Δ-ERGODIC THEORY AND RELIABILITY THEORY

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We show that the natural framework for the Markov chains considered in [5] is the Δ-ergodic theory (in a more general context, the general Δ-ergodic theory (see [8] and [11])) but not ergodic theory (for Δ-ergodic theory see, e.g., [8], [9], [10], and [11] and for ergodic theory see, e.g., [2], [3], [4], and [12]). For this, we give in Section 1 the notions, notation, and results from Δ-ergodic theory we need for the study of Markov chains from reliability theory. In Section 2 we then define and study these Markov chains.

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1. Δ-ERGODIC THEORY

In this section we deal with Δ-ergodic theory which appears to be the natural framework for the study of Markov chains from [5].

Consider a finite Markov chain $(X_n)_{n\geq 0}$ with state space $S = \{1, 2, \ldots, r\}$, initial distribution π_0, and transition matrices $(P_n)_{n\geq 1}$. We frequently shall refer to it as the (finite) Markov chain $(P_n)_{n\geq 1}$. For all integers $m \geq 0$, $n > m$, define

$$P_{m,n} = P_{m+1}P_{m+2}\cdots P_n = ((P_{m,n})_{ij})_{i,j\in S}\,.$$

(The entries of a matrix Z will be denoted Z_{ij}.)

Set

$$\mathrm{Par}(E) = \{\Delta \mid \Delta \text{ is a partition of } E\}\,,$$

where E is a nonempty set. We shall agree that the partitions do not contain the empty set, except for some cases (if needed) where this will be specified.

Definition 1.1. Let $\Delta_1, \Delta_2 \in \mathrm{Par}\,(E)$. We say that Δ_1 *is finer than* Δ_2 if $\forall V \in \Delta_1$, $\exists W \in \Delta_2$ such that $V \subseteq W$.

Write $\Delta_1 \preceq \Delta_2$ when Δ_1 is finer than Δ_2.

In Δ-ergodic theory the natural space is $S \times \mathbf{N}$, called *state-time space*. Let $\emptyset \neq A \subseteq S$ and $\emptyset \neq B \subseteq \mathbf{N}$. Let $\Sigma \in \mathrm{Par}(A)$. Frequently, when we only use a partition Σ of A we shall omit to mention this. Also we can omit Σ if $\Sigma = (\{i\})_{i\in A}\,$.

Definition 1.2 ([11]). Let $i, j \in S$. We say that i and j are in the same *weakly ergodic class on* $A \times B$ (or *on* $A \times B$ *with respect to* Σ, or *on* $(A \times B, \Sigma)$ when confusion can arise) if $\forall K \in \Sigma$, $\forall m \in B$ we have

$$\lim_{n\to\infty} \sum_{k\in K} [(P_{m,n})_{ik} - (P_{m,n})_{jk}] = 0.$$

Write $i \overset{A\times B}{\sim} j$ (with respect to Σ) (or $i \overset{(A\times B,\Sigma)}{\sim} j$) when i and j are in the same weakly ergodic class on $A \times B$. Then $\overset{A\times B}{\sim}$ is an equivalence relation and determines a partition $\Delta = \Delta(A \times B, \Sigma) = (C_1, C_2, \ldots, C_s)$ of S. The sets $C_1, C_2, \ldots, C_s$ are called *weakly ergodic classes on* $A \times B$.

Definition 1.3 ([11]). Let $\Delta = (C_1, C_2, \ldots, C_s)$ be the partition of weakly ergodic classes on $A \times B$ of a Markov chain. We say that the chain is *weakly* Δ*-ergodic on* $A \times B$. In particular, a weakly (S)-ergodic chain on $A \times B$ is called *weakly ergodic on* $A \times B$ for short.

Definition 1.4 ([11]). Let $(C_1, C_2, \ldots, C_s)$ be the partition of weakly ergodic classes on $A \times B$ of a Markov chain with state space S and $\Delta \in \mathrm{Par}(S)$. We say that the chain is *weakly* $[\Delta]$*-ergodic on* $A \times B$ if $\Delta \preceq (C_1, C_2, \ldots, C_s)$.

In connection with the above notions and notation we mention some special cases ($\Sigma \in \mathrm{Par}(A)$):

1. $A \times B = S \times \mathbf{N}$. In this case we can write $\sim$ instead of $\overset{S\times\mathbf{N}}{\sim}$ (or $\overset{\Sigma}{\sim}$ instead of $\overset{(S\times\mathbf{N},\Sigma)}{\sim}$) and can omit '*on* $S \times \mathbf{N}$' in Definitions 1.2, 1.3, and 1.4.

2. $A = S$. In this case we can write $\overset{B}{\sim}$ instead of $\overset{S\times B}{\sim}$ (or $\overset{(B,\Sigma)}{\sim}$ instead of $\overset{(S\times B,\Sigma)}{\sim}$) and can replace '$S \times B$' by '(*time set*) B (*with respect to* Σ)' (or by '(B, Σ)') in Definitions 1.2, 1.3, and 1.4. A special subcase is $B = \{m\}$ $(m \geq 0)$; in this situation we can write $\overset{m}{\sim}$ (or $\overset{(m,\Sigma)}{\sim}$) and can replace '*on* (*time set*) $\{m\}$' by '*at time* m' in Definitions 1.2, 1.3, and 1.4 .

3. $B = \mathbf{N}$. In this case we can set $\overset{A}{\sim}$ instead of $\overset{A\times\mathbf{N}}{\sim}$ (or $\overset{(A,\Sigma)}{\sim}$ instead of $\overset{(A\times\mathbf{N},\Sigma)}{\sim}$) and can replace '$A \times \mathbf{N}$' by '(*state set*) A (*with respect to* Σ)' (or by '(A, Σ)') in Definitions 1.2, 1.3, and 1.4.

Definition 1.5 ([11]). Let C be a weakly ergodic class on $A \times B$. Let $\emptyset \neq A_0 \subseteq A$ for which $\exists K_1, K_2, \ldots, K_p \in \Sigma$ such that $A_0 = \bigcup_{u=1}^{p} K_u$. Let $\emptyset \neq B_0 \subseteq B$. We say that C is a *strongly ergodic class on* $A_0 \times B_0$ *with respect to* $A \times B$ (*and* Σ) if $\forall i \in C$, $\forall K \in \Sigma$ with $K \subseteq A_0$, $\forall m \in B_0$ the limit

$$\lim_{n\to\infty} \sum_{j\in K} (P_{m,n})_{ij} := \sigma_{m,K} = \sigma_{m,K}(C)$$

exists and does not depend on i.

In connection with the last definition we mention some special cases:

1. $A \times B = A_0 \times B_0$. In this case we can say that C is a *strongly ergodic class on* $A \times B$. A special subcase is $A \times B = A_0 \times B_0 = S \times \mathbf{N}$ and $C = S$ when we can say that the Markov chain itself is *strongly ergodic.*

2. $A = A_0 = S$. In this case we can say that C is a *strongly ergodic class on* (*time set*) B_0 *with respect to* (*time set*) B. If $B = B_0$, then we can say that C is a *strongly ergodic class on* (*time set*) B. A special subcase of the case $A = A_0 = S$ and $B = B_0$ is $B = B_0 = \{m\}$ when we can say that C is a *strongly ergodic class at time* m.

3. $B = B_0 = \mathbf{N}$. In this case we can say that C is a *strongly ergodic class on* (*state set*) A_0 *with respect to* (*state set*) A. If $A = A_0$, then we can say that C is a *strongly ergodic class on* (*state set*) A.

Definition 1.6 ([11]). Consider a weakly Δ-ergodic chain on $A \times B$ (with respect to Σ). We say that the chain is *strongly* Δ-*ergodic on* $A \times B$ if any $C \in \Delta$ is a strongly ergodic class on $A \times B$. In particular, a strongly (S)-ergodic chain on $A \times B$ is called *strongly ergodic on* $A \times B$ for short.

Definition 1.7 ([11]). Consider a weakly $[\Delta]$-ergodic chain on $A \times B$. We say that the chain is *strongly* $[\Delta]$-*ergodic on* $A \times B$ if any $C \in \Delta$ is included in a strongly ergodic class on $A \times B$.

Also, in these definitions we can simplify the language when referring to A and B (and Σ). These are left to the reader.

Let

$$R_{m,n} = \{P \mid P \text{ is a real } m \times n \text{ matrix}\},$$

$$N_{m,n} = \{P \mid P \text{ is a nonnegative } m \times n \text{ matrix}\},$$

$$S_{m,n} = \{P \mid P \text{ is a stochastic } m \times n \text{ matrix}\},$$

and, for $m = n := r$,

$$R_r = R_{r,r}, \quad N_r = N_{r,r}, \quad \text{and} \quad S_r = S_{r,r}.$$

Let $T = (T_{ij}) \in R_{m,n}$. Let $\emptyset \neq U \subseteq \{1, 2, \ldots, m\}$ and $\emptyset \neq V \subseteq \{1, 2, \ldots, n\}$. Let $\Delta \in \mathrm{Par}(\{1, 2, \ldots, m\})$. Define

$$T_U = (T_{ij})_{i \in U,\, j \in \{1,2,\ldots,n\}}, \quad T^V = (T_{ij})_{i \in \{1,2,\ldots,m\},\, j \in V}, \quad T_U^V = (T_{ij})_{i \in U,\, j \in V}$$

(in particular, $T_{\{i\}}, T^{\{j\}}$, and $T_{\{i\}}^{\{j\}}$ are the ith row of T, the jth column of T, and the entry T_{ij} of T, respectively; if $Z \in R_{n,p}$, then $(TZ)_{ij} = (TZ)_{\{i\}}^{\{j\}} = T_{\{i\}} Z^{\{j\}}$),

$$\mu(T) = \max_{1 \leq j \leq n} \min_{1 \leq i \leq m} T_{ij}$$

(if $T \in S_{m,n}$, then $\mu(T)$ is called *Markov's ergodicity coefficient of* T (see, e.g., [4, p. 56])),

$$\mu_\Delta(T) = \min_{K \in \Delta} \mu(T_K)$$

(see [7]),

$$\alpha(T) = \min_{1 \le i,j \le m} \sum_{k=1}^{n} \min(T_{ik}, T_{jk})$$

(if $T \in S_{m,n}$, then $\alpha(T)$ is called *Dobrushin's ergodicity coefficient of* T (see, e.g., [2] or [4, p. 56])),

$$\overline{\alpha}(T) = \frac{1}{2} \max_{1 \le i,j \le m} \sum_{k=1}^{n} |T_{ik} - T_{jk}|,$$

$$\gamma_\Delta(T) = \min_{K \in \Delta} \alpha(T_K), \quad \overline{\gamma}_\Delta(T) = \max_{K \in \Delta} \overline{\alpha}(T_K)$$

(see [7] for γ_Δ and $\overline{\gamma}_\Delta$), and

$$|||T|||_\infty = \max_{1 \le i \le m} \sum_{j=1}^{n} |T_{ij}|$$

(*the* ∞*-norm of* T).

If $T \in S_{m,n}$ and $\Delta \in \text{Par}(\{1, 2, \ldots, m\})$, then we have

$$\overline{\alpha}(T) = 1 - \alpha(T) \quad \text{and} \quad \overline{\gamma}_\Delta(T) = 1 - \gamma_\Delta(T)$$

(see [7]).

The following result shows that weak and strong ergodicity are kept back (good at a time, good in past) while weak and strong Δ-ergodicity with $\Delta \ne (S)$ are kept forward (bad at a time, bad in future).

THEOREM 1.8. *Let* $(P_n)_{n\ge1}$ *be a Markov chain. Let* $\Sigma = (\{i\})_{i \in S}$.

(i) *If the chain is weakly ergodic at time* m_0 ($m_0 \ge 0$), *then it is weakly ergodic at any time* $m \le m_0$ ($m \ge 0$).

(ii) *If the chain is weakly* Δ*-ergodic at time* m_0 *with* $\Delta \ne (S)$, *then* $\forall m \ge m_0$, $\exists \Delta_m \in \text{Par}(S)$ *with* $\Delta_m \ne (S)$ *such that it is weakly* Δ_m*-ergodic at time* m.

(iii) *If the chain is strongly ergodic at time* m_0, *then it is strongly ergodic at any time* $m \le m_0$.

(iv) *If the chain is strongly* Δ'*-ergodic* ($A \times B = S \times \mathbf{N}$) *and strongly* Δ*-ergodic at time* m_0 *with* $\Delta \ne (S)$, *then* $\forall m \ge m_0$, $\exists \Delta_m \in \text{Par}(S)$ *with* $\Delta_m \ne (S)$ *such that it is strongly* Δ_m*-ergodic at time* m.

Proof. (i) See, e.g., [4, p. 218].

(ii) See [3, p. 242] (where different notions and notation are used) or the argument below. Suppose that $\exists m_1 > m_0$ such that the chain is weakly ergodic ($\Delta_{m_1} = (S)$) at time m_1. By (i), the chain is weakly ergodic at any time $m \leq m_1$, and we reached a contradiction.

(iii) Let $\Pi_{m_0} = \lim_{n\to\infty} P_{m_0,n}$ (Π_{m_0} is a stable stochastic matrix, i.e., a stochastic matrix with identical rows).

Case 1. $m_0 = 0$. Obvious.

Case 2. $m_0 > 0$. Let $0 \leq m < m_0$. Then

$$P_{m,n} = P_{m,m_0} P_{m_0,n} \to P_{m,m_0} \Pi_{m_0} = \Pi_{m_0} \text{ as } n \to \infty,$$

i.e., the chain is strongly ergodic at time m (with the same limit matrix Π_{m_0}).

(iv) If the chain is strongly Δ'-ergodic, then $\forall m \geq 0$, $\exists \Delta_m \in \operatorname{Par}(S)$ such that it is strongly Δ_m-ergodic at time m. Let us show that $\Delta_m \neq (S)$, $\forall m > m_0$. Suppose that $\exists m_1 > m_0$ such that the chain is strongly ergodic at time m_1. By (iii), the chain is strongly ergodic at any time $m \leq m_1$, and we reached a contradiction. □

Definition 1.9 ([9]). Let $(P_n)_{n\geq 1}$ and $(P'_n)_{n\geq 1}$ be two (finite) Markov chains. We say that $(P'_n)_{n\geq 1}$ is a *perturbation of the first type of* $(P_n)_{n\geq 1}$ if

$$\sum_{n\geq 1} \left|\left|\left| P_n - P'_n \right|\right|\right|_\infty < \infty.$$

THEOREM 1.10. *Let* $(P_n)_{n\geq 1}$ *be a Markov chain and* $(P'_n)_{n\geq 1}$ *a perturbation of the first type of it. Let* $\Sigma = (\{i\})_{i\in S}$ ($\Sigma \in \operatorname{Par}(S)$).

(i) $(P_n)_{n\geq 1}$ *is weakly ergodic if and only if* $(P'_n)_{n\geq 1}$ *is weakly ergodic.*

(ii) $(P_n)_{n\geq 1}$ *is weakly* Δ*-ergodic with* $\Delta \neq (S)$ *if and only if* $(P'_n)_{n\geq 1}$ *is weakly* Δ'*-ergodic with* $\Delta' \neq (S)$.

(iii) $(P_n)_{n\geq 1}$ *is strongly ergodic if and only if* $(P'_n)_{n\geq 1}$ *is strongly ergodic.*

(iv) $(P_n)_{n\geq 1}$ *is strongly* Δ*-ergodic with* $\Delta \neq (S)$ *if and only if* $(P'_n)_{n\geq 1}$ *is strongly* Δ'*-ergodic with* $\Delta' \neq (S)$.

Proof. (i) See Theorem 1.33 in [9].

(ii) This follows from (i).

(iii) See Theorem 1.47 in [9].

(iv) This follows from (iii). It also follows from Theorem 6 in [3] (where different notions and notation are used). □

THEOREM 1.11. *Let* $P \in S_r$ *with a single closed set* R (*see, e.g.*, [4, p. 81] ($\emptyset \neq R \subseteq \{1, 2, \ldots, r\}$)). *Suppose that* R *is aperiodic and* $\exists i \in R$ *such that* $P_{ii} > 0$. *Then*

(i) $(P^{r-1})^{\{i\}} > 0$;

(ii) $\mu(P^{r-1}) > 0$ *and* $\alpha\left(P^{r-1}\right) > 0$.

Proof. (i) Let $j \in \{1, 2, \ldots, r\}$. By hypothesis, $\exists i_1, i_2, \ldots, i_p \in \{1, 2, \ldots, r\}$ for which $i_k \neq i_l$, $\forall k, l \in \{1, 2, \ldots, p\}$ with $k \neq l$ such that $j = i_1$, $i = i_p$, and

$P_{i_s i_{s+1}} > 0$, $\forall s \in \{1, 2, \ldots, p-1\}$. Obviously, $1 \leq p \leq r$. Now,

$$(P^{r-1})_{ji} \geq P_{i_1 i_2} P_{i_2 i_3} \cdots P_{i_{p-1} i_p} P_{i_p i_p} \cdots P_{i_p i_p} > 0.$$

Therefore, $(P^{r-1})^{\{i\}} > 0$.

(ii) By (i), $\mu(P^{r-1}) > 0$. By $\mu \leq \alpha$ and $\mu(P^{r-1}) > 0$, we have $\alpha(P^{r-1}) > 0$. $\square$

Define

$$\mathcal{R}_{m,n}^{ij} = \{P \mid P \in S_{m,n} \text{ and } P_{ij} = 1\},$$

where $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$. In particular, if $i = j := k$ and $m = n := r$, set

$$\mathcal{R}_r^k = \mathcal{R}_{r,r}^{kk} \ (\mathcal{R}_r^k = \{P \mid P \in S_r \text{ and } P_{kk} = 1\}).$$

THEOREM 1.12. *The following statements hold.*

(i) *If* $P \in \mathcal{R}_{m,n}^{ij}$ *and* $Q \in \mathcal{R}_{n,p}^{jk}$ ($i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$, $k \in \{1, 2, \ldots, p\}$), *then* $PQ \in \mathcal{R}_{m,p}^{ik}$.

(ii) $\forall i \in S_r$, $\mathcal{R}_r^i$ *is closed under multiplication.*

Proof. (i) We have

$$(PQ)_{ik} = P_{\{i\}} Q^{\{k\}} = P_{ij} Q_{jk} = 1,$$

i.e., $PQ \in \mathcal{R}_{m,p}^{ik}$.

(ii) This follows from (i). $\square$

THEOREM 1.13. *Let* $P \in \mathcal{R}_{m,n}^{ij}$. *Then*

$$\alpha(P) = \mu(P) = \mu\big(P^{\{j\}}\big).$$

Proof. We know that $\mu(P) \leq \alpha(P)$ since $\mu \leq \alpha$. Further, let us show that $\mu(P) \geq \alpha(P)$. From $P \in S_{m,n}$ and $P_{ij} = 1$ we have $P_{ik} = 0$, $\forall k \in \{1, 2, \ldots, n\}$, $k \neq j$. It follows that

$$\mu(P) = \min_{1 \leq l \leq m} P_{lj}.$$

Let $u \in \{1, 2, \ldots, m\}$ such that $\mu(P) = P_{uj}$. Further,

$$\sum_{k=1}^{n} \min(P_{uk}, P_{ik}) = \min(P_{uj}, P_{ij}) = P_{uj} = \mu(P).$$

Therefore,

$$\alpha(P) \leq \mu(P).$$

By $\mu(P) \leq \alpha(P)$ and $\mu(P) \geq \alpha(P)$, we have

$$\alpha(P) = \mu(P).$$

Now,

$$\mu(P) = \min_{1\le l\le m} P_{lj} = \mu\big(P^{\{j\}}\big).$$

Therefore,

$$\mu(P) = \mu\big(P^{\{j\}}\big). \quad \square$$

More generally, we have the following result.

THEOREM 1.14. *Let* $P \in S_{m,n}$ *and* $\Delta \in \operatorname{Par}(\{1,2,\dots,m\})$. *Suppose that* $\forall K \in \Delta$, $\exists i = i(K) \in K$, $\exists j = j(K) \in \{1,2,\dots,n\}$ *such that* $P_K \in \mathcal{R}^{lj}_{|K|,n}$, *where* l $(l \in \{1,2,\dots,|K|\})$ *is the row of* P_K $(P_K \in S_{|K|,n})$ *corresponding to* i $((P_K)_{lj} = P_{ij} = 1)$. *Then*

$$\gamma_\Delta(P) = \mu_\Delta(P) = \min_{K\in\Delta,\, u=u(K)} \mu\big(P_K^{\{u\}}\big),$$

where $u = u(K) \in U_K := \{w \mid w \in \{1,2,\dots,n\}$ *and* $\exists v \in K$ *such that* $P_{vw} = 1\}$ *is given,* $\forall K \in \Delta$.

Proof. By Theorem 1.13, we have

$$\gamma_\Delta(P) = \min_{K\in\Delta} \alpha\left(P_K\right) = \min_{K\in\Delta} \mu\left(P_K\right) = \mu_\Delta(P).$$

The second equality follows from Theorem 1.13 and $\mu_\Delta(P) = \min_{K\in\Delta} \mu\left(P_K\right)$. $\square$

A vector $x \in \mathbf{R}^n$ will be understood as a row vector and x' is its transpose. Consider also the canonical basis $(e_i(n))_{i\in\{1,2,\dots n\}}$ of $\mathbf{R}^n$ and $e(n) = \sum_{i=1}^{n} e_i(n)$ (i.e., $e(n) = (1,1,\dots,1)$).

THEOREM 1.15. *Let* $(P_n)_{n\ge1}$ *be a Markov chain.*

(i) *If* $P_n \in \mathcal{R}^i_r$ $(i \in S = \{1,2,\dots,r\})$, $\forall n \ge 1$, *then* $\exists \Delta \in \operatorname{Par}(S)$ *such that the chain is strongly* Δ*-ergodic on* $\{i\}$ $(A = \{i\}$, $B = \mathbf{N}$, $\Sigma = (\{i\}) \in \operatorname{Par}(\{i\}))$.

(ii) (*a generalization of* (i)) *Let*

$$G = \left\{i \mid i \in S \text{ and } P_n \in \mathcal{R}^i_r,\ \forall n \ge 1\right\}.$$

If $G \ne \emptyset$, *then* $\exists \Delta \in \operatorname{Par}(S)$ *such that the chain is strongly* Δ*-ergodic on* G $(A = G$, $B = \mathbf{N}$, $\Sigma = (\{i\})_{i\in G} \in \operatorname{Par}(G))$.

(iii) (*another generalization of* (i)) *If* $\exists H$, $\emptyset \ne H \subseteq S$, *such that* $(P_n)^H_H = Q_n$, $\forall n \ge 1$, *where* $Q_n \in S_{|H|}$, $\forall n \ge 1$, *then* $\exists \Delta \in \operatorname{Par}(S)$ *such that the chain is strongly* Δ*-ergodic on* H *with respect to* (H) $(\Sigma = (H) \in \operatorname{Par}(H))$.

Proof. (i) This follows from (iii).

(ii) Obviously, $\exists \Delta \in \operatorname{Par}(S)$ such that the chain $(P_n)_{n\ge1}$ is strongly Δ-ergodic on G if and only if $\forall i \in G$, $\exists \Delta' = \Delta'(i)$ such that it is strongly Δ'-ergodic on $\{i\}$. Now, (ii) follows from (i).

(iii) Let $i \in S$. We have

$$(P_{m,n+1})^H_{\{i\}} = (P_{m,n})_{\{i\}} (P_{n+1})^H \geq (P_{m,n})^H_{\{i\}} (P_{n+1})^H_H =$$

$$= (P_{m,n})^H_{\{i\}} Q_{n+1}, \quad \forall m, n, \ 0 \leq m < n.$$

Set $t = |H|$. Further,

$$(P_{m,n+1})^H_{\{i\}} e'(t) \geq (P_{m,n})^H_{\{i\}} Q_{n+1} e'(t) = (P_{m,n})^H_{\{i\}} e'(t), \quad \forall m, n, \ 0 \leq m < n.$$

Therefore, $\left((P_{m,n})^H_{\{i\}} e'(t)\right)_{n>m}$ is an increasing sequence, $\forall m \geq 0$. This implies that $\left((P_{m,n})^H_{\{i\}} e'(t)\right)_{n>m}$ is convergent, $\forall m \geq 0$. Consequently, $\exists \Delta \in \operatorname{Par}(S)$ such that the chain $(P_n)_{n\geq 1}$ is strongly Δ-ergodic on H with respect to (H). □

THEOREM 1.16. *Let $(P_n)_{n\geq 1}$ be a Markov chain.*

(i) *If $\exists \Delta \in \operatorname{Par}(S)$ such that the chain is strongly Δ-ergodic on $A \times B$ with respect to Σ, then $\exists \Delta' \in \operatorname{Par}(S)$ with $\Delta \preceq \Delta'$ such that it is strongly Δ'-ergodic on $\mathcal{C}A \times B$ with respect to $(\mathcal{C}A)$, where $\mathcal{C}A$ is the complement of A.*

(ii) *The chain is strongly Δ-ergodic on $A \times B$ with respect to (A) if and only if it is strongly Δ-ergodic on $\mathcal{C}A \times B$ with respect to $(\mathcal{C}A)$. In particular, the chain is strongly ergodic on A with respect to (A) if and only if it is strongly ergodic on $\mathcal{C}A$ with respect to $(\mathcal{C}A)$.*

Proof. Obvious. □

The following two results show that Δ-ergodic theory can be used to obtain results in ergodic theory (for other examples see, e.g., Theorems 3.6, 3.10, and 3.12 in [10] (the latter is Theorem 1.23 below)).

THEOREM 1.17. *Let $(P_n)_{n\geq 1}$ be a Markov chain. Let $\Sigma = (\{i\})_{i\in S}$, $\emptyset \neq K \subseteq S$, $w \in S$, and $s \geq 0$. Suppose that $\exists t > s$, $\exists u = u(t) \in K$ such that $(P_{s,t})_{\{u\}} > 0$. Then the following statements are equivalent.*

(i) *The chain is strongly ergodic at any time m, $0 \leq m \leq t$, with*

$$\lim_{n\to\infty} P_{m,n} = e'(r) e_w(r) := \Pi, \quad \forall m, \ 0 \leq m \leq t.$$

(Obviously, $\Pi^{\{w\}} = e'(r)$.)

(ii) *K is included in a strongly ergodic class at time s with*

$$\lim_{n\to\infty} (P_{s,n})_K = e'(v) e_w(r),$$

where $v := |K|$.

Proof. (i)⇒(ii) Obvious.

(ii)⇒(i) Let $0 \leq m < t$ and $n > t$. We have

$$\left(P_{s,n}\right)_{\{u\}}^{\mathcal{C}\{w\}} = \left(P_{s,t}P_{t,n}\right)_{\{u\}}^{\mathcal{C}\{w\}} = \left(P_{s,t}\right)_{\{u\}}\left(P_{t,n}\right)^{\mathcal{C}\{w\}}.$$

Because $\lim_{n\to\infty}\left(P_{s,n}\right)_{\{u\}}^{\mathcal{C}\{w\}} = 0$ (the zero vector) and $\left(P_{s,t}\right)_{\{u\}} > 0$, we have

$$\lim_{n\to\infty}\left(P_{t,n}\right)^{\mathcal{C}\{w\}} = 0 \text{ (the zero matrix)}.$$

Therefore,

$$\lim_{n\to\infty} P_{t,n} = \Pi.$$

Hence the chain is strongly ergodic at time t. Further,

$$P_{m,n} = P_{m,t}P_{t,n} \to P_{m,t}\Pi = \Pi \text{ as } n \to \infty,$$

i.e., the chain is strongly ergodic at time m with $\lim_{n\to\infty} P_{m,n} = \Pi$. □

THEOREM 1.18. *Let $(P_n)_{n\geq 1}$ be a Markov chain. Let $\Sigma = (\{i\})_{i\in S}$, $\emptyset \neq K \subseteq S$, $w \in S$, and $s \geq 0$. Suppose that there exist two sequences $s < t_1 < t_2 < \cdots$ and $u_1 = u_1(t_1) \in K$, $u_2 = u_2(t_2) \in K, \ldots$ such that $\left(P_{s,t_l}\right)_{\{u_l\}} > 0$, $\forall l \geq 1$. Then the following statements are equivalent.*

(i) *The chain is strongly ergodic with*

$$\lim_{n\to\infty} P_{m,n} = e'(r)e_w(r) := \Pi, \quad \forall m \geq 0.$$

(ii) *K is included in a strongly ergodic class at time s with*

$$\lim_{n\to\infty}\left(P_{s,n}\right)_K = e'(v)e_w(r),$$

where $v := |K|$.

Proof. (i)⇒(ii) Obvious.

(ii)⇒(i) As in the proof of (ii)⇒(i) in Theorem 1.17, the chain is strongly ergodic at time t_l with $\lim_{n\to\infty} P_{t_l,n} = \Pi$, $\forall l \geq 1$. Now, let $m \geq 0$. Obviously, $\exists l \geq 1$ such that $m < t_l$. Further, for $n > t_l$,

$$P_{m,n} = P_{m,t_l}P_{t_l,n} \to P_{m,t_l}\Pi = \Pi \text{ as } n \to \infty,$$

i.e., the chain is strongly ergodic at time m with $\lim_{n\to\infty} P_{m,n} = \Pi$. It follows that the chain is strongly ergodic with $\lim_{n\to\infty} P_{m,n} = \Pi$, $\forall m \geq 0$. □

A special case of the hypothesis of Theorem 1.18 is $u_1 = u_2 = \cdots := u \in K$. Below we give a sufficient condition for $\left(P_{s,t_l}\right)_{\{u\}} > 0$, $\forall l \geq 1$. For this, we need the following notion.

Definition 1.19 (see, e.g., [12, p. 80]). Let $T \in N_{m,n}$. We say that T is a *column-allowable matrix* if it has at least one positive entry in each column.

THEOREM 1.20. *Let* $(P_n)_{n\geq 1}$ *be a Markov chain. Let* $\emptyset \neq K \subseteq S$ *and* $s \geq 0$. *If* $\exists t > s$, $\exists u = u(t) \in K$ *such that* $(P_{s,t})_{\{u\}} > 0$ *and there exists a sequence* $t < t_1 < t_2 < \cdots$ *such that* P_{t,t_l} *is a column-allowable matrix,* $\forall l \geq 1$, *then* $(P_{s,t_l})_{\{u\}} > 0$, $\forall l \geq 1$.

Proof. Obvious. □

The following result is closely related to Theorem 2.1, (1)⇔(3), in [6].

THEOREM 1.21. *Let* $(P_n)_{n\geq 1}$ *be a Markov chain. Let* $\emptyset \neq K \subseteq S$ *included in a strongly ergodic class on* $A \times B$ *with respect to* Σ. *Then the following statements are equivalent.*

(i) *The chain is strongly ergodic on* $A \times B$ *with respect to* Σ.

(ii) *The chain is weakly ergodic on* $A \times B$ *with respect to* Σ.

Proof. Obvious. □

THEOREM 1.22 ([3]). *Let* $(P_n)_{n\geq 1}$ *be a Markov chain. Then it is weakly ergodic if and only if there exists a strictly increasing sequence* $0 \leq n_1 < n_2 < \cdots$ (*of natural integers*) *such that* $\sum_{s\geq 1} \alpha\left(P_{n_s,n_{s+1}}\right) = \infty$.

Proof. See, e.g., [3] or [4, p. 219]. □

THEOREM 1.23 ([10]). *Let*

$$P_n = \begin{pmatrix} Q_n & 0 \\ R_n & T_n \end{pmatrix}, \quad n \geq 1,$$

be a Markov chain and $(K_1, K_2) \in \mathrm{Par}(S)$, *where* $(P_n)_{K_1}^{K_1} = Q_n$, $\forall n \geq 1$. *If*

(i) K_1 *is included in a weakly* (*respectively, strongly*) *ergodic class*;

(ii) T_n *is lower triangular,* $\forall n \geq 1$, *or it is upper triangular,* $\forall n \geq 1$;

(iii) $\prod_{n\geq t} (P_n)_{ii} = 0, \ \forall t \geq 1, \ \forall i \in K_2$,

then $(P_n)_{n\geq 1}$ *is weakly* (*respectively, strongly*) *ergodic.*

Proof. See [10]. □

2. RELIABILITY THEORY

In this section we consider the Markov chains from [5]. Generally speaking, the matrices corresponding to a reliability structure depend on the number of structure components and time (time in the sense of Markov chain theory); [1] studies especially the case where the matrices depend on both the number of structure components and time while [5] the case where these only depend on time. Here we do not set forth the Markov chain approach of reliability structures (for this see [5]), but only we define and study the Markov chains (without their meaning) corresponding to reliability structures from [5].

The underlying Markov chain we need is $(X_n)_{n\geq 0}$ with state space $S = \{0, 1, \ldots, r\}$, initial distribution π_0 with supp $\pi_0 \subseteq \{0, 1, \ldots, r-1\}$, where supp $\pi_0 := \{i \mid i \in S \text{ and } (\pi_0)_i > 0\}$ (*the support of* π_0), and transition matrices $(P_n)_{n\geq 1}$ with $P_n \in \mathcal{R}^r_{r+1}$ (the rows and columns of these matrices are labelled $0, 1, \ldots, r$), $\forall n \geq 1$. We call it *the reliability Markov chain* and frequently shall refer to it as the reliability (finite) Markov chain $(P_n)_{n\geq 1}$

Further, we define reliability Markov chains corresponding to reliability structures from [5] (where they were considered without naming them). We divide them into three classes as follows.

Class I. This contains:

1. *Series Markov chains.* These are the Markov chains with state space $S = \{0, 1\}$ $(r = 1)$, initial distribution $\pi_0 = (1, 0)$, and transition matrices

$$P_n = \begin{pmatrix} p_n & q_n \\ & 1 \end{pmatrix}, \quad n \geq 1$$

$(p_n + q_n = 1$ and we can omit 0, $\forall n \geq 1)$.

2. *k-out-of-∞: F Markov chains.* These are the Markov chains with state space $S = \{0, 1, \ldots, k\}$ $(k \geq 1)$, initial distribution π_0 with supp $\pi_0 \subseteq \{0, 1, \ldots, k-1\}$, and transition matrices

$$P_n = \begin{pmatrix} p_n & q_n & & & & \\ & p_n & q_n & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & \\ & & & & p_n & q_n \\ & & & & & 1 \end{pmatrix}, \quad n \geq 1.$$

We note that a series Markov chain is in fact a 1-out-of-∞: F Markov chain.

3. *Weighted k-out-of-∞: F Markov chains.* These are the Markov chains with state space $S = \{0, 1, \ldots, k\}$, initial distribution π_0 with supp $\pi_0 \subseteq \{0, 1, \ldots, k-1\}$, and transition matrices

$$P_n = \begin{pmatrix} p_n & \cdots & q_n & & & & & \\ & p_n & & q_n & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & & & \\ & & & & p_n & & & q_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & & p_n & q_n \\ & & & & & & & 1 \end{pmatrix}, \quad n \geq 1,$$

with $(P_n)_{i,i+w_n} = q_n$, $\forall n \geq 1$, $\forall i \in S, i + w_n \leq k$, where w_n is a natural number, $1 \leq w_n \leq k$, $\forall n \geq 1$. We call w_n *the weight of* P_n, $\forall n \geq 1$.

We note that a k-out-of-∞: F Markov chain is in fact a weighted k-out-of-∞: F Markov chain with $w_n = 1$, $\forall n \geq 1$.

Class II. This contains:

1. *Consecutive-k-out-of-∞: F Markov chains.* These are the Markov chains with state space $S = \{0, 1, \ldots, k\}$, initial distribution π_0 with $\operatorname{supp} \pi_0 \subseteq \{0, 1, \ldots, k-1\}$, and transition matrices

$$P_n = \begin{pmatrix} p_n & q_n & & & & \\ p_n & & q_n & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & \\ p_n & & & & & q_n \\ & & & & & 1 \end{pmatrix}, \quad n \geq 1.$$

2. *Weighted consecutive-k-out-of-∞: F Markov chains.* These are the Markov chains with state space $S = \{0, 1, \ldots, k\}$, initial distribution π_0 with $\operatorname{supp} \pi_0 \subseteq \{0, 1, \ldots, k-1\}$, and transition matrices

$$P_n = \begin{pmatrix} p_n & \cdots & q_n & & & & & \\ p_n & & & q_n & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & & & \\ p_n & & & & & & & q_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_n & & & & & & & q_n \\ & & & & & & & 1 \end{pmatrix}, \quad n \geq 1,$$

with $(P_n)_{i,i+w_n} = q_n$, $\forall n \geq 1$, $\forall i \in S$, $i + w_n \leq k$, where w_n is a natural number, $1 \leq w_n \leq k$, $\forall n \geq 1$. We call w_n *the weight of* P_n, $\forall n \geq 1$.

3. *m-consecutive-k-out-of-∞: F Markov chains.* These are the Markov chains with state space $S = \{0, 1, \ldots, mk\}$, initial distribution π_0 with $\operatorname{supp} \pi_0 \subseteq \{0, 1, \ldots, mk-1\}$, and transition matrices

$$P_n = \begin{pmatrix} V_n & Y_n & & & & \\ & V_n & Y_n & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & V_n & Y_n & \\ & & & & V_n & q_n e'_k(k) \\ & & & & & 1 \end{pmatrix}, \quad n \geq 1,$$

where

$$V_n = \begin{pmatrix} p_n & q_n & & & & \\ p_n & & q_n & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & \\ p_n & & & & & q_n \\ p_n & & & & & \end{pmatrix}, \quad Y_n = \begin{pmatrix} & & & & & \\ & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & \\ & & & & & \\ q_n & & & & & \end{pmatrix}$$

are $k \times k$ matrices, $\forall n \geq 1$.

Obviously, a consecutive-k-out-of-∞: F Markov chain is both a weighted consecutive-k-out-of-∞: F Markov chain with $w_n = 1$, $\forall n \geq 1$, and a 1-consecutive-k-out-of-∞: F Markov chain.

4. *Weighted m-consecutive-k-out-of-*∞*: F Markov chains*. These are the Markov chains with state space $S = \{0, 1, \ldots, mk\}$, initial distribution π_0 with $\operatorname{supp} \pi_0 \subseteq \{0, 1, \ldots, mk-1\}$, and transition matrices

$$P_n = \begin{pmatrix} V_n & Y_n & & & & \\ & V_n & Y_n & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & V_n & Y_n & \\ & & & & V_n & q_n e'_k(k) \\ & & & & & 1 \end{pmatrix}, \quad n \geq 1,$$

where

$$V_n = \begin{pmatrix} p_n & \cdots & q_n & & & & \\ p_n & & & q_n & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_n & & & & & & q_n \\ & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_n & & & & & & \end{pmatrix}, \quad Y_n = \begin{pmatrix} & & & & & & \\ & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & & \\ q_n & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q_n & & & & & & \end{pmatrix}$$

are $k \times k$ matrices with $(V_n)_{i,i+w_n} = (Y_n)_{k-w_n+j,0} = q_n$ (we label rows and columns of V_n and Y_n as $0, 1, \ldots, k-1$), $\forall n \geq 1$, $\forall i \in \{0, 1, \ldots, k-1\}$, $i + w_n \leq k-1$, $\forall j \in \{0, 1, \ldots, w_n - 1\}$, where w_n is a natural number, $1 \leq w_n \leq k$, $\forall n \geq 1$. We call w_n *the weight of* P_n, $\forall n \geq 1$. Note that this subclass of Markov chains does not exists in [5]. Also, note that an m-consecutive-k-out-of-∞: F Markov chain is a weighted m-consecutive-k-out-of-∞: F Markov chain with $w_n = 1$, $\forall n \geq 1$.

Class III. This contains *u-within-consecutive-k-out-of-*∞*: F Markov chains*. For ease of the illustration (following [5]) for $u = 2$ these are the Markov chains with state space $S = \{0, 1, \ldots, k+1\}$, initial distribution π_0

with $\operatorname{supp}\pi_0 \subseteq \{0, 1, \ldots, k\}$, and transition matrices

$$P_n = \begin{pmatrix} p_n & q_n & & & & & \\ & & p_n & & & & q_n \\ & & & p_n & & & q_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & p_n & q_n \\ p_n & q_n & & & & & 0 \\ 0 & 0 & & & & & 1 \end{pmatrix}, \quad n \geq 1.$$

Note that in this paper the above case is only considered.

Let $(P_n)_{n\geq1}$ be a reliability Markov chain. Let $f(r+1) = e(r+1) - e_{r+1}(r+1) \in \mathbf{R}^{r+1}$ (for $e(r+1)$ and $e_{r+1}(r+1)$ see Section 1). Following [5, Theorem 3.1] the reliability R_v and unreliability F_v of a v-component system linearly arranged and labelled $1, 2, \ldots, v$ and which correspond, in the Markovian approach, to initial the distribution π_0 and matrices $P_1, P_2, \ldots, P_v$ are given by

$$R_v = \pi_0 P_{0,v} f'(r+1) \text{ and } F_v = \pi_0 P_{0,v} e'_{r+1}(r+1), \text{ respectively, } \forall v \geq 1.$$

Setting $R_0 = 1$ and $F_0 = 0$ we can call R_v and F_v *the reliability* and *the unreliability at time v* ($v \geq 0$) *of reliability Markov chain* $(P_n)_{n\geq1}$, respectively. Obviously,

$$F_v = 1 - R_v, \quad \forall v \geq 0.$$

THEOREM 2.1. *Let $(P_n)_{n\geq1}$ be a reliability Markov chain. Then*

$$R_v = \sum_{i\in\operatorname{supp}\pi_0} (\pi_0)_i \sum_{j=0}^{r-1} (P_{0,v})_{ij} \quad \textit{and} \quad F_v = \sum_{i\in\operatorname{supp}\pi_0} (\pi_0)_i (P_{0,v})_{ir}, \ \forall v \geq 1.$$

In particular, if $\pi_0 = (1, 0, \ldots, 0)$ (*this is the usual case in the reliability theory*), *then*

$$R_v = \sum_{j=0}^{r-1} (P_{0,v})_{0,j} \quad \textit{and} \quad F_v = (P_{0,v})_{0,r}, \quad \forall v \geq 1.$$

Proof. Obvious. □

THEOREM 2.2. *Let $(P_n)_{n\geq1}$ be a reliability Markov chain. Then*

$$\exists \lim_{v\to\infty} R_v, \quad \lim_{v\to\infty} F_v.$$

Proof. This follows from Theorems 1.15(i) and 2.1. □

Set

$$R_\infty = \lim_{v\to\infty} R_v \quad \text{and} \quad F_\infty = \lim_{v\to\infty} F_v.$$

We call R_∞ and F_∞ *the limit reliability* and *the limit unreliability*, respectively.

Further, we give necessary and/or sufficient conditions for $R_\infty = 0$ (equivalently, for $F_\infty = 1$). The following result gives necessary and sufficient conditions for $R_\infty = 0$.

THEOREM 2.3. *Let $(P_n)_{n\geq 1}$ be a reliability Markov chain. Let $\Sigma_1 = (\{i\})_{i\in S}$ and $\Sigma_2 = (\{0,1,\ldots,r-1\},\{r\})$. Then the following statements are equivalent.*

(i) $R_\infty = 0$.

(ii) $\operatorname{supp}\pi_0 \cup \{r\}$ *is included in a weakly ergodic class at time* 0 *with respect to* Σ_1.

(iii) $\operatorname{supp}\pi_0 \cup \{r\}$ *is included in a strongly ergodic class at time* 0 *with respect to* Σ_1.

(iv) $\operatorname{supp}\pi_0 \cup \{r\}$ *is included in a weakly ergodic class at time* 0 *with respect to* Σ_2.

(v) $\operatorname{supp}\pi_0 \cup \{r\}$ *is included in a strongly ergodic class at time* 0 *with respect to* Σ_2.

Proof. (i)⇔(iii) See Theorem 2.1.

(ii)⇒(iii) Obvious, since $\{r\}$ is included in a strongly ergodic class with respect to Σ_1.

(iii)⇒(ii) Obvious.

(iv)⇒(v) Obvious, since $\{r\}$ is included in a strongly ergodic class with respect to Σ_2.

(v)⇒(iv) Obvious.

(ii)⇔(iv) Obvious. □

THEOREM 2.4. *Let $(P_n)_{n\geq 1}$ be a reliability Markov chain. Let $\Sigma_1 = (\{i\})_{i\in S}$ and $\Sigma_2 = (\{0,1,\ldots,r-1\},\{r\})$. Then the following statements are equivalent.*

(i) *The chain is weakly ergodic on* (*time set*) *B with respect to* Σ_1.

(ii) *The chain is strongly ergodic on B with respect to* Σ_1.

(iii) *The chain is weakly ergodic on B with respect to* Σ_2.

(iv) *The chain is strongly ergodic on B with respect to* Σ_2.

Proof. (i)⇔(ii) See Theorem 1.21.

(i)⇔(iii) Obvious.

(iii)⇔(iv) See Theorem 1.21. □

The following result gives a sufficient condition for $R_\infty = 0$. (For other sufficient conditions, see Theorem 2.3.)

THEOREM 2.5. *Let $(P_n)_{n\geq 1}$ be a reliability Markov chain. Let $\emptyset \neq B \subseteq \mathbf{N}$ with $0 \in B$ and $\Sigma = (\{i\})_{i\in S}$. If the chain is weakly ergodic on B with respect to Σ, then $R_\infty = 0$.*

Proof. Obvious. □

A problem is the use of Theorems 1.17 and 1.18. We note that in any reliability Markov chain $(P_n)_{n\geq 1}$ the set $\{r\}$ is included in a strongly ergodic class at time 0.

THEOREM 2.6. *Let $(P_n)_{n\geq 1}$ be a k-out-of-∞: F, or consecutive-k-out-of-∞: F, or 2-within-consecutive-k-out-of-∞: F Markov chain (with initial distribution π_0 (the case $\pi_0 = (1, 0, \dots, 0)$ is closely related to the conclusions below)).*

(i) *If $p_1, q_1, p_2, q_2, \dots, p_k, q_k > 0$, then $(P_{0,k})_{\{0\}} > 0$.*

(ii) *If $p_l, q_l > 0$, $\forall l \geq 1$, then $(P_{0,n})_{\{0\}} > 0$, $\forall n \geq k$.*

(iii) *If $(P_n)_{n\geq 1}$ is a k-out-of-∞: F Markov chain, $p_l > 0$, $\forall l \geq 1$, and there exists a strictly increasing sequence $1 \leq l_1 < l_2 < \cdots$ (of natural integers) with $q_{l_h} > 0$, $\forall h \geq 1$, then $\exists n_0 \geq k$ such that $(P_{0,n})_{\{0\}} > 0$, $\forall n \geq n_0$.*

Proof. (i) Obvious (by induction).

(ii) We have

$$(P_{0,n})_{\{0\}} = (P_{0,k})_{\{0\}} P_{k,n} > 0, \quad \forall n > k,$$

because $(P_{0,k})_{\{0\}} > 0$ and $P_{k,n}$ is a column-allowable matrix, $\forall n > k$.

(iii) We have $P_l = I_{k+1}$, if $q_l = 0$. Now, cf. (i) and (ii). □

Remark 2.7. In general, we have no results similar to Theorem 2.6 for weighted Markov chains. We give an example for weighted k-out-of-∞: F Markov chains. Let $k = 3$ and $w_n = 2$, $\forall n \geq 1$. Then

$$P_n = \begin{pmatrix} p_n & 0 & q_n & 0 \\ 0 & p_n & 0 & q_n \\ 0 & 0 & p_n & q_n \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \forall n \geq 1.$$

Obviously, $(P_{m,n})_{0,1} = 0$, $\forall m, n, 0 \leq m < n$, $\forall l \geq 1$, $\forall p_l, q_l \in [0, 1]$.

THEOREM 2.8. *Let $(P_n)_{n\geq 1}$ be an m-consecutive-k-out-of-∞: F Markov chain (with initial distribution π_0 (the case $\pi_0 = (1, 0, \dots, 0)$ is closely related to the conclusions below)).*

(i) *If $p_1, q_1, p_2, q_2, \dots, p_{mk}, q_{mk} > 0$, then $(P_{0,mk})_{\{0\}} > 0$.*

(ii) *If $p_l, q_l > 0$, $\forall l \geq 1$, then $(P_{0,n})_{\{0\}} > 0$, $\forall n \geq mk$.*

Proof. (i) Obvious (by induction; P_n has $(P_n)_{0,1} = (P_n)_{1,2} = \cdots = (P_n)_{mk-1,mk} = q_n$, $\forall n \geq 1$).

(ii) This is similar to the proof of Theorem 2.6(ii). □

Further, we give necessary and/or sufficient conditions for strong ergodicity (equivalently, cf. Theorem 2.4, for weak ergodicity) (also, remember that in this case $A = S$, $B = \mathbf{N}$, and $\Sigma = (\{i\})_{i \in S}$ (see Section 1)).

THEOREM 2.9. *Let $(P_n)_{n\geq 1}$ be a reliability Markov chain belonging to the union of Classes* I, II, *and* III. *If $\exists a > 0$ such that $p_n, q_n > a$, $\forall n \geq 1$, then the chain is strongly ergodic.*

Proof. If $(P_n)_{n\geq 1}$ is a k-out-of-∞: F Markov chain, then

$$P_n = \begin{pmatrix} p_n & q_n & & & & \\ & p_n & q_n & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & p_n & q_n \\ & & & & & 1 \end{pmatrix} \geq \begin{pmatrix} a & a & & & & \\ & a & a & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & a & a \\ & & & & & 1 \end{pmatrix} := P, \quad \forall n \geq 1.$$

Now, we can use Theorem 1.11 for P even if $P \in N_{k+1}$. It follows that $\alpha\left(P^k\right) := b > 0$. This implies $\alpha\left(P_{m,m+k}\right) \geq b$, $\forall m \geq 0$. Now, by Theorem 1.22, the chain $(P_n)_{n\geq 1}$ is weakly ergodic. Strong ergodicity now follows from Theorem 2.4.

The others cases have similar proofs. □

THEOREM 2.10. *Let $(P_n)_{n\geq 1}$ be a reliability Markov chain belonging to the union of Classes* I, II, *and* III.

(i) *If $\sum_{n\geq 1} p_n < \infty$, then $(P_n)_{n\geq 1}$ is strongly ergodic.*

(ii) *If $(P_n)_{n\geq 1}$ is strongly ergodic, then $\sum_{n\geq 1} q_n = \infty$.*

Proof. (i) Let $(P'_n)_{n\geq 1}$ be a perturbation of the first type of $(P_n)_{n\geq 1}$, where

$$\left(P'_n\right)_{ij} = \begin{cases} 0 & \text{if } q_n \text{ is not assigned to entry } (i,j) \text{ of } P_n, \\ 1 & \text{if } q_n \text{ is assigned to entry } (i,j) \text{ of } P_n, \end{cases}$$

$\forall n \geq 1$, $\forall i, j \in S$ (S is state space of $(P_n)_{n\geq 1}$ and $(P'_n)_{n\geq 1}$).

Case 1. $(P_n)_{n\geq 1}$ belongs to the union of Classes I and II. It follows from Theorem 1.23 that $(P'_n)_{n\geq 1}$ is strongly ergodic. Now, by Theorem 1.10, $(P_n)_{n\geq 1}$ is strongly ergodic.

Case 2. $(P_n)_{n\geq 1}$ belongs to Class III (only the case $u = 2$). First, it follows that $(P'_n)_{n\geq 1}$ is weakly ergodic, because

$$P'_n P'_{n+1} = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad \forall n \geq 1.$$

Second, by Theorem 1.10, $(P_n)_{n\geq 1}$ is weakly ergodic. Third, by Theorem 1.21, $(P_n)_{n\geq 1}$ is strongly ergodic.

(ii) For $(P_n)_{n\geq 1}$ belonging to Class I, see Theorem 2.11 below. Further, for any $(P_n)_{n\geq 1}$ belonging to the union of Classes II and III, the proof is similar to "⇒" from the proof of Theorem 2.11 below. □

THEOREM 2.11. *Let $(P_n)_{n\geq 1}$ be a weighted k-out-of-∞: F Markov chain. Then the chain is strongly ergodic if and only if*

$$\sum_{n\geq 1} q_n = \infty.$$

Proof. " ⇒ " Suppose that $\sum\limits_{n\geq 1} q_n < \infty$. Then the chain $(P'_n)_{n\geq 1}$, where $P'_n = I_{k+1}$, $\forall n \geq 1$, is a perturbation of the first type of $(P_n)_{n\geq 1}$. Because $(P'_n)_{n\geq 1}$ is strongly $(\{i\})_{i\in S}$-ergodic, it follows from Theorem 1.10 that $(P_n)_{n\geq 1}$ is strongly Δ-ergodic with $\Delta \neq (S)$ (moreover, using Theorem 1.45 in [9], we have $\Delta = (\{i\})_{i\in S}$), and we reached a contradiction.

" ⇐ " See Theorem 1.23. □

Remark 2.12. If $(P_n)_{n\geq 1}$ is a consecutive-k-out-of-∞: F Markov chain, then condition $\sum\limits_{n\geq 1} q_n = \infty$ is not sufficient for strong ergodicity. Indeed, for $k \geq 2$, let

$$P_{2n-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ . & . & . & . & . \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} := E, \ P_{2n} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} := F, \ \forall n \geq 1.$$

(A generalization: $P_n \in \{E, F\}$, $\forall n \geq 1$, E appears a least once in any l consecutive matrices belonging to $(P_n)_{n\geq 1}$, where $2 \leq l \leq k$ ($k \geq 2$), and there exists a sequence $1 \leq n_1 < n_2 < \cdots$ such that $P_{n_t} = F$, $\forall t \geq 1$.) We have $\sum\limits_{n\geq 1} q_n = \infty$ and

$$EF = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ . & . & . & . & . \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad EFE = E.$$

Therefore, the chain $(P_n)_{n\geq 1}$ is not strongly ergodic.

THEOREM 2.13. *Let $(P_n)_{n\geq 1}$ be a consecutive-k-out-of-∞: F Markov chain. Then*

$$\alpha(P_{m,m+k}) = \mu(P_{m,m+k}) = \mu\big((P_{m,m+k})^{\{k\}}\big) = q_{m+1}q_{m+2}\cdots q_{m+k},\ \forall m\geq 0.$$

Proof. The equalities $\alpha(P_{m,m+k}) = \mu(P_{m,m+k}) = \mu\big((P_{m,m+k})^{\{k\}}\big)$ follow from Theorem 1.13. Further, let us prove that

$$\mu\big((P_{m,m+k})^{\{k\}}\big) = q_{m+1}q_{m+2}\cdots q_{m+k}, \quad \forall m\geq 0.$$

We have

$$P_n = \begin{pmatrix} p_n & q_n & 0 & \cdots & 0 \\ p_n & 0 & q_n & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p_n & 0 & 0 & \cdots & q_n \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} p_n & 0 & 0 & \cdots & 0 \\ p_n & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p_n & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} 0 & q_n & 0 & \cdots & 0 \\ 0 & 0 & q_n & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & q_n \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad \forall n\geq 1.$$

Setting

$$G_n = \begin{pmatrix} p_n & 0 & 0 & \cdots & 0 \\ p_n & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p_n & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad H_n = \begin{pmatrix} 0 & q_n & 0 & \cdots & 0 \\ 0 & 0 & q_n & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & q_n \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad \forall n\geq 1,$$

we have

$$P_n = G_n + H_n, \quad \forall n\geq 1.$$

First, we prove that

$$(H_{m,m+t})_{0,k} = 0, \quad \forall m\geq 0,\ \forall t,\ 1\leq t\leq k-1.$$

Indeed,

$$(H_{m,m+t})_{0,k} =$$

$$= (H_{m+1})_{0,1}\,(H_{m+2})_{1,2}\cdots(H_{m+t})_{t-1,k} = 0, \quad \forall m\geq 0,\ \forall t,\ 1\leq t\leq k-1.$$

Second, we prove that

$$(P_{m+k-v,m+k})^{\{k\}} = (H_{m+k-v,m+k})^{\{k\}}, \quad \forall m\geq 0,\ \forall v,\ 1\leq v\leq k.$$

Indeed, this follows by induction with respect to v.

Step 1. $v=1$. We have

$$(P_{m+k-1,m+k})^{\{k\}} = (P_{m+k})^{\{k\}} = (G_{m+k} + H_{m+k})^{\{k\}} =$$

$$= (G_{m+k})^{\{k\}} + (H_{m+k})^{\{k\}} = (H_{m+k})^{\{k\}} = (H_{m+k-1,m+k})^{\{k\}}, \quad \forall m \geq 0.$$

Step 2. $v \mapsto v+1$ $(1 \leq v+1 \leq k)$. We have

$$(P_{m+k-v-1,m+k})^{\{k\}} = P_{m+k-v} (P_{m+k-v,m+k})^{\{k\}} =$$
$$= (G_{m+k-v} + H_{m+k-v}) (H_{m+k-v,m+k})^{\{k\}} =$$
$$(G_{m+k-v} (H_{m+k-v,m+k})^{\{k\}} = 0 \text{ since } (H_{m+k-v,m+k})_{0,k} = 0, \ \forall m \geq 0)$$
$$= H_{m+k-v} (H_{m+k-v,m+k})^{\{k\}} = (H_{m+k-v} H_{m+k-v,m+k})^{\{k\}} =$$
$$= (H_{m+k-v-1,m+k})^{\{k\}}, \ \forall m \geq 0.$$

Third, we show that

$$(H_{m,m+k})^{\{k\}} = \begin{pmatrix} q_{m+1} \cdots q_{m+k} \\ q_{m+1} \cdots q_{m+k-1} \\ \vdots \\ q_{m+1} \\ 1 \end{pmatrix}, \quad \forall m \geq 0.$$

For this, setting

$$Z_n = \begin{pmatrix} 0 & q_n & 0 & \cdots & 0 \\ 0 & 0 & q_n & \cdots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \cdots & q_n \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \forall n \geq 1,$$

and

$$Z = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

we have $H_n = Z_n + Z$, $\forall n \geq 1$. Further, because

$$ZZ_n = 0, \ Z_n Z = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \cdots & q_n \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ \forall n \geq 1, \text{ and } ZZ = Z,$$

we have

$$H_{m,m+k} = H_{m+1} H_{m+2} \cdots H_{m+k} = (Z_{m+1} + Z)(Z_{m+2} + Z) \cdots (Z_{m+k} + Z) =$$
$$= Z_{m+1} [(Z_{m+2} + Z) \cdots (Z_{m+k} + Z)] + Z [(Z_{m+2} + Z) \cdots (Z_{m+k} + Z)] =$$
$$= Z_{m+1} [(Z_{m+2} + Z) \cdots (Z_{m+k} + Z)] + Z =$$

$$= Z_{m+1}\left\{Z_{m+2}[(Z_{m+3}+Z)\cdots(Z_{m+k}+Z)]+Z[(Z_{m+3}+Z)\cdots(Z_{m+k}+Z)]\right\}+$$

$$+Z = Z_{m+1}Z_{m+2}\left[(Z_{m+3}+Z)\cdots(Z_{m+k}+Z)\right] + Z_{m+1}Z + Z =$$

(by induction)

$$= Z_{m+1}Z_{m+2}\cdots Z_{m+k} + Z_{m+1}Z_{m+2}\cdots Z_{m+k-1}Z + \cdots + Z_{m+1}Z + Z =$$

$$= \begin{pmatrix} 0 & \cdots & 0 & q_{m+1}q_{m+2}\cdots q_{m+k} \\ 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \\ 0 & \cdots & 0 & 0 \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & q_{m+1}q_{m+2}\cdots q_{m+k-1} \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \\ 0 & \cdots & 0 & 0 \end{pmatrix} +$$

$$+ \cdots + \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \\ 0 & \cdots & 0 & q_{m+1} \\ 0 & \cdots & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & \cdots & 0 & q_{m+1}q_{m+2}\cdots q_{m+k} \\ 0 & \cdots & 0 & q_{m+1}q_{m+2}\cdots q_{m+k-1} \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ 0 & \cdots & 0 & q_{m+1} \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \quad \forall m \geq 0.$$

Consequently,

$$(H_{m,m+k})^{\{k\}} = \begin{pmatrix} q_{m+1}q_{m+2}\cdots q_{m+k} \\ q_{m+1}q_{m+2}\cdots q_{m+k-1} \\ \vdots \\ q_{m+1} \\ 1 \end{pmatrix}, \quad \forall m \geq 0.$$

Hence

$$\mu\big((P_{m,m+k})^{\{k\}}\big) = \mu\big((H_{m,m+k})^{\{k\}}\big) = q_{m+1}q_{m+2}\cdots q_{m+k}, \quad \forall m \geq 0. \quad \square$$

THEOREM 2.14. *Let* $(P_n)_{n\geq 1}$ *be a consecutive-*k*-out-of-*∞*: F Markov chain.*

(i) *If* $(P_n)_{n\geq 1}$ *is strongly ergodic, then* $\forall l,\ \left[\frac{k-2}{2}\right] < l < k$, *we have*

$$\sum_{s\geq 0} \min\left(q_{s(k-l)+1}, q_{s(k-l)+2}, \dots, q_{s(k-l)+k-l}\right) = \infty.$$

(ii) *If* $\exists t,\ 0 \leq t \leq k-1$, *such that*

$$\sum_{s\geq 0} q_{sk+t+1} q_{sk+t+2} \cdots q_{sk+t+k} = \infty,$$

then $(P_n)_{n\geq 1}$ *is strongly ergodic.*

Proof. (i) Let $\left[\frac{k-2}{2}\right] < l < k$. Suppose that

$$\sum_{s\geq 0} \min\left(q_{s(k-l)+1}, q_{s(k-l)+2}, \dots, q_{s(k-l)+k-l}\right) < \infty.$$

Let $n_0 < n_1 < n_2 < \cdots$, where n_s is a natural integer such that

$$\min\left(q_{s(k-l)+1}, q_{s(k-l)+2}, \dots, q_{s(k-l)+k-l}\right) = q_{n_s}, \quad \forall s \geq 0.$$

(Therefore, $n_s \in \{s(k-l)+1, s(k-l)+2, \dots, s(k-l)+k-l\},\ \forall s \geq 0$.) Let

$$P'_n = \begin{cases} P_n & \text{if } n \notin \{n_0, n_1, n_2, \dots\}, \\ Q & \text{if } n \in \{n_0, n_1, n_2, \dots\}, \end{cases}$$

where

$$Q = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 1 & 0 & \cdots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \cdots & & 0 \\ 0 & 0 & \cdots & & 1 \end{pmatrix}.$$

We have $\sum_{n\geq 1} \left|\left\|P_n - P'_n\right\|\right|_\infty < \infty$. Because

$$n_{s+1} - n_s \leq (s+1)(k-l) + k - l - s(k-l) - 1 = 2(k-l) - 1 < k+1,\ \forall s \geq 0,$$

we have

$$n_{s+1} - 1 - n_s < k, \quad \forall s \geq 0.$$

Consequently, the sequence $(P'_n)_{n\geq 1}$ contains at most $k-1$ consecutive matrices belonging to $(P_n)_{n\geq 1}$. Further, $\forall m \geq 0,\ \forall u, v,\ 1 \leq u \leq k-1,\ 2 \leq v \leq k$, we have

$$(P_{m,m+u}Q)_{\{0\}} = (P_{m,m+u})_{\{0\}} Q = (1, 0, \dots, 0),$$

$$(P_{m,m+u}QP_{m+u+1,m+u+v})_{\{0\}} = (P_{m,m+u}Q)_{\{0\}} P_{m+u+1,m+u+v} =$$
$$= (1, 0, \dots, 0) P_{m+u+1,m+u+v} = (P_{m+u+1,m+u+v})_{\{0\}},$$

$$(P_{m,m+u}QP_{m+u+1,m+u+v}Q)_{\{0\}} = (P_{m,m+u}QP_{m+u+1,m+u+v})_{\{0\}} Q =$$
$$= (P_{m+u+1,m+u+v})_{\{0\}} Q = (1, 0, \dots, 0)$$

etc. and

$$(P_{m,m+u}Q)_{\{k\}} = (P_{m,m+u})_{\{k\}} Q = (0,0,\ldots,1),$$
$$(P_{m,m+u}QP_{m+u+1,m+u+v})_{\{k\}} = (P_{m,m+u}Q)_{\{k\}} P_{m+u+1,m+u+v} =$$
$$= (0,0,\ldots,1) P_{m+u+1,m+u+v} = (P_{m+u+1,m+u+v})_{\{k\}},$$
$$(P_{m,m+u}QP_{m+u+1,m+u+v}Q)_{\{k\}} = (P_{m,m+u}QP_{m+u+1,m+u+v})_{\{k\}} Q =$$
$$= (P_{m+u+1,m+u+v})_{\{k\}} Q = (0,0,\ldots,1)$$

etc. It follows that $(P'_n)_{n\geq 1}$ is not strongly ergodic. By Theorem 1.10, $(P_n)_{n\geq 1}$ is not strongly ergodic, and we reached a contradiction.

(ii) See Theorems 1.21, 1.22, and 2.13. □

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