r_{(ji)3}, r_{(ji)4}) \), where \( r_{(ji)1} < r_{(ji)2} \leq r_{(ji)3} < r_{(ji)4} \). Let \( \tilde{b}_i = (b_{1i}, b_{2i}, b_{3i}, b_{4i}) \) be the trapezoidal fuzzy number for VaR level and \( \lambda_i > 0 \), with \( i = 1, \ldots, k \). Then for the
possibilistic mean VaR portfolio selection model an efficient portfolio is an optimal solution of the problem

\[
\max_{x \in \mathbb{R}^n} \sum_{i=1}^{n} \lambda_i \left[ \frac{\sum_{j=1}^{n} r_{ji}^2 x_j + \sum_{j=1}^{n} r_{ji}^3 x_j}{3} + \frac{\sum_{j=1}^{n} r_{ji} x_j + \sum_{j=1}^{n} r_{ji}^4 x_j}{6} - \sum_{j=1}^{n} c_{ji} x_j \right]
\]

subject to

\[
(3.18) \quad (1 - \beta_i) \left( \sum_{j=1}^{n} r_{ji} x_j - b_i^4 \right) + \beta_i \left( \sum_{j=1}^{n} r_{ji}^2 x_j - b_i^3 \right) \geq 0, \quad i = 1, \ldots, k,
\]

\[
(3.19) \quad \sum_{j=1}^{n} x_j = 1,
\]

\[
(3.20) \quad M_{1j} \leq x_j \leq M_{2j}, \quad j = 1, \ldots, n.
\]

Proof. From equation (3.8) we get

\[
\tilde{E} \left( \sum_{j=1}^{n} \tilde{r}_{ji} x_j \right) = \frac{\sum_{j=1}^{n} r_{ji}^2 x_j}{3} + \frac{\sum_{j=1}^{n} r_{ji} x_j + \sum_{j=1}^{n} r_{ji}^4 x_j}{6},
\]

\( i = 1, \ldots, k. \) By Lemma 3.1,

\[
\text{Pos} \left( \sum_{i=1}^{n} \tilde{r}_{jk} x_j < \tilde{b}_i \right) \leq \beta_i, \quad i = 1, \ldots, k,
\]

is equivalent to

\[
(1 - \beta_i) \left( \sum_{j=1}^{n} r_{ji} x_j - b_i^4 \right) + \beta_i \left( \sum_{j=1}^{n} r_{ji}^2 x_j - b_i^3 \right) \geq 0.
\]

Therefore, Theorem 3.3 follows from Theorem 3.2. \( \square \)

Problem (3.17)–(3.20) is a standard multi-objective linear programming problem. For finding the optimal solution, we can use several algorithms [12, 14].

4. WEIGHTED POSSIBILISTIC MEAN VALUE APPROACH

In this section, after introducing a weighting function that measures the importance of the \( \lambda \)-level sets of fuzzy numbers, we define the weighted lower possibilistic and upper possibilistic mean values as well as the crisp possibilistic
mean value of fuzzy numbers, which is consistent with the extension principle and with the well-known definition of expectation in probability theory. We also show that the weighted interval-valued possibilistic mean is always a subset (moreover, a proper subset excluding some special cases) of the interval-valued probabilistic mean for any weighting function.

A trapezoidal fuzzy number \( \tilde{r} = (r_1, r_2, r_3, r_4) \) is a fuzzy set of the real line \( \mathbb{R} \) with a normal, fuzzy convex and continuous membership function of bounded support. The family of fuzzy numbers will be denoted by \( \mathcal{F} \). A \( \lambda \)-level set of a fuzzy number \( \tilde{r} = (r_1, r_2, r_3, r_4) \) is defined as \( [\tilde{r}]^\lambda = \{ x \mid \mu(x) \geq \lambda, \ x \in \mathbb{R} \} \), so that

\[
[\tilde{r}]^\lambda = \{ x \mid \mu(x) \geq \lambda, \ x \in \mathbb{R} \} = [r_1 + \lambda(r_2 - r_1), r_4 - \lambda(r_1 - r_4)],
\]

if \( \lambda > 0 \) and \( [\tilde{r}]^\lambda = \text{cl}\{x \in \mathbb{R} \mid \mu(x) \geq 0 \} \) (the closure of the support of \( \tilde{r} \)) if \( \lambda = 0 \). It is well-known that if \( \tilde{r} \) is a fuzzy number, then \( [\tilde{r}]^\lambda \) is a compact subset of \( \mathbb{R} \) for all \( \lambda \in [0, 1] \).

**Definition 4.1** ([6]). Let \( \tilde{r} \in \mathcal{F} \) be a fuzzy number with \( [\tilde{r}]^\lambda = [a_1(\lambda), a_2(\lambda)] \), \( \lambda \in [0, 1] \). A function \( w : [0, 1] \rightarrow \mathbb{R} \) is said to be a weighting function if it is nonnegative, increasing, and satisfies the normalization condition

\[
(4.1) \quad \int_0^1 w(\lambda)d\lambda = 1.
\]

We define the \( w \)-weighted possibilistic mean (or expected) value of a fuzzy number \( \tilde{r} \) as

\[
(4.2) \quad \overline{E}_w(\tilde{r}) = \int_0^1 \frac{a_1(\lambda) + a_2(\lambda)}{3} w(\lambda)d\lambda.
\]

Note that if \( w(\lambda) = 2\lambda, \ \lambda \in [0, 1] \), then

\[
\overline{E}_w(\tilde{r}) = \int_0^1 [a_1(\lambda) + a_2(\lambda)]\lambda d\lambda.
\]

Thus, the \( w \)-weighted possibilistic mean value defined by (4.2) can be considered as a generalization of possibilistic mean value in [6]. From the definition of a weighting function, it can be seen that \( w(\lambda) \) might be zero for certain (nonimportant) \( \lambda \)-level sets of \( \tilde{r} \). So, by introducing different weighting functions we can give different (case-dependent) importance to \( \lambda \)-levels sets of fuzzy numbers.

Let \( \tilde{r} = (r_1, r_2, \alpha, \beta) \) be a trapezoidal fuzzy number with peak \( [r_1, r_2] \), left-width \( \alpha > 0 \) and right-width \( \beta > 0 \), and let

\[
(4.3) \quad w(\lambda) = (2q - 1)(1 - \lambda)^{-1/2q} - 1,
\]
where \( q \geq 1 \). It is clear that \( w \) is a weighting function with \( w(0) = 0 \) and \( \lim_{q \to 1} w(\lambda) = \infty \). Then the \( w \)-weighted lower and upper possibilistic mean values of \( \tilde{r} \) are given by
\[
\tilde{E}_w^-(\tilde{r}) = \int_0^1 [r_1 - (1 - \lambda)\alpha(2q - 1)](1 - \lambda)^{-1/2q - 1}d\lambda = r_1 - \frac{\alpha(2q - 1)}{2(4q - 1)}
\]
and
\[
\tilde{E}_w^+(\tilde{r})(\tilde{r}) = \int_0^1 [r_2 + (1 - \lambda)\beta(2q - 1)](1 - \lambda)^{-1/2q - 1}d\lambda = r_2 + \frac{\beta(2q - 1)}{2(4q - 1)}.
\]
Therefore,
\[
\tilde{E}_w(\tilde{r}) = \left[ r_1 - \frac{\alpha(2q - 1)}{2(4q - 1)}, \frac{r_2 + \beta(2q - 1)}{2(4q - 1)} \right],
\]

This remark along with Theorem 3.1 leads to the following result.

**Theorem 4.1.** The mean VaR efficient portfolio model is
\[
\max_{x \in \mathbb{R}^n} \sum_{i=1}^k \lambda_i \left[ \tilde{E}_w \left( \sum_{j=1}^n \tilde{r}_{ji}x_j \right) - \sum_{i=1}^k c_{ji}x_j \right]
\]
subject to
\[
\text{Pos} \left( \sum_{j=1}^n \tilde{r}_{ji}x_j < \tilde{b}_i \right) \leq \beta_i, \quad i = 1, \ldots, k,
\]
\[
\sum_{i=1}^n x_j = 1,
\]
\[
M_{1j} \leq x_j \leq M_{2j}, \quad j = 1, \ldots, n.
\]

The next result extends Theorem 3.3 to the case of weighted possibilistic mean value approach, with the special weighting function (4.3).

**Theorem 4.2.** Let \( w(\lambda) = (2q - 1)(1 - \lambda)^{-3q - 1}, q \geq 1 \), and let the rate of return on security \( j \), \( j = 1, \ldots, n \), be represented by the trapezoidal number \( \tilde{r}_{ji} = (r_{(ji)1}, r_{(ji)2}, r_{(ji)3}, r_{(ji)4}) \), where \( r_{(ji)1} < r_{(ji)2} \leq r_{(ji)3} < r_{(ji)4} \). Let \( \tilde{b}_i = (b_{i1}, b_{i2}, b_{i3}, b_{i4}) \) be the trapezoidal fuzzy number for \( (VaR)_i \), \( i = 1, \ldots, k \).
Then the possibilistic mean VaR portfolio selection model is

\[
\max_{\mathbf{x} \in \mathbb{R}^n} \sum_{i=1}^{k} \lambda_i \left[ \frac{\sum_{j=1}^{n} r_{(ji)}1x_j + \sum_{j=1}^{n} r_{(ji)}2x_j}{2} + \frac{(2q-1)\left( \sum_{j=1}^{n} r_{(ji)}4x_j - \sum_{j=1}^{n} r_{(ji)}3x_j \right)}{4(4q-1)} - \sum_{j=1}^{n} c_{ji}x_j \right]
\]

subject to

\[
(1 - \beta_i) \left( \sum_{j=1}^{n} r_{(ji)}1x_j - b_{i4} \right) + \beta_i \left( \sum_{j=1}^{n} r_{(ji)}2x_j - b_{i3} \right) \geq 0, \quad i = 1, \ldots, k,
\]

\[
\sum_{j=1}^{n} x_j = 1,
\]

\[
M_{1j} \leq x_j \leq M_{2j}, \quad j = 1, \ldots, n.
\]

**Proof.** From equation (4.2) we get

\[
\widetilde{E}\left( \sum_{j=1}^{n} \tilde{r}_{ji}x_j \right) = \frac{\sum_{j=1}^{n} r_{(ji)}1x_j + \sum_{j=1}^{n} r_{(ji)}2x_j}{2} + \frac{(2q-1)\left( \sum_{j=1}^{n} r_{(ji)}4x_j - \sum_{j=1}^{n} r_{(ji)}3x_j \right)}{4(4q-1)},
\]

\[i = 1, \ldots, k, \quad q \geq 1.\] By Lemma 3.1,

\[\text{Pos}\left( \sum_{i=1}^{n} \tilde{r}_{ij}x_j < \tilde{b}_i \right) \leq \beta_i, \quad i = 1, \ldots, k,
\]

is equivalent to

\[(1 - \beta_i) \left( \sum_{j=1}^{n} r_{(ji)}1x_j - b_{i4} \right) + \beta_i \left( \sum_{j=1}^{n} r_{(ji)}2x_j - b_{i3} \right) \geq 0.
\]

Therefore, Theorem 4.2 follows from Theorem 3.2. \(\square\)

Problem (4.9)–(4.12) is a standard multi-objective linear programming problem. For finding its optimal solution, we can use several algorithms [12, 15].

Letting \(q \to \infty\) in (4.4), we obtain

\[
\lim_{q \to \infty} \widetilde{E}_w(\tilde{r}) = \frac{r_1 + r_2}{2} + \frac{\beta - \alpha}{8}.\]

Thus, we get
COROLLARY 4.1. For $q \to \infty$, the weighted possibilistic mean VaR efficient portfolio selection model can be reduced to the linear programming problem

$$
\max_{x \in \mathbb{R}^n} \sum_{i=1}^{k} \lambda_i \begin{bmatrix}
\sum_{j=1}^{n} r_{(ji)1} x_j + \sum_{j=1}^{n} r_{(ji)2} x_j \\
2
\end{bmatrix} + \begin{bmatrix}
\sum_{j=1}^{n} r_{(ji)4} x_j - \sum_{j=1}^{n} r_{(ji)3} x_j \\
8
\end{bmatrix} - \sum_{j=1}^{n} c_{ji} x_j
$$

subject to

$$(1 - \beta_i) \left( \sum_{j=1}^{n} r_{(ji)1} x_j - b_{i4} \right) + \beta_i \left( \sum_{j=1}^{n} r_{(ji)2} x_j - b_{i3} \right) \geq 0,$$

$$\sum_{j=1}^{n} x_j = 1,$$

$$M_{1j} \leq x_j \leq M_{2j}, \quad j = 1, \ldots, n.$$


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