ON FUZZY LEAST SQUARES

CIPRIAN COSTIN POPESCU

An improvement an optimization algorithm with application to LR and LL fuzzy numbers, is given. Two situations are presented: the case of LR fuzzy numbers and the special LL case.

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1. INTRODUCTION

We discuss and extend an optimization method [2] based on the fuzzy least squares approach. The improvement consists in the introduction of the weighted model which substantially enlarge the range of the applications. In Section 2 we recall some basic definitions and results about fuzzy sets. Section 3 contains the general model setted up on \( n \) statistical units. In fact, our aim is to discuss the problems which occur when the weighted algorithm is applied to the particular case of LL fuzzy numbers, as in Section 5. Till then it is necessary to recall some results on more general LR fuzzy numbers. These preliminary results are presented in Section 4.

2. SOME DEFINITIONS AND PRELIMINARIES

We start with

Definition 2.1. Let \( X \) be a universal set [9, 11], a collection of objects denoted by \( x \) [1]. Then a fuzzy set \( A \subset X \) is defined as a set of ordered pairs, namely, \( A = \{(x, \mu_A(x)) \mid x \in X\} \). Here, \( \mu_A(x) \) is a function called membership function of \( A \) in \( X \) with the properties

(i) \( \mu_A(x) : X \to [0, 1] \);

(ii) \( \mu_A \) assigns to each element \( x \in X \) a number \( \mu_A(x) \in [0, 1] \) called the grade of membership (or “degree of compatibility” or “degree of truth”, see [1]) of \( x \) in \( A \).

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Definition 2.2. Denote by $A_\alpha \subset A$ the subset of elements $x \in A$ whose degree of membership is at least $\alpha$. Then $A_\alpha$ is called the $\alpha$-level set [9] (or the $\alpha$-level cut [1]) and, clearly, $A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$. The set $A'_{\alpha} = \{x \in X \mid \mu_A(x) > \alpha\}$ is called the strong $\alpha$-level cut [1].

Definition 2.3. A fuzzy set $A \subset X$ is said to be convex if

$$\mu_A(\lambda x_1 + (1 - \lambda) x_2) \geq \min \{\mu_A(x_1), \mu_A(x_2)\},$$

for any $x_1, x_2 \in X$ and $\lambda \in [0, 1]$ [1].

Definition 2.4. If $A$ and $B$ are two fuzzy sets, we say that $A \subset B$ if and only if $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$. For a fuzzy set $A$, its complement $\overline{A}$ is defined as a fuzzy set with membership function $\mu_{\overline{A}}(x) = 1 - \mu_A(x)$ for all $x \in X$ see [9]).

Definition 2.5. A fuzzy number $M$ is of LR type if there exist two functions $L$ (left), $R$ (right) and the scalars $a, b > 0$ such that $\mu_M(x) = L(m - x/a)$ if $x \leq m$, and $\mu_M(x) = L(x - m/b)$ if $x > m$, where $m \in R$ is the mean value of $M$ and $a, b$ are called the left and right spreads respectively [1]. A triangular fuzzy number is a particular type of semisymmetric LR fuzzy number [1, 8, 13].

3. THE MODEL

We start with the model given by Coppi et al. [2]. Consider input crisp variables $X_1, \ldots, X_m$ and a fuzzy output variable, $\tilde{Y}$, on a sample of size $n$. The data will be denoted by $(\tilde{y}_i, x_i)$, $i = 1, \ldots, n$.

Assume that $\tilde{Y}$ is an LR fuzzy variable: $\tilde{Y} = (m, a, b)_{LR}$, $a, b > 0$, where $\mu_{\tilde{Y}}(y) = L(m - y/a)$ if $y \leq m$, and $\mu_{\tilde{Y}}(y) = L(y - m/b)$ if $y \geq m$. Consider the theoretical values $\mathbf{m}_t, \mathbf{a}_t, \mathbf{b}_t$ and the errors $\varepsilon, \varepsilon_A, \varepsilon_B$. Thus, see [2], we can write

$$\mathbf{m} = \mathbf{m}_t + \varepsilon, \quad \mathbf{m} - \mathbf{a} = (\mathbf{m}_t - \mathbf{a}_t) + \varepsilon_A, \quad \mathbf{m} + \mathbf{b} = (\mathbf{m}_t + \mathbf{b}_t) + \varepsilon_B.$$

We replace the parameters $\alpha, \beta_A, \beta_B, \gamma_A, \gamma_B$ which allow us to transform these equations into a regression model, namely,

$$\mathbf{m}_t = F \mathbf{a}, \quad \mathbf{a}_t = \beta_A \mathbf{m}_t + \gamma_A \mathbf{1}, \quad \mathbf{b}_t = \beta_B \mathbf{m}_t + \gamma_B \mathbf{1},$$

where $F$ is a matrix with rows $\mathbf{f}_i^T = [f_1(x_{i1}), \ldots, f_p(x_{ip})]$ (see again [2]).

The theoretical values of the output variables are

$$\tilde{y}_i^* = ((\mathbf{m}_t)_i, (\mathbf{a}_t)_i, (\mathbf{b}_t)_i)_{LR}, \quad i = 1, n.$$

Next, we will generalize the distance $d^2$ and the theoretical results from [2].
4. WEIGHTED LR CASE

Definition 4.1. For $w = (w_1, w_2, w_3)$, where $w_1, w_2, w_3 \in \mathbb{R}_+^*$, define

$$d^2 (w; \tilde{y}, \tilde{y}^*) = d^2 (w; (m, a, b)_{LR}, \tilde{(m_t, a_t, b_t)_{LR}}) =$$

$$= w_1 \|m - m_t\|^2 + w_2 \|(m - \lambda a) - (m_t - \lambda a_t)\|^2 + w_3 \|(m + \rho b) - (m_t + \rho b_t)\|^2,$$

where $\lambda = \int_0^1 L^{-1} (r) \, dr$ and $\rho = \int_0^1 R^{-1} (r) \, dr$.

Theorem 4.1. The relation

$$d^2 (w; \tilde{y}, \tilde{y}^*) = (w_1 + w_2 + w_3) \left[ (m - m_t)^T (m - m_t) \right] -$$

$$- 2w_2 \lambda (m - m_t)^T (a - a_t) + w_3 \lambda^2 (a - a_t)^T (a - a_t) +$$

$$+ 2w_3 \rho (m - m_t)^T (b - b_t) + w_3 \rho^2 (b - b_t)^T (b - b_t)$$

holds.

Proof. We have

$$d^2 (w; \tilde{y}, \tilde{y}^*) = w_1 \|m - m_t\|^2 + w_2 \|(m - \lambda a) - (m_t - \lambda a_t)\|^2 +$$

$$+ w_3 \|(m + \rho b) - (m_t + \rho b_t)\|^2 = w_1 (m - m_t)^T (m - m_t) +$$

$$+ w_2 \lambda (m - m_t)^T (a - a_t) +$$

$$+ w_3 \lambda^2 (a - a_t)^T (a - a_t) +$$

$$+ w_3 \rho (m - m_t)^T (b - b_t) +$$

$$+ w_3 \rho^2 (b - b_t)^T (b - b_t)$$

$$= (w_1 + w_2 + w_3) \left[ (m - m_t)^T (m - m_t) \right] - 2w_2 \lambda (m - m_t)^T (a - a_t) +$$

$$+ 2w_2 \lambda (a - a_t)^T (a - a_t) + 2w_3 \rho (m - m_t)^T (b - b_t) +$$

$$+ w_3 \rho^2 (b - b_t)^T (b - b_t).$$

We will discuss how the model given in [2] is affected after the introduction of the weighted distance, $d^2 (w; \tilde{y}, \tilde{y}^*)$. For $w_1 = w_2 = w_3 = 1$ we obtain the results from [2].

A procedure of finding $\alpha, \beta_A, \beta_B, \gamma_A, \gamma_B$ is to minimize the weighted distance $d^2 (w; \cdot, \cdot)$ between the experimental measurements of the response variable $\tilde{y}_i, i = 1, n$ and the theoretical values $\tilde{y}_i^*$. In other words, we have to solve the problem

$$\min_{\alpha, \beta_A, \beta_B, \gamma_A, \gamma_B} d^2 (w; \tilde{y}, \tilde{y}^*) = \min_{\alpha, \beta_A, \beta_B, \gamma_A, \gamma_B} D^2_{LR} (w; \alpha, \beta_A, \beta_B, \gamma_A, \gamma_B).$$
Theorem 4.2. The problem \( \min_{\alpha, \beta_A, \beta_B, \gamma_A, \gamma_B} D^2_{LR}(w; \alpha, \beta_A, \beta_B, \gamma_A, \gamma_B) \) admits local solutions which can be improved using an iterative estimation algorithm as follows:

\[
\alpha = \left[ \sum_{i=1}^{3} w_i - w_2 \lambda \beta_A (2 - \lambda \beta_A) + w_3 \rho \beta_B (2 + \rho \beta_B) \right]^{-1} \times \\
\times \left( \mathbf{F}^T \mathbf{F} \right)^{-1} \mathbf{F}^T [(w_1 + w_2 + w_3) \mathbf{m} - w_2 \lambda (m \beta_A + a - 1 \gamma_A) + \\
w_2 \lambda^2 (a \beta_A - 1 \beta_A \gamma_A) + w_3 \rho (m \beta_B + b - 1 \gamma_B) + w_3 \rho^2 (b \beta_B - 1 \beta_B \gamma_B)] ,
\]

\[
\beta_A = \lambda^{-1} \left( \mathbf{a}^T \mathbf{F}^T \mathbf{a} - \alpha^T \mathbf{F}^T \mathbf{1} \gamma_A \right) - \left( \alpha^T \mathbf{F}^T \mathbf{m} - \alpha^T \mathbf{F}^T \mathbf{F} \alpha \right) ,
\]

\[
\beta_B = \rho^{-1} \left( \mathbf{b}^T \mathbf{F}^T \mathbf{b} - \alpha^T \mathbf{F}^T \mathbf{1} \gamma_B \right) + \left( \alpha^T \mathbf{F}^T \mathbf{m} - \alpha^T \mathbf{F}^T \mathbf{F} \alpha \right) ,
\]

\[
\gamma_A = (n \lambda)^{-1} \left( \mathbf{1}^T \mathbf{a} - \mathbf{F} \alpha \beta_A - 1 \mathbf{1}^T (\mathbf{m} - \mathbf{F} \alpha) \right) ,
\]

\[
\gamma_B = (n \rho)^{-1} \left[ \rho \mathbf{1}^T (\mathbf{b} - \mathbf{F} \alpha \beta_B) + \mathbf{1}^T (\mathbf{m} - \mathbf{F} \alpha) \right] .
\]

Proof. We have

\[
D^2_{LR}(w; \alpha, \beta_A, \beta_B, \gamma_A, \gamma_B) = (w_1 + w_2 + w_3) \left[ \mathbf{m} - \mathbf{F} \alpha \right]^T (\mathbf{m} - \mathbf{F} \alpha) - \\
2w_2 \lambda \left( \mathbf{m} - \mathbf{F} \alpha \right)^T \left( \mathbf{a} - \mathbf{F} \alpha \beta_A - 1 \gamma_A \right) + \\
w_2 \lambda^2 \left( \mathbf{a} - \mathbf{F} \alpha \beta_A - 1 \gamma_A \right)^T \left( \mathbf{a} - \mathbf{F} \alpha \beta_A - 1 \gamma_A \right) + \\
w_3 \rho \left( \mathbf{m} - \mathbf{F} \alpha \right)^T \left( \mathbf{b} - \mathbf{F} \alpha \beta_B - 1 \gamma_B \right) + \\
w_3 \rho^2 \left( \mathbf{b} - \mathbf{F} \alpha \beta_B - 1 \gamma_B \right)^T \left( \mathbf{b} - \mathbf{F} \alpha \beta_B - 1 \gamma_B \right) = \\
= (w_1 + w_2 + w_3) \left( \mathbf{m}^T \mathbf{m} - 2 \mathbf{m}^T \mathbf{F} \alpha + \alpha^T \mathbf{F}^T \mathbf{F} \alpha \right) - \\
-2w_2 \lambda \left( \mathbf{a}^T \mathbf{m} - \mathbf{F} \alpha \beta_A - \mathbf{m}^T \gamma_A - \alpha^T \mathbf{F}^T \mathbf{a} + \alpha^T \mathbf{F}^T \mathbf{F} \alpha \beta_A + \alpha^T \mathbf{F}^T \mathbf{1} \gamma_A \right) + \\
w_2 \lambda^2 \left( \mathbf{a} - 2 \mathbf{a}^T \mathbf{F} \alpha \beta_A - 2 \mathbf{a}^T \gamma_A + \alpha^T \mathbf{F}^T \mathbf{a} \beta_A + 2 \alpha^T \mathbf{F}^T \mathbf{1} \beta_A \gamma_A + n \gamma_A \right) + \\
2w_3 \rho \left( \mathbf{m}^T \mathbf{b} - \mathbf{m}^T \mathbf{F} \alpha \beta_B - \mathbf{m}^T \gamma_B - \alpha^T \mathbf{F}^T \mathbf{b} + \alpha^T \mathbf{F}^T \mathbf{F} \alpha \beta_B + \alpha^T \mathbf{F}^T \mathbf{1} \gamma_B \right) + \\
w_3 \rho^2 \left( \mathbf{b}^T \mathbf{b} - 2 \mathbf{b}^T \mathbf{F} \alpha \beta_B - 2 \mathbf{b}^T \gamma_B + \alpha^T \mathbf{F}^T \mathbf{F} \alpha \beta_B + 2 \alpha^T \mathbf{F}^T \mathbf{1} \beta_B \gamma_B + n \gamma_B \right).
\]

Equating to zero the partial derivatives of \( D^2_{LR}(w; \alpha, \beta_A, \beta_B, \gamma_A, \gamma_B) \): and obtain the equations

\[
\mathbf{F}^T \mathbf{F} \alpha = \left[ (w_1 + w_2 + w_3) - 2w_2 \lambda \beta_A + w_2 \lambda^2 + w_3 \rho \beta_B + w_3 \rho^2 \beta_B \right] = \\
= (w_1 + w_2 + w_3) \mathbf{F}^T \mathbf{m} - w_2 \lambda (\mathbf{F}^T \mathbf{m} \beta_A + \mathbf{F}^T \mathbf{a} - \mathbf{F}^T \mathbf{1} \gamma_A) + \\
w_2 \lambda^2 (\mathbf{F}^T \mathbf{a} - \mathbf{F}^T \mathbf{1} \beta_A \gamma_A) + w_3 \rho (\mathbf{F}^T \mathbf{m} \beta_B + \mathbf{F}^T \mathbf{b} - \mathbf{F}^T \mathbf{1} \gamma_B) + \\
w_3 \rho^2 (\mathbf{F}^T \mathbf{b} - \mathbf{F}^T \mathbf{1} \beta_B \gamma_B) ,
\]

\[
w_2 [\alpha^T \mathbf{F}^T \mathbf{m} - \alpha^T \mathbf{F}^T \mathbf{F} \alpha - \lambda (\alpha^T \mathbf{F}^T \mathbf{a} - \alpha^T \mathbf{F}^T \mathbf{F} \alpha \beta_A - \alpha^T \mathbf{F}^T \mathbf{1} \gamma_A)] = 0,
\]

\[
w_3 [-\alpha^T \mathbf{F}^T \mathbf{m} + \alpha^T \mathbf{F}^T \mathbf{F} \alpha - \rho (\alpha^T \mathbf{F}^T \mathbf{b} - \alpha^T \mathbf{F}^T \mathbf{F} \alpha \beta_B - \alpha^T \mathbf{F}^T \mathbf{1} \gamma_B)] = 0,
\]

\[
w_2 [\alpha^T \mathbf{F}^T \mathbf{1} - \mathbf{m}^T \mathbf{1} + \lambda (\mathbf{1}^T \mathbf{a} - \alpha^T \mathbf{F}^T \mathbf{1} \beta_A - n \gamma_A)] = 0,
\]
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An iterative solution is given by the relations

\[
\alpha = [(w_1 + w_2 + w_3) - w_2\lambda\beta_A (2 - \lambda\beta_A) + w_3\rho\beta_B (2 + \rho\beta_B)]^{-1} \times
\]

\[
(F^T F)^{-1} F^T [(w_1 + w_2 + w_3) m - w_2\lambda (m\beta_A + a - \mathbf{1}\gamma_A) + w_2\lambda^2 (\mathbf{a}\beta_A - \mathbf{1}\beta_A\gamma_A) + w_3\rho (m\beta_B + b - \mathbf{1}\gamma_B) + w_3\rho^2 (b\beta_B - \mathbf{1}\beta_B\gamma_B)],
\]

\[
\beta_A = \lambda^{-1} (\alpha^T F^T F \alpha)^{-1} [\lambda (\alpha^T F^T a - \alpha^T F^T m - \alpha^T F^T \mathbf{a})],
\]

\[
\beta_B = \rho^{-1} (\alpha^T F^T F \alpha)^{-1} [\rho (\alpha^T F^T b - \alpha^T F^T m - \alpha^T F^T \mathbf{a})],
\]

\[
\gamma_A = (n\lambda)^{-1} [\lambda T (a - F\alpha\beta_A) - \mathbf{1}^T (m - F\alpha)],
\]

\[
\gamma_B = (n\rho)^{-1} [\rho T (b - F\alpha\beta_B) + \mathbf{1}^T (m - F\alpha)],
\]

5. THE WEIGHTED LL SYMMETRICAL CASE

In this case \( L = R, \ a = b. \) Consequently, we have \( \tilde{Y} = (m, a)_{LL}. \) Moreover, \( \rho = \lambda, \ \beta_A = \beta_B \) and \( \gamma_A = \gamma_B \) (see [2]). We have the following results.

**Theorem 5.1.** In the special LL symmetrical case the distance becomes

\[
d^2 (w; \tilde{y}, \tilde{y}') = (w_1 + w_2 + w_3) \left[ (m - m_1)^T (m - m_l) \right] + 2 (-w_2 + w_3) \lambda (m - m_l)^T (a - a_l) + (w_2 + w_3) \lambda^2 (a - a_l)^T (a - a_l).
\]

**Proof.** We have

\[
d^2 (w; \tilde{y}, \tilde{y}') = w_1 \| m - m_l \|^2 +
\]

\[
+ w_2 \| (m - \lambda a) - (m_l - \lambda a_l) \|^2 + w_3 \| (m + \rho b) - (m_l + \rho b_l) \|^2 =
\]

\[
w_1 \| m - m_l \|^2 + w_2 \| (m - \lambda a) - (m_l - \lambda a_l) \|^2 +
\]

\[
+ w_3 \| (m + \lambda a) - (m_l + \lambda a_l) \|^2 =
\]

\[
w_1 \| m - m_l \|^2 + w_2 \| (m - m_l) - \lambda (a - a_l) \|^2 +
\]

\[
+ w_3 \| (m - m_l) + \lambda (a - a_l) \|^2 = w_1 (m - m_l)^T (m - m_l) +
\]

\[
+ \left[ w_2 (m - m_l)^T (m - m_l) - 2w_2 \lambda (m - m_l)^T (a - a_l) + w_2 \lambda^2 (a - a_l)^T (a - a_l) \right] +
\]

\[
+ \left[ w_3 (m - m_l)^T (m - m_l) + 2w_3 \lambda (m - m_l)^T (a - a_l) + w_3 \lambda^2 (a - a_l)^T (a - a_l) \right] =
\]

\[
= (w_1 + w_2 + w_3) \left[ (m - m_l)^T (m - m_l) \right] + 2 (-w_2 + w_3) \lambda (m - m_l)^T (a - a_l) + (w_2 + w_3) \lambda^2 (a - a_l)^T (a - a_l). \]
Thus we have the problem
\[
\min_{\alpha, \beta, \gamma} d^2(w; \tilde{y}, \tilde{y}^*) = \min_{\alpha, \beta, \gamma} D_{LL}^2(w; \alpha, \beta, \gamma_A).
\]

**Theorem 5.2.** The problem \( \min_{\alpha, \beta, \gamma} D_{LL}^2(w; \alpha, \beta, \gamma_A) \) admits local solutions which can be improved using an iterative estimation algorithm as follows:

\[
\alpha = \left( (w_1 + w_2 + w_3) + 2 (-w_2 + w_3) \lambda \beta_A + (w_2 + w_3) \lambda^2 \beta_A^2 \right)^{-1} \times \\
\times (F^T F)^{-1} F^T [(w_1 + w_2 + w_3) m + (-w_2 + w_3) \lambda (m \beta_A + a - 1 \gamma_A) + \\
+ (w_2 + w_3) \lambda^2 (a \beta_A - 1 \beta_A \gamma_A)],
\]

\[
\beta_A = \left( m^T m - 2 m^T F \alpha + \alpha^T F^T F \alpha \right)^{-1} \frac{\lambda^{-1} (-w_2 + w_3)}{w_2 + w_3} \left( \alpha^T F^T 1 + m^T 1 \right),
\]

\[
\gamma_A = \frac{1}{n} \left( 1^T (a - F \alpha \beta_A) + \frac{\lambda^{-1} (-w_2 + w_3)}{w_2 + w_3} \left( -\alpha^T F^T 1 + m^T 1 \right) \right).
\]

**Proof.** We have

\[
D_{LL}^2(w; \alpha, \beta, \gamma_A) = (w_1 + w_2 + w_3) \left[ (m - F \alpha)^T (m - F \alpha) \right] + \\
+ 2 (-w_2 + w_3) \lambda (m - F \alpha)^T (a - F \alpha \beta_A - 1 \gamma_A) + \\
+ (w_2 + w_3) \lambda^2 (a - F \alpha \beta_A - 1 \gamma_A)^T (a - F \alpha \beta_A - 1 \gamma_A) = \\
= (w_1 + w_2 + w_3) (m^T m - 2 m^T F \alpha + \alpha^T F^T F \alpha) + \\
+ 2 (-w_2 + w_3) \lambda (m^T a - m^T F \alpha \beta_A - m^T 1 \gamma_A - \alpha^T F^T a + \alpha^T F^T F \alpha \beta_A + \alpha^T F^T 1 \gamma_A) + \\
+ (w_2 + w_3) \lambda^2 (a^T a - 2 a^T F \alpha \beta_A - 2 a^T 1 \gamma_A + \alpha^T F^T F \alpha \beta_A^2 + 2 \alpha^T F^T 1 \beta_A \gamma_A + n \gamma_A^2).
\]

Similarly to Theorem 4.2, we obtain the equations

\[
F^T F \alpha^T [(w_1 + w_2 + w_3) + 2 (-w_2 + w_3) \lambda \beta_A + (w_2 + w_3) \lambda^2 \beta_A^2] = \\
= (w_1 + w_2 + w_3) F^T m + (-w_2 + w_3) \lambda (F^T m \beta_A + F^T a - F^T 1 \gamma_A) + \\
+ (w_2 + w_3) \lambda^2 (F^T a \beta_A - F^T 1 \beta_A \gamma_A),
\]

\[
- (w_2 + w_3) (\alpha^T F^T 1 - m^T 1) + (w_2 + w_3) \lambda (-a^T 1 + \alpha^T F^T 1 \beta_A + n \gamma_A) = 0.
\]

Thus,

\[
\alpha = \left( (w_1 + w_2 + w_3) + \\
+ 2 (-w_2 + w_3) \lambda \beta_A + (w_2 + w_3) \lambda^2 \beta_A^2 \right)^{-1} (F^T F)^{-1} F^T [(w_1 + w_2 + w_3) m + \\
+ (-w_2 + w_3) \lambda (m \beta_A + a - 1 \gamma_A) + (w_2 + w_3) \lambda^2 (a \beta_A - 1 \beta_A \gamma_A)],
\]

\[
\beta_A = \left( m^T m - 2 m^T F \alpha + \alpha^T F^T F \alpha \right)^{-1} \frac{\lambda^{-1} (-w_2 + w_3)}{w_2 + w_3} \left( \alpha^T F^T 1 + m^T 1 \right),
\]

\[
\gamma_A = \frac{1}{n} \left( 1^T (a - F \alpha \beta_A) + \frac{\lambda^{-1} (-w_2 + w_3)}{w_2 + w_3} \left( -\alpha^T F^T 1 + m^T 1 \right) \right).
\]
\[ \beta_A = \left( \alpha^T F^T F \alpha \right)^{-1} \left[ \alpha^T F^T (a - 1 \gamma_A) + \frac{\lambda^{-1} (-w_2 + w_3)}{w_2 + w_3} \left( \alpha^T F^T m - \alpha^T F^T F \alpha \right) \right], \]

\[ \gamma_A = \frac{1}{n} \left[ 1^T (a - F \beta_A) + \frac{\lambda^{-1} (-w_2 + w_3)}{w_2 + w_3} \left( -\alpha^T F^T 1 + m^T 1 \right) \right], \]

and the proof is complete. \( \square \)

For \( w_1 = w_2 = w_3 = 1 \) we obtain the solutions from [2].

REFERENCES


