THE CORE OF A REINSURANCE MARKET

MANUELA GHICA

We propose a game-theoretic model for a market of pure exchange. Some properties of the core of a reinsurance market are given.

AMS 2000 Subject Classification: 62P05, 91B30, 90A12.

Key words: reinsurance market, market core, optimal allocations.

1. INTRODUCTION

The reinsurance problem appears at first sight to be a problem that can be analyzed in terms of classical economic theory if the objectives of the companies are formulated in an operational manner by means of Bernoulli's utility concept: one should not maximize the expected gain, but the expected utility of the gain [3]. However, closer investigations show that the economic theory is only relevant part of the way. Then the problem becomes a problem of cooperation between parties that have conflicting interests, and that are free to form and break any coalitions which may serve their particular interests [14]. Classical economic theory is powerless when it comes to analyze such problems. The only theory which at present seems to hold some promise of being able to sort out and explain this apparently chaotic situation, is Game Theory.

The concept of core of a game was introduced in 1959 by Gillies [12]. It provides a very attractive solution, if any, of a general game. However, it has the unpleasant property of being empty for a large class of games. The concept of the core of a market game has proved very useful in economic applications of game theory because it is nonempty. This fact was proved by Debreu and Scarf [9] in 1963. Later, in 1981, Baton and Lemaire [4] introduced the collective rationality condition and characterized the core for important special cases of negative exponential utility functions. These types of utility functions are characterized by a constant risk aversion and possess very desirable properties, proved by Gerber [10] in 1974.

In the following we present a brief game-theoretic approach to risk allocation problem.

Let $N = \{1, 2, \ldots, n\}$ be a finite set of agents and let S an arbitrary subset of N. The characteristic function $v : 2^N \to \mathbb{R}$ of the game gives the

MATH. REPORTS **10(60)**, *2* (2008), 155–164

total payoff which the players that belong to a coalition $S \in 2^N$ obtain by cooperating.

Let z_i be the payoff to player i that cooperates in the game. So,

$$\sum_{i=1}^{n} z_i = v\left(N\right), \tag{1}$$

that represents the "collective rationalit", meaning that the players that cooperate will obtain the maximum total payoff [13].

If we see N as a group of **n** reinsurers having preferences $\geq_i$, $i \in N$ over a suitable set of random variables denoted by $\mathcal{R}$, or gambles with realizations (outcomes) in some $A \subseteq R$, then we represent these preferences by the von Neumann-Morgenstern expected utility. This means that there is a set of continuous utility functions $u_i : \mathcal{R} \to \mathbb{R}$, such that $X \geq_i Y$ if and only if $Eu_i(X) \geq Eu_i(Y)$, where E stands for the mean operator. We assume monotonic preferences, and risk aversion, so that we have $u_i'(w) > 0$, $u_i''(w) \leq 0$ for all w in the relevant domains [7]. In some cases we shall also require strict risk aversion, to mean strict concavity for some u_i. For a better understanding we assume that each agent is endowed with a random variable payoff X_i called initial portfolio. More precisely, there exists a probability space $(\Omega, \mathcal{K}, P)$ such that we have the payoff $X_i\left(\omega\right)$ when $\omega \in \Omega$ occurs and, moreover both expected values and variances exist for all these initial portfolios, to mean that all $X_i \in L^2\left(\Omega, \mathcal{K}, P\right)$ [8]. Because every agent can negotiate any affordable contracts, we will have a new set of random variables Y_i, $i \in N$, representing the final portfolios.

Assumption (1) corresponds to Pareto optimality in our reinsurance market [2], i.e., the optimal solution Y solves

$$\sum_{i=1}^{n} \lambda_i E u_i\left(Y_i\right) = E u_{\lambda_N}\left(X_N\right), \tag{2}$$

where $\lambda_N = \left(\lambda_1, \lambda_2, \ldots, \lambda_n\right)$, $\lambda_i \in \mathbb{R}_+^*$, $i = \overline{1,n}$ are the agent weights, $X_N = \sum\limits_{i \in N} X_i$ gives agent pricing and the last element is given by $Eu_{\lambda_N}\left(X_N\right) := \sup\limits_{Z_1,\ldots,Z_n} \sum\limits_{i \in N} \lambda_i E u_i\left(Z_i\right)$ s.t. $\sum\limits_{i \in N} Z_i \leq X_N$, a.s..

Next, the condition $z_i \geq v\left(\{i\}\right)$ represents "individual rationality" and corresponds to $Eu_i\left(Y_i\right) \geq Eu_i\left(X_i\right)$, $i \in N$, which implies that no player will participate in the game if it can obtain more by himself [1]. This rationality assumption is natural to be imposed because it corresponds to any coalition of the players, i.e., for any $S \in 2^N$. So, we can write

$$\sum_{i \in S} z_i \geq v\left(S\right), \quad \forall S \in 2^N. \tag{3}$$

We can see condition (3) as: "social stability". It corresponds in a reinsurance market to a further restriction [2] on the investor weights $\lambda \neq 0$ such that

$$\sum_{i \in S} \lambda_i E u_i (Y_i) \geq E u_{\lambda_S} (X_S), \quad X_S = \sum_{i \in S} X_i, \tag{4}$$

where

$$E u_{\lambda_S} (X_S) := \sup_{Z_1,\ldots,Z_n} \sum_{i \in S} \lambda_i E u_i (Z_i) \text{ s.t. } \sum_{i \in S} Z_i \leq X_S, \text{ a.s.} \tag{5}$$

and $\lambda_S = (\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_k})$, $\lambda_{i_k} \in \mathbb{R}_+^*$, $S = \{i_1, i_2, \ldots, i_k\}$.

The set of vectors Z that satisfy (5) is called the core of the game. It yields a very attractive solution when not empty, but for a large class of games it is empty. The concept is very useful in economic applications.

The paper is organized as follows. In Section 2 we present the basic game model for a reinsurance market. In Section 3 we define, in a original way, the core of a reinsurance market and give some properties of it. The paper ends with Section 4, where we find a form for the market core when the reinsurers have negative exponential utility functions.

2. A GAME MODEL FOR A REINSURANCE MARKET

In this section we present the structure of a game model applied in the case of a reinsurance market.

Definition 1. *A competitive reinsurance market* is an ordered pair $\langle N, \{E u_{\lambda_S} (X_S)\}_{S \subseteq N} \rangle$ consisting of the agent set $N = \{1, 2, \ldots, n\}$ interpreted as (re)insurers, where $u_\lambda (\cdot) : \mathcal{R} \to \mathbb{R}$ is the von Neumann-Morgenstern expected utility, and $E u_{\lambda_\varnothing} (X_\varnothing) = 0$.

Let $\mathcal{RM}(u)$ be a competitive reinsurance market and $\mathcal{RM}(N, X) = \{\mathcal{RM}(u) \mid u \in \mathcal{U}\}$ the set of all reinsurance markets, where $N = \{1, 2, \ldots, n\}$ is the set of the players, $X = (X_1, X_2, \ldots, X_n)$ the initial random vectors, $X_i \in L^2 (\Omega, \mathcal{K}, P)$, and $\mathcal{U} = \{u \mid u : \mathbb{R} \to \mathbb{R} \text{ is concave and increasing}\}$.

Definition 2. A competitive reinsurance market is said to be *monotonie* if $E u_{\lambda_S} (X_S) \leq E u_{\lambda_T} (X_T)$ for $S \subset T \subseteq N$.

Definition 3. A competitive reinsurance market is *additive* if

$$E u_{\lambda_{S \cup T}} (X_{S \cup T}) = E u_{\lambda_S} (X_S) + E u_{\lambda_T} (X_T)$$

for all $S, T \subset N$ and $S \cap T = \varnothing$.

Definition 4. A competitive reinsurance market is *superadditive* if

$$E u_{\lambda_S} (X_S) + E u_{\lambda_T} (X_T) \leq E u_{\lambda_{S \cup T}} (X_{S \cup T})$$

for all $S, T \subset N$ and $S \cap T = \varnothing$.

Definition 5. A competitive reinsurance market is said to be an *essential* reinsurance market if $Eu_{\lambda_N}(X_N) > \sum\limits_{i=1}^{n} \lambda_i Eu_i(X_i)$.

For each subset $S \subset N$ we denote by 1_S the characteristic vector of S:

$$(1_S)^i = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \in N \backslash S. \end{cases}$$

Definition 6. A map $\gamma : 2^N \backslash \{\varnothing\} \to \mathbb{R}_+$ is said to be *balanced* if $\sum\limits_{S \in 2^N \backslash \{\varnothing\}} \gamma(S) 1_S = 1_N$.

Definition 7. A collection $\mathcal{B}$ of coalitions is called *balanced* if there is a balanced map γ such that $\mathcal{B} = \left\{S \in 2^N \mid \gamma(S) > 0\right\}$.

Definition 8. A competitive reinsurance market is balanced if

$$\sum_{S; S \subset N} \gamma(S) Eu_{\lambda_S}(X_S) \leq Eu_{\lambda_N}(X_N) \tag{6}$$

for each balanced map $\gamma : 2^N \backslash \{\varnothing\} \to \mathbb{R}_+$.

Let us consider two reinsurance markets. We shall answer the question "When can we say that the first market is essentially the same the second market?"

Definition 9. Let two reinsurance markets. Then the first reinsurance market $\left\langle N, \{Eu_{\lambda_S}(X_S)\}_{S \subseteq N} \right\rangle$ is *strategically equivalent* to the reinsurance market $\left\langle N, \{E\bar{u}_{\lambda_S}(X_S)\}_{S \subseteq N} \right\rangle$ if there exist $k > 0$ and a payoff a_i, $i \in S$, such that

$$Eu_{\lambda_S}(X_S) = kE\bar{u}_{\lambda_S}(X_S) + \sum_{i \in S} a_i, \quad \forall S;\ S \subseteq N. \tag{7}$$

Definition 10. Let $\alpha, \beta \in \mathbb{R}$. A competitive reinsurance market $\langle N, \{Eu_{\lambda_S}(X_S)\}_{S \subseteq N} \rangle$ is called a reinsurance market *in* (α, β)*-form* if $Eu_{\lambda_{\{i\}}}(X_i) = \alpha$, $\forall i \in N$ and $Eu_{\lambda_N}(X_N) = \beta$.

Now we shall prove

Proposition 1. *Any essential reinsurance market* $\left\langle N, \{Eu_{\lambda_S}(X_S)\}_{S \subseteq N} \right\rangle$ *is strategically equivalent to the reinsurance market* $\left\langle N, \{E\bar{u}_{\lambda_S}(X_S)\}_{S \subseteq N} \right\rangle$ *in* $(0,1)$*-form.*

Proof. We can take

$$k=\frac{1}{Eu_{\lambda_N}(X_N)-\sum\limits_{i=1}^{n}\lambda_i Eu_i(X_i)}$$

and

$$a_i=\frac{Eu_i(X_i)}{Eu_{\lambda_N}(X_N)-\sum\limits_{i=1}^{n}\lambda_i Eu_i(X_i)}.$$

So, if we replace the values of k and a_i in (7) we have

$$E\bar{u}_{\lambda_S}(X_S)=\frac{Eu_{\lambda_S}(X_S)-\sum\limits_{i\in S}\lambda_i Eu_i(X_i)}{Eu_{\lambda_N}(X_N)-\sum\limits_{i=1}^{n}\lambda_i Eu_i(X_i)}.\quad\square$$

Definition 11. A reinsurance market $\left\langle N,\{Eu_{\lambda_S}(X_S)\}_{S\subseteq N}\right\rangle$ is called *zero-normalized* if $Eu_{\lambda_{\{i\}}}(X_i)=0$ for $\forall i\in N$.

Definition 12. The initial portfolios are $X_1,X_2,\ldots,X_n$ and the market portfolio $X_N=\sum\limits_{i=1}^{n}X_i$. Let $f:\mathcal{RM}(N,X)\to\mathbb{R}^n$ be a map. Then f satisfies:

1. *individual rationality* if $f_i(\mathcal{RM}(u))\geq\lambda_i Eu_i(X_i),\ i\in N$;

2. *efficiency* if $Eu_{\lambda_N}(X_N)=\sum\limits_{i=1}^{n}f_i(\mathcal{RM}(u))$;

3. *social stability* if additive reinsurance market, $\forall\mathcal{RM}(u_1),\mathcal{RM}(u_2)\in\mathcal{RM}(N,X)$, $a\in\mathcal{RM}(N,X)$, and $k>0$, the equation $E(u_1)_{\lambda_N}(X_N)=kE(u_2)_{\lambda_N}(X_N)+a$ implies $f(\mathcal{RM}(u_2))=kf(\mathcal{RM}(u_1))+a$;

4. *the dummy agent property* if $f_i(\mathcal{RM}(u))=\lambda_i Eu_i(X_i)$, $\forall\mathcal{RM}(u)\in\mathcal{RM}(N,X)$ and all dummy players $i\in N$ implies

$$Eu_{\lambda_{S\cup\{i\}}}\left(X_{S\cup\{i\}}\right)=Eu_{\lambda_S}(X_S)+Eu_{\lambda_{\{i\}}}(X_i),$$

for all $S\in 2^{N\setminus\{i\}}$;

5. *the anonymity property* if $f(\mathcal{RM}(u)^{\sigma})=\sigma^*(f(\mathcal{RM}(u)))$, $\forall\sigma\in\Pi(N)$, where $\Pi(N)$ is the set of all permutations of N. Here, $\mathcal{RM}(u)^{\sigma}$ is the reinsurance market with $Eu^{\sigma}_{\lambda_{\sigma(S)}}\left(X_{\sigma(S)}\right)=Eu_{\lambda_S}(X_S)$ for all $S\subseteq N$ or $Eu^{\sigma}_{\lambda_S}(X_S)=Eu_{\lambda_{\sigma^{-1}(S)}}\left(X_{\sigma^{-1}(S)}\right)$ for all $S\subseteq N$, and $\sigma^*:\mathbb{R}^n\to\mathbb{R}^n$ is defined by $(\sigma^*(x))_{\sigma(k)}:=x_k$ for all $x\in\mathbb{R}^n$;

6. *additivity* if $\forall\mathcal{RM}(u_1)$, $\mathcal{RM}(u_2)\in\mathcal{RM}(N,X)$ we have $f\big(\mathcal{RM}(u_1)+\mathcal{RM}(u_2)\big)=f(\mathcal{RM}(u_1))+f(\mathcal{RM}(u_2))$.

3. THE CORE OF A REINSURANCE MARKET

In the following we define the market core and give a characterization theorem. We take $f_i(\mathcal{RM}(u)) = \lambda_i Eu_i(Y_i)$, denote the set of "investor weights" λ_i, $i = \overline{1,n}$, by

$$I^{**} = \left\{\lambda \in \mathbb{R}^n \mid \sum_{i=1}^{n} \lambda_i Eu_i(Y_i) \leq Eu_{\lambda_N}(X_N)\right\} \tag{8}$$

and the set of efficient "investor weights" vectors in the reinsurance market $\left\langle N, \{Eu_{\lambda_S}(X_S)\}_{S\subseteq N} \right\rangle$ by I^*, i.e.,

$$I^{*} = \left\{\lambda \in \mathbb{R}^n \mid \sum_{i=1}^{n} \lambda_i Eu_i(Y_i) = Eu_{\lambda_N}(X_N)\right\}. \tag{9}$$

Obviously we have $I^* \subset I^{**}$.

The "individual rationality" condition, $Eu_i(Y_i) \geq Eu_i(X_i)$, $i \in N$, should hold in order that a weight vector λ have a real chance to be realized in the reinsurance market.

Definition 13. A weight vector $\lambda \in \mathbb{R}^n$ is an *imputation* for the reinsurance market $\left\langle N, \{Eu_{\lambda_S}(X_S)\}_{S\subseteq N} \right\rangle$ if it is efficient and enjoys the property of individual rationality, i.e.,

1. $\sum\limits_{i=1}^{n} \lambda_i Eu_i(Y_i) = Eu_{\lambda_N}(X_N)$,
2. $Eu_i(Y_i) \geq Eu_i(X_i)$, $\forall i \in N$.

Denote by I the set of imputations λ. Clearly, I is empty if and only if $\sum\limits_{i=1}^{n} \lambda_i Eu_i(Y_i) > Eu_{\lambda_N}(X_N)$. Actually, I is a simplex with extreme points $f^1, f^2, \ldots, f^n$, where $f^i = (f^i_1, f^i_2, \ldots, f^i_n)$, $i \in N$, is given by $f^i_j = \lambda_i Eu_i(X_i)$, if $i \neq j$ and $f^i_i = Eu_\lambda(X_N) - \sum\limits_{k\in N-\{i\}} \lambda_k Eu_k(X_k)$.

THEOREM 1. *Let a reinsurance market* $\left\langle N, \{Eu_{\lambda}(X_S)\}_{S\subseteq N} \right\rangle$. *If this market is essential, then*

1. I *is an infinite set.*
2. I *is the convex hull of the points* $f^1, f^2, \ldots, f^n$ *defined above.*

Proof. 1. Since $\left\langle N, \{Eu_{\lambda_S}(X_S)\}_{S\subseteq N} \right\rangle$ is essential, we have $a = Eu_{\lambda_N}(X_N) - \sum\limits_{k\in N} \lambda_k Eu_k(Y_k) > 0$. For any n-tuple $b = (b_1, b_2, \ldots, b_n)$ of nonnegative numbers such that $\sum\limits_{i\in N} b_i = a$, the weights vector $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_n)$ with $\lambda'_i Eu_i(Y_i) > \lambda_i Eu_i(X_i) + b_i$, $\forall i \in N$, is an imputations.

2. This result follows from the characterization theorem of the extreme points of a polyhedral set [15]. We note that $I = \left\{\lambda \in \mathbb{R}^n \mid \lambda^{\mathrm{T}} A \geq b^{\mathrm{T}}\right\}$, where $A \in \mathcal{M}_{n,n+2}(\mathbb{R})$ with the vectors $e^1, \ldots, e^n, 1^n, -1^n$, as columns, where e^i is the ith vector of the standard basis in $\mathbb{R}^n$and 1^n is the vector with all coordinates equal to 1. To complete the proof we take the vector b of the form $b = (Eu_{\lambda_1}(X_1), \ldots, Eu_{\lambda_n}(X_n), Eu_{\lambda_N}(X_N), -Eu_{\lambda_N}(X_N))$. □

It follows from the above theorem that the set of imputations is too large for an essential reinsurance market. So, we need some criteria to single out those imputations that are able to appear. We could obtain some subsets of I as solution concepts. One of them is the core of a reinsurance market.

Definition 14. The market core, denoted by MC, of a reinsurance market $\left\langle N, \{Eu_{\lambda_S}(X_S)\}_{S\subseteq N}\right\rangle$ is the set

$$MC = \left\{\lambda \in I \mid \sum_{i\in S} \lambda_i Eu_i(Y_i) \geq Eu_{\lambda_S}(X_S),\ \forall S \subseteq N\right\}. \tag{10}$$

If $MC \neq \varnothing$ then its elements can be easily obtained because the core is defined by means of a finite system of linear inequalities.

The next result gives a characterization of the reinsurance market with a nonempty market core.

Theorem 2. *Let a reinsurance market* $\left\langle N, \{Eu_\lambda(X_S)\}_{S\subseteq N}\right\rangle$. *Then the following two assertions are equivalent:*

1. $MC \neq \varnothing$.
2. *The reinsurance market is balanced.*

Proof. From equation (10) we deduce that $MC \neq \varnothing$ if and only if

$$Eu_\lambda(X_N) = \min\left\{\sum_{i\in N} \lambda_i Eu_i(Y_i) \mid \sum_{i\in S} \lambda_i Eu_i(Y_i) \geq Eu_{\lambda_S}(X_S),\ \forall S\subseteq N\right\}. \tag{11}$$

By the duality theorem from linear programming theory [15], equation (11) holds iff

$$Eu_\lambda(X_N) = \max\left\{\sum_{i\in N} \lambda_i Eu_i(Y_i) \mid \sum_{S\in 2^N\setminus\{\varnothing\}} \gamma(S)\, 1_S = 1_N,\ \gamma > 0\right\}. \tag{12}$$

Now, (12) holds iff equation (6) from Definition 8 does hold. So, we can conclude that 1) and 2) are equivalent. □

4. A MARKET CORE WITH EXPONENTIAL UTILITY

In general the core will be characterized by the Pareto optimal allocations corresponding to investor weights λ_i in some region restricted by inequalities as in the following result.

The initial portfolios are denoted by $X_1, X_2, \dots, X_n$, the "market portfolio" is $X_N = \sum_{i=1}^{n} X_i$ and the reinsurers have negative exponential utility functions given by $u_i(x) = 1 - a_i \mathrm{e}^{-\frac{x}{a_i}}$, $x \in \mathbb{R}$, $i \in N$.

The Pareto optimal allocations that result from coalition [11] are $Y_i = \frac{a_i}{A} X_N + b_i$, where $b_i = a_i \ln \lambda_i - a_i \frac{K}{A}$, $A = \sum_{i=1}^{n} a_i$ and $K = \sum_{i=1}^{n} a_i \ln \lambda_i$. For any subset $S \subseteq N$ the corresponding formulas are $Y_i = \frac{a_i}{A_S} X_S + b_i$, where $b_i = a_i \ln \lambda_i - a_i \frac{K_S}{A_S}$, $A_S = \sum_{i \in S} a_i$, $K = \sum_{i \in S} a_i \ln \lambda_i$, $X_S = \sum_{i \in S} X_i$.

Theorem 3. *The market core is characterized by the Pareto optimal allocations corresponding to investor weights λ_i that are solutions of the system of inequalities*

$$\sum_{i \in S} \lambda_i \frac{a_i}{A_S} E\left[\mathrm{e}^{-\frac{Y_i}{a_i}}\right] \leq \mathrm{e}^{\frac{K_S}{A_S}} E\left[\mathrm{e}^{-\frac{X_S}{A_S}}\right].$$

Proof. By (10), the market core is characterized by the inequalities

$$\sum_{i \in S} \lambda_i E\left[1 - \mathrm{e}^{-\frac{Y_i}{a_i}}\right] \geq E\left[\sum_{i \in S} \lambda_i - A_S \mathrm{e}^{\frac{K_S - X_S}{A_S}}\right]$$

for any $S \subseteq N$. This is equivalent to $\sum_{i \in S} \lambda_i \left[1 - a_i E\left[\mathrm{e}^{-\frac{Y_i}{a_i}}\right]\right] \geq \sum_{i \in S} \lambda_i - A_S E\left[\mathrm{e}^{\frac{K_S - X_S}{A_S}}\right]$ for any $S \subseteq N$ and, finally, $\sum_{i \in S} \lambda_i a_i E\left[\mathrm{e}^{-\frac{Y_i}{a_i}}\right] \leq A_S E\left[\mathrm{e}^{\frac{K_S - X_S}{A_S}}\right]$ for any $S \subseteq N$. $\square$

Theorem 4. *In the case of a reinsurance market with $n = 3$ agents, the market core is characterized by the Pareto optimal allocations corresponding to investor weights λ_i that are solutions of the system of inequalities*

$$c_{i,j} E\left(\mathrm{e}^{-\frac{X_i + X_j}{a_i + a_j}}\right) d_{i,j} \geq a_i E\left(\mathrm{e}^{-\frac{X_N}{A}}\right) + a_j E\left(\mathrm{e}^{-\frac{X_N}{A}}\right),$$

$$2AE\left(\mathrm{e}^{-\frac{X_M}{A}}\right) \leq \sum_{i,j \in \{1,\ 2,3\},\ i \neq j} c_{i,j} \mathrm{e}^{\left\{\frac{b_i + b_j}{a_i + a_j}\right\}} E\left(\mathrm{e}^{-\frac{X_i + X_j}{a_i + a_j}}\right),$$

for $i,j \in \{1,2,3\}$, $i \neq j$, *where* $c_{i,j} = (a_i + a_j)$ *and* $d_{i,j} = \mathrm{e}^{\left\{\frac{b_i+b_j}{a_i+a_j}\right\}}$.

Proof. Let $N = \{1,2,3\}$. The conditions $\sum_{i\in S} \lambda_i E u_i(Y_i) \geq E u_{\lambda_S}(X_S)$, $\forall S \subseteq N$, seen as social stability, for $S \in \{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$ become

$$\lambda_i E[u_i(Y_i)] + \lambda_j E[u_j(Y_j)] \geq E\left[\lambda_i + \lambda_j - c_{i,j}\mathrm{e}^{\frac{K - a_k \ln\lambda_k - (X_i+X_j)}{a_i+a_j}}\right]$$

$$2Eu_{\lambda_N}[X_N] \geq \sum_{i,j\in N, i\neq j} Eu_{\lambda_{\{i,j\}}}\left[X_{\{i,j\}}\right]$$

for $i,j,k \in N,\ i \neq j \neq k$.

If we replace in this system the negative exponential utility function of, then we obtain

$$\lambda_i E\left[a_i \mathrm{e}^{-\frac{Y_i}{a_i}}\right] + \lambda_j E\left[a_j \mathrm{e}^{-\frac{Y_j}{a_j}}\right] \geq -c_{i,j} E\left[\mathrm{e}^{-\frac{X_i+X_j}{a_i+a_j}}\right] d_{i,j}\mathrm{e}^{\frac{K}{A}}$$

$$2E\left[\sum_{i\in N}\lambda_i - A\mathrm{e}^{\frac{K-X_M}{A}}\right] \geq \sum_{i,j\in N, i\neq j} E\left[\lambda_i + \lambda_j - c_{i,j}\mathrm{e}^{\frac{K - a_k\ln\lambda_k - (X_i+X_j)}{a_i+a_j}}\right]$$

for $i,j \in N$, $i \neq j$. This is equivalent to

$$c_{i,j}E\left[\mathrm{e}^{-\frac{X_i+X_j}{a_i+a_j}}\right] d_{i,j}\mathrm{e}^{\frac{K}{A}} \geq \lambda_i a_i E\left[\mathrm{e}^{-\left(\frac{1}{A}X_N + \frac{b_i}{a_i}\right)}\right] + \lambda_j a_j E\left[\mathrm{e}^{-\left(\frac{1}{A}X_N + \frac{b_j}{a_j}\right)}\right]$$

$$\sum_{i,j\in N, i\neq j} c_{i,j}E\left[\mathrm{e}^{\frac{-(X_i+X_j)}{a_i+a_j}}\right] d_{i,j}\mathrm{e}^{\frac{K}{A}} \geq 2A\mathrm{e}^{\frac{K}{A}}E\left[\mathrm{e}^{-\frac{X_N}{A}}\right]$$

for $i,j \in N$, $i \neq j$.

If we simplify by $\lambda_i \mathrm{e}^{\frac{b_i}{a_i}} = \mathrm{e}^{\frac{K}{A}}$, then we obtain

$$c_{i,j}E\left[\mathrm{e}^{-\frac{X_i+X_j}{a_i+a_j}}\right]\mathrm{e}^{\frac{b_i+b_j}{(a_i+a_j)}} \geq a_i E\left[\mathrm{e}^{-\frac{X_N}{A}}\right] + a_j E\left[\mathrm{e}^{-\frac{X_N}{A}}\right]$$

$$2AE\left(\mathrm{e}^{-\frac{X_M}{A}}\right) \leq \sum_{i,j\in N, i\neq j} c_{i,j}d_{i,j}E\left[\mathrm{e}^{-\frac{X_i+X_j}{a_i+a_j}}\right],$$

for $i,j \in N,\ i \neq j$. $\square$

We see that the market core is characterized by the Pareto optimal allocations corresponding to investor weights λ_i in some region restricted by inequalities of the above kind, in general a polyhedron in $\mathrm{int}\left(\mathbb{R}^n_+\right)$.

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Received 1st November 2007

Spiru Haret University
Faculty of Mathematics and Computer Science
Str. Ion Ghica 13
030045 Bucharest, Romania