EQUILIBRIUM IN ABSTRACT ECONOMIES WITH
WEAKLY CONVEX GRAPH CORRESPONDENCES

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We establish new existence results for the equilibrium of the abstract economies. The constraint correspondences have the WCG property. This notion was proposed in 1998 by Ding and He [18]. Basically, our proofs are based on a continuous selection theorem for correspondences with WCG property. We use also Wu’s fixed point theorem (see [21]).

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1. INTRODUCTION

In game theory, most authors studied the existence of equilibria for abstract economies with preferences represented as correspondences which have continuity properties. Thus, the results obtained by Shafer and Sonnenschein [15] concern economies with finite dimensional commodity space and preference correspondences having an open graph. Yannelis and Prahbakar [23] used selection theorems and fixed-point theorems for correspondences with open lower sections defined on infinite dimensional strategy spaces. Then, a problem of great interest was to weaken the assumptions to lower semi-continuity. Yuan [25] developed an approximation method for economies with lower semi-continuous correspondences defined on locally convex spaces. Some authors developed the theory of continuous selections of correspondences and gave numerous applications in game theory. Michael’s selection theorem [12] is well known and basic in many applications. Browder [4, 5] first used a continuous selection theorem to prove the Fan-Browder fixed point theorem. Later, Yannelis and Prabhakar [23], Ben-El-Mechaiekh [2], Ding, Kim and Tan [7], Horvath [10], Wu [21], Park [14], Yu and Lin [24] and many others established several continuous selection theorems with applications.

Other general existence results appeared in the literature dealing with economies with upper semi-continuous correspondences representing the constraints and preferences of each agent. For example, Tan and Wu [17], Ding [6] obtained some results in this area.

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In this paper, we use the concept of weakly convex graph for correspondences to show that an equilibrium for an abstract economy exists without continuity assumptions. This concept was proposed in 1998 by Ding and He [18] who obtained some new coincidence, best approximation and fixed point theorems for correspondences without compact convex values and continuity in topological vector spaces.

We introduce the notions of correspondences with the WCGS or e-WCGS property. Then, using a technique based on a continuous selection, we prove new equilibrium existence theorem for an abstract economy.

The paper is organized as follows Section 2 contains preliminaries and notation. The selection theorem is presented in Section 3. The equilibrium theorems are stated in Section 4 and their proofs are collected in Section 5.

2. PRELIMINARIES AND NOTATION

Throughout this paper we shall use the following notation and definitions.

Let $A$ be a subset of a topological space $X$.
1. $2^A$ denotes the family of all subsets of $A$.
2. $\text{cl} A$ denotes the closure of $A$ in $X$.
3. If $A$ is a subset of a vector space, $\text{co} A$ denotes the convex hull of $A$.
4. If $F, T : A \to 2^X$ are correspondences, then $\text{co} T, \text{cl} T, T \cap F : A \to 2^X$ are correspondences defined by $\text{(co} T)(x) = \text{co} T(x)$, $(\text{cl} T)(x) = \text{cl} T(x)$ and $(T \cap F)(x) = T(x) \cap F(x)$ for each $x \in A$, respectively.
5. The graph of $T : X \to 2^Y$ is the set $\text{Gr}(T) = \{(x, y) \in X \times Y \mid y \in T(x)\}$
6. The correspondence $\overline{T}$ is defined by $\overline{T}(x) = \{y \in Y : (x, y) \in \text{cl}_{X \times Y} \text{Gr}(T)\}$ (the set $\text{cl}_{X \times Y} \text{Gr}(T)$ is called the adherence of the graph of $T$).

It is easy to see that $\text{cl} T(x) \subset \overline{T}(x)$ for each $x \in X$.

Lemma 1 ([25]). Let $X$ be a topological space, $Y$ a non-empty subset of a topological vector space $E$, $\mathcal{B}$ a base of the neighborhoods of 0 in $E$ and $A : X \to 2^Y$. For each $V \in \mathcal{B}$, let $A_V : X \to 2^Y$ be defined by $A_V(x) = (A(x) + V) \cap Y$ for each $x \in X$. If $\hat{x} \in X$ and $\hat{y} \in Y$ are such that $\hat{y} \in \bigcap_{V \in \mathcal{B}} \overline{A_V(\hat{x})}$, then $\hat{y} \in \overline{A(\hat{x})}$.

Definition 1. Let $X, Y$ be topological spaces and $T : X \to 2^Y$ a correspondence.

1. $T$ is said to be upper semicontinuous if for each $x \in X$ and each open set $V$ in $Y$ with $T(x) \subset V$ there exists an open neighborhood $U$ of $x$ in $X$ such that $T(x) \subset V$ for each $y \in U$. 
2. $T$ is said to be lower semicontinuous if for each $x \in X$ and each open set $V$ in $Y$ with $T(x) \cap V \neq \emptyset$ there exists an open neighborhood $U$ of $x$ in $X$ such that $T(y) \cap V \neq \emptyset$ for each $y \in U$.

3. $T$ is said to have open lower sections if $T^{-1}(y) := \{ x \in X : y \in T(x) \}$ is open in $X$ for each $y \in Y$.

**Lemma 2 ([19]).** Let $X$ be a topological space, $Y$ a topological linear space, and let $A : X \to 2^Y$ be an upper semicontinuous correspondence with compact values. Assume that the sets $C \subset Y$ and $K \subset Y$ are closed and respectively compact. Then $T : X \to 2^Y$ defined by $T(x) = (A(x) + C) \cap K$ for all $x \in X$ is upper semicontinuous.

Ding Xieping and He Yiran [18] proposed the concept of weakly convex graph for a correspondence.

**Definition 2 ([18]).** Let $X$ be a nonempty convex subset of a topological vector space $E$ and $Y$ a nonempty subset of $E$. The correspondence $T : X \to 2^Y$ is said to have weakly convex graph (it is a WCG correspondence for short) if for each finite set $\{x_1, x_2, \ldots, x_n\} \subset X$ there exists $y_i \in T(x_i)$, $i = 1, 2, \ldots, n$, such that

\[
\text{co}(\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}) \subset \text{Gr}(T).
\]

Let $\Delta_{n-1} = \{ (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n : \sum_{i=1}^n \lambda_i = 1, \lambda_i \geq 0, i = 1, 2, \ldots, n \}$ be the standard $(n-1)$-dimensional simplex. Then relation (1.1) is equivalent to

\[
\sum_{i=1}^n \lambda_i y_i \in T\left( \sum_{i=1}^n \lambda_i x_i \right), \quad \forall (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma.
\]

It is clear that if either the graph $\text{Gr}(T)$ is convex or $\bigcap\{T(x) : x \in X\} \neq \emptyset$, then $T$ has a weakly convex graph.

**Remark 1.** A WCG correspondence has non-empty values.

**Remark 2.** A WCG correspondence $T : X \to 2^Y$ may not be convex-valued.

**Example 1.** $T_1 : [0, 2] \to [0, 2], T_1(x) = \begin{cases} [0, \frac{1}{2}] \cup [\frac{3}{2}, 2] & \text{if } x = 1, \\ [0, 2 - \frac{1}{2}x] & \text{if } x \in [0, 2] \setminus \{1\} \end{cases}$ is a WCG correspondence (since $\bigcap\{T_1(x) : x \in [0, 2]\} = \{0\} \neq \emptyset$), but $T_1(1)$ is not convex and $\text{Gr}(T_1)$ is not convex either.

**Remark 3.** A WCG correspondence may not enjoy topological properties.
Example 2. \( T_2 : [0, 2] \to [0, 2], \) \( T_2(x) = \begin{cases} [0, 2] & \text{if } x \in [0, 1], \\ \{1\} & \text{if } x \in (1, 2] \end{cases} \) is a WCG correspondence, but it is not lower semicontinuous and it has not open lower sections.

Example 3. \( T_3 : [0, 2] \to [0, 2], \) \( T_3(x) = \begin{cases} \{1\} & \text{if } x \in [0, 1], \\ [0, 2] & \text{if } x \in (1, 2] \end{cases} \) is a WCG correspondence, but it is not upper semicontinuous.

Example 4. \( T_1 \) is not upper semicontinuous at \( x = 1. \)

Following Ding and He [18], we can formulate the selection theorem which we shall use to prove our equilibrium theorem.

Theorem 1. Let \( Y \) be a non-empty subset of a topological vector space \( E \) and \( K \) an \((n-1)\)-dimensional simplex in a topological vector space \( F \). Let \( T : K \to 2^Y \) be a WCG correspondence. Then \( T \) has a continuous selection on \( K \).

Proof. Let \( a_1, a_2, \ldots, a_n \) be the vertices of \( K \). Since Gr\((T)\) is weakly convex, there exist \( b_i \in T(a_i), \ i = 1, 2, \ldots, n, \) such that

\[
\sum_{i=1}^{n} \lambda_i b_i \in T \left( \sum_{i=1}^{n} \lambda_i a_i \right) 
\]

for all \((\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Delta_{n-1}.

Define a mapping \( f : K \to Y \) by

\[
f \left( \sum_{i=1}^{n} \lambda_i a_i \right) = \sum_{i=1}^{n} \lambda_i b_i \in T \left( \sum_{i=1}^{n} \lambda_i a_i \right) \quad \forall (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Delta_{n-1}. 
\]

We show that \( f \) is continuous. Since \( K \) is a \((n-1)\)-dimensional simplex with vertices \( a_1, a_2, \ldots, a_n, \) there exist unique continuous functions \( \lambda_i : K \to R, \ i = 1, 2, \ldots, n, \) such that for each \( x \in K \) we have \((\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x)) \in \Delta_{n-1} \) and \( x = \sum_{i=1}^{n} \lambda_i(x) a_i. \) Hence \( f(x) \in T(x) \) for each \( x \in K. \) Let \( (x_m)_{m \in N} \) be a sequence which converges to \( x_0 \in K, \) where \( x_m = \sum_{i=1}^{n} \lambda_i(x_m) a_i \) and \( x_0 = \sum_{i=1}^{n} \lambda_i(x_0) a_i. \) It follows from the continuity of \( \lambda_i \) that \( \lambda_i(x_m) \to \lambda_i(x_0) \) for each \( i = 1, 2, \ldots, n \) as \( m \to \infty. \) Hence we must have \( f(x_m) \to f(x_0) \) as \( m \to \infty, \) i.e., \( f \) is continuous. \( \square \)
Remark 4. Let $T : X \rightarrow 2^Y$ be a WCG correspondence and $X_0$ be a non-empty convex subset of $X$. Then the restriction $T_{|X_0} : X_0 \rightarrow 2^Y$ of $T$ to $X_0$ is a WCG correspondence, too.

Now, we introduce the definitions below.

Let $I$ be an index set. For each $i \in I$, let $X_i$ be a non-empty convex subset of a topological linear space $E_i$ and denote $X = \prod_{i \in I} X_i$.

**Definition 3.** Let $K_i$ be a subset of $X$. The correspondence $A_i : X \rightarrow 2^{X_i}$ is said to have the WCGS-property on $K_i$ if there is a WCG correspondence $T_i : K_i \rightarrow 2^{X_i}$ such that $x_i /\notin T_i(x)$ and $T_i(x) \subset A_i(x)$ for all $x \in K_i$.

**Definition 4.** Let $K_i$ be a subset of $X$. The correspondence $A_i : X \rightarrow 2^{X_i}$ is said to have the e-WCGS-property on $K_i$ if for each convex neighbourhood $V$ of 0 in $X_i$ there is a WCG correspondence $T^V_i : K_i \rightarrow 2^{X_i}$ such that $x_i /\notin T^V_i(x)$ and $T^V_i(x) \subset A_i(x) + V$ for all $x \in K_i$.

To prove our theorems, we need

**Theorem 2 ([21]).** Let $I$ be an index set. For each $i \in I$, let $X_i$ be a non-empty convex subset of a Hausdorff locally convex topological vector space $E_i$, $D_i$ a non-empty compact metrizable subset of $X_i$, and $S_i, T_i : X := \prod_{i \in I} X_i \rightarrow 2^{D_i}$ two correspondences such that

1. $\overline{\operatorname{co}} S_i(x) \subseteq T_i(x)$ and $S_i(x) \neq \emptyset$ for each $x \in X$;
2. $S_i$ is lower semicontinuous.

Then there exists a point $\bar{x} = \prod_{i \in I} x_i \in D = \prod_{i \in I} D_i$ such that $\bar{x}_i \in T_i(\bar{x})$ for each $i \in I$.

3. **EQUILIBRIUM THEOREMS**

First, we present the model of an abstract economy and the definition of an equilibrium.

Let $I$ be a non-empty set (the set of agents). For each $i \in I$, let $X_i$ be a non-empty topological vector space representing the set of actions and define $X := \prod_{i \in I} X_i$. Let $A_i, B_i : X \rightarrow 2^{X_i}$ be the constraint correspondences and $P_i$ the preference correspondence.

**Definition 5.** The family $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$ is said to be an abstract economy.

**Definition 6.** An equilibrium for $\Gamma$ is defined as a point $\bar{x} \in X$ such that $\bar{x}_i \in B_i(\bar{x})$ and $A_i(\bar{x}_i) \cap P_i(\bar{x}) = \emptyset$ for each $i \in I$. 

Remark 5. When $A_i(x) = B_i(x)$ for all $x \in X$ and each $i \in I$, this abstract economy model coincides with the classical one introduced by Borglin and Keiding [3]. If in addition, $B_i(x) = \text{cl}_{X_i} B_i(x)$ for each $x \in X$, which is the case if $B_i$ has a closed graph in $X \times X_i$, the definition of an equilibrium coincides with that used by Yannelis and Prabhakar [23].

Yannelis and Prabhakar [23] proved the existence of equilibrium for abstract economies with the correspondences $A_i$ and $P_i$ having open lower sections. They used a selection theorem for the correspondences $A_i \cap P_i$, which also have open lower sections. Zhou [20] gave existence equilibrium theorems and the approach he adopted to prove these theorems is an extension of Yannelis and Prabhakar’s argument in [23]. When the strategy spaces are metrizable, Theorems 5 and 6 in [20] are strict generalizations of the results of Shafer and Sonnenschein, Yannelis and Prabhakar.

The theorems we will state use the selection theorem mentioned in Section 3 and the same technique based on a continuous selection, as in [20] and [23]. We show the existence of equilibrium for an abstract economy without assuming the continuity of the constraint and preference correspondences $A_i$ and $P_i$.

Since a WCG correspondence may fail to have continuity properties or convex values and has a continuous selection on a simplex, our results are related to the above mentioned theorems only through the technique of proof.

First, we prove a new equilibrium existence theorem for a noncompact abstract economy with constraint and preference correspondences $A_i$ and $P_i$, which have the property that their intersection $A_i \cap P_i$ contains a WCG selector on the domain $W_i$ of $A_i \cap P_i$ and $W_i$ must be a simplex. To find the equilibrium point, we use Wu’s fixed point theorem [21].

**Theorem 3.** Let $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$ be an abstract economy, where $I$ is a (possibly uncountable) set of agents such that for each $i \in I$: 

1. $X_i$ is a non-empty convex set in a locally convex space $E_i$ and there exists a compact subset $D_i$ of $X_i$ containing all the values of the correspondences $A_i, P_i$ and $B_i$ such that $D = \prod_{i \in I} D_i$ is metrizable;

2. $\text{cl}_{B_i} B_i$ is lower semicontinuous, has non-empty convex values and $A_i(x) \subset B_i(x)$ for each $x \in X$;

3. $W_i = \{x \in X \mid (A_i \cap P_i)(x) \neq \emptyset\}$ is an $(n_i - 1)$-dimensional simplex in $X$ such that $W_i \subset \text{co} D$;

4. there exists a WCG correspondence $S_i : W_i \to 2^{D_i}$ such that $S_i(x) \subset (A_i \cap P_i)(x)$ for each $x \in W_i$;

5. $x_i \notin (A_i \cap P_i)(x)$ for each $x \in W_i$.

Then there exists an equilibrium point $\bar{x} \in D$ for $\Gamma$, i.e., $\bar{x} \in \text{cl}_{B_i} \bar{x}$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ for each $i \in I$. 

Since a correspondence $T : X \to 2^Y$ with a convex graph or having the property that $\bigcap \{T(x) : x \in X\} \neq \emptyset$ is a WCG correspondence, we obtain the following corollaries.

**Corollary 1.** Let $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$ be an abstract economy, where $I$ is a (possibly uncountable) set of agents such that for each $i \in I$:

1. $X_i$ is a non-empty convex set in a locally convex space $E_i$ and there exists a compact subset $D_i$ of $X_i$ containing all the values of the correspondences $A_i, P_i$ and $B_i$ such that $D = \prod_{i \in I} D_i$ is metrizable;
2. $\text{cl}B_i$ is lower semicontinuous, has non-empty convex values and $A_i(x)$ ⊂ $B_i(x)$ for each $x \in X$;
3. $W_i = \{x \in X \mid (A_i \cap P_i)(x) \neq \emptyset\}$ is an $(n_i - 1)$-dimensional simplex in $X$ such that $W_i \subset \text{co}D$;
4. there exists a correspondence $S_i : W_i \to 2^{D_i}$ such that $S_i$ has a convex graph and $S_i(x) \subset (A_i \cap P_i)(x)$ for each $x \in W_i$;
5. $x_i \notin (A_i \cap P_i)(x)$ for each $x \in W_i$.

Then there exists an equilibrium point $\overline{x} \in D$ for $\Gamma$, i.e., $\overline{x}_i \in \text{cl}B_i(\overline{x})$ and $A_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset$ for each $i \in I$.

**Corollary 2.** Let $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$ be an abstract economy, where $I$ is a (possibly uncountable) set of agents such that for each $i \in I$:

1. $X_i$ is a non-empty compact convex metrizable set in a locally convex space $E_i$;
2. $\text{cl}B_i$ is lower semicontinuous, has non-empty convex values and $A_i(x)$ ⊂ $B_i(x)$ for each $x \in X$;
3. $W_i = \{x \in X \mid (A_i \cap P_i)(x) \neq \emptyset\}$ is an $(n_i - 1)$-dimensional simplex in $X$;
4. there exists a correspondence $S_i : W_i \to 2^{X_i}$ with closed values such that $S_i$ has the property that for any finite set $\{x_1, x_2, \ldots, x_n\} \subset X$, $\bigcap_{i=1}^n S(x_i) \neq \emptyset$ and $S_i(x) \subset (A_i \cap P_i)(x)$ for each $x \in W_i$;
5. $x_i \notin (A_i \cap P_i)(x)$ for each $x \in W_i$.

Then there exists an equilibrium point $\overline{x} \in X$ for $\Gamma$, i.e., $\overline{x}_i \in \text{cl}B_i(\overline{x})$ and $A_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset$ for each $i \in I$.

To prove Theorem 4 we use an approximation method, to mean that we obtain a continuous selection $f_i^{V_i}$ of $(A_i + V_i) \cap P_i$ for each $i \in I$, where $V_i$ is a convex neighborhood of 0 in $X_i$. For every $V = \prod_{i \in I} V_i$, we obtain an equilibrium point for the associated approximate abstract economy $\Gamma_V = (X_i, A_i, P_i, B_i^{V_i})_{i \in I}$, i.e., a point $\overline{x} \in X$ such that $A_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset$ and $\overline{x}_i \in B_i^{V_i}(\overline{x})$, where the correspondence $B_i^{V_i} : X \to 2^{X_i}$ is defined by $B_i^{V_i}(x) = \text{cl}(B_i(x) + V_i) \cap X_i$ for each $x \in X$ and each $i \in I$. Finally, we use Lemma 1
to get an equilibrium point for $\Gamma$ in $X$. The compactness assumption for $X_i$ is essential in the proof.

**Theorem 4.** Let $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$ be an abstract economy, where $I$ is a (possibly uncountable) set of agents such that for each $i \in I$:

1. $X_i$ is a non-empty compact convex set in a locally convex space $E_i$;
2. $\overline{cl} B_i$ is upper semicontinuous, has non-empty convex values and $A_i(x) \subset B_i(x)$ for each $x \in X$;
3. the set $W_i := \{ x \in X \mid (A_i \cap P_i)(x) \neq \emptyset \}$ is non-empty, open and $K_i = \overline{cl} W_i$ is an $(n_i - 1)$-dimensional simplex in $X$;
4. for each convex neighbourhood $V$ of 0 in $X_i$, $(A_i + V) \cap P_i : K_i \to 2^{X_i}$ is a WCG correspondence;
5. $x_i \notin P_i(x)$ for each $x \in K_i$.

Then there exists an equilibrium point $\pi \in X$ for $\Gamma$, i.e., $\pi_i \in \overline{cl} B_i(\pi)$ and $A_i(\pi) \cap P_i(\pi) = \emptyset$ for each $i \in I$.

Theorem 5 uses the idea of Zhen’s Theorem 3.2 in [19]. These two results differ by the nature of the correspondence selectors $\overline{cl} B_i$. To find the equilibrium point, we use Wu’s fixed point theorem for correspondences $\overline{cl} B_i$ which are lower semicontinuous and we need a non-empty compact metrizable set $D_i$ in $X_i$ for each $i \in I$. The spaces $X_i$ are not compact.

**Theorem 5.** Let $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$ be an abstract economy, where $I$ is a (possibly uncountable) set of agents such that for each $i \in I$:

1. $X_i$ is a non-empty convex set in a Hausdorff locally convex space $E_i$ and there exists a nonempty compact metrizable subset $D_i$ of $X_i$ containing all values of the correspondences $A_i$, $P_i$ and $B_i$;
2. $\overline{cl} B_i$ is lower semicontinuous with non-empty convex values;
3. there exists an $(n_i - 1)$-dimensional simplex $K_i$ in $X$ and

$$W_i := \{ x \in X \mid (A_i \cap P_i)(x) \neq \emptyset \} \subset \text{int}_X(K_i);$$

4. $\overline{cl} B_i$ has the (WCGS)-property on $K_i$.

Then there exists an equilibrium point $\pi \in D$ for $\Gamma$, i.e., $\pi_i \in \overline{cl} B_i(\pi)$ and $A_i(\pi) \cap P_i(\pi) = \emptyset$ for each $i \in I$.

Theorem 6 follows the idea of Zheng’s Theorem 3.1 in [19], but the correspondences $\overline{cl} B_i$ have e-WCGS property on a simplex $K_i$ in $X$ which contains the domain of $A_i \cap P_i$. The sets $X_i$ are non-empty, compact, convex in locally convex spaces $E_i$. As in Theorem 4, we first obtain an equilibrium for $\Gamma_V$, and the proof then coincides with the proof of Theorem 4.

**Theorem 6.** Let $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$ be an abstract economy, where $I$ is a (possibly uncountable) set of agents such that for each $i \in I$:

1. $X_i$ is a non-empty compact convex set in a locally convex space $E_i$;
(2) $\text{cl } B_i$ is upper semicontinuous with non-empty convex values;

(3) the set $W_i := \{x \in X \mid (A_i \cap P_i)(x) \neq \emptyset\}$ is open and there exists an $(n_i - 1)$-dimensional simplex $K_i$ in $X$ such that $W_i \subset \text{int}_X(K_i)$;

(4) $\text{cl } B_i$ has the e-WCGS-property on $K_i$.

Then there exists an equilibrium point $\bar{\pi} \in X$ for $\Gamma$, i.e., $\bar{\pi}_i \in \overline{B_i}(\bar{\pi})$ and $A_i(\bar{\pi}) \cap P_i(\bar{\pi}) = \emptyset$ for each $i \in I$.

4. PROOFS OF THE THEOREMS

Proof of Theorem 3. Let $i \in I$. It follows from assumption (4) and the selection theorem that there exists a continuous function $f_i : W_i \rightarrow D_i$ such that $f_i(x) \in S_i(x) \subset A_i(x) \cap \text{cl } B_i(x)$ for each $x \in W_i$.

Define the correspondence $T_i : X \rightarrow 2^{D_i}$ by

$$T_i(x) := \begin{cases} \{f_i(x)\} & \text{if } x \in W_i, \\ \text{cl } B_i(x) & \text{if } x \notin W_i; \end{cases}$$

$T_i$ is lower semicontinuous on $X$. Let $V$ be an closed subset of $X_i$. Then

$$U := \{x \in X \mid T_i(x) \subset V\} = \{x \in W_i \mid T_i(x) \subset V\} \cup \{x \in X \setminus W_i \mid T_i(x) \subset V\}$$

$$= \{x \in W_i \mid f_i(x) \in V\} \cup \{x \in X \mid \text{cl } B_i(x) \subset V\}$$

$$= (f_i^{-1}(V) \cap W_i) \cup \{x \in X \mid \text{cl } B_i(x) \subset V\}.$$

$U$ is a closed set because $W_i$ is closed, $f_i$ is a continuous map on $\text{int}_X K_i$ and the set $\{x \in X \mid \text{cl } B_i(x) \subset V\}$ is closed since $\text{cl } B_i$ is l.s.c. Let $D = \prod_{i \in I} D_i$. Then by Tychonoff’s theorem, $D$ is compact in the convex set $X$. By Theorem 2 (Wu’s fixed-point theorem) applied to the correspondences $S_i = T_i$ and $T_i : X \rightarrow 2^{D_i}$, there exists $\bar{\pi} \in D$ such that $\bar{\pi}_i \in T_i(\bar{\pi})$ for each $i \in I$. If $\bar{\pi} \in W_i$ for some $i \in I$, then $\bar{\pi}_i = f_i(\bar{\pi})$, which is a contradiction. Therefore, $\bar{\pi} \notin W_i$, hence $(A_i \cap P_i)(\bar{\pi}) = \emptyset$. Also, for each $i \in I$, we have $\bar{\pi}_i \in T_i(\bar{\pi})$, and then $\bar{\pi}_i \in \text{cl } B_i(\bar{\pi})$. 

Remark 6. In Theorem 3, the correspondences $A_i \cap P_i$ don’t verify continuity assumptions and do not have convex or compact values.

Remark 7. In assumption (3), $W_i$ must be a proper subset of $X$. In fact, if $W_i = X_i$ then, by applying Himmelberg’s fixed point theorem to $\prod_{i \in I} f_i(x)$, where $f_i$ is a continuous selection of $S_i \subset A_i \cap P_i$, we can get a fixed point $\bar{\pi} \in \prod_{i \in I} (A_i \cap P_i)(\bar{\pi})$, which contradicts assumption (5).
Proof of Corollary 1. Since a correspondence with a convex graph is a WCG correspondence, $S_i$ verifies assumption (4) from Theorem 3, so that we can apply this theorem. □

Proof of Corollary 2. $X$ is a compact space and for each $i \in I$ the closed sets $S_i(x)$, $x \in X$ have the finite intersection property. Then $\bigcap \{S_i(x) : x \in X\} \neq \emptyset$. It follows that $S_i$ is a WCG correspondence and the conclusion comes from Theorem 3. □

Proof of Theorem 4. For each $i \in I$, let $\beta_i$ denote the family of all open convex neighborhoods of zero in $E_i$. Let $V = (V_i)_{i \in I} \in \prod_{i \in I} \beta_i$. Since $(A_i + V_i) \cap P_i$ is a WCG correspondence on $K_i$, from the selection theorem there exists a continuous function $f_i^{V_i} : K_i \rightarrow X_i$ such that

$$f_i^{V_i}(x) \in (A_i(x) + V_i) \cap P_i(x) \subset (A_i(x) + V_i) \cap X_i,$$

for each $x \in K_i$. It follows that $f_i^{V_i}(x) \in \text{cl}(B_i(x) + V_i)$ for $x \in K_i$. Since $X_i$ is compact, $\text{cl}(B_i(x))$ is compact for every $x \in X$ and $\text{cl}(B_i(x) + V_i) = \text{cl}(B_i(x)) + \text{cl} V_i$ for every $V_i \subset E_i$.

Define the correspondence $T_i^{V_i} : X \rightarrow 2^{X_i}$ by

$$T_i^{V_i}(x) := \begin{cases} \{f_i^{V_i}(x)\} & \text{if } x \in \text{int}_X K = W_i, \\ \text{cl}(B_i(x) + V_i) \cap X_i & \text{if } x \in X \setminus \text{int}_X K_i. \end{cases}$$

The correspondence $B_i^{V_i} : X \rightarrow 2^{X_i}$, defined by $B_i^{V_i}(x) := \text{cl}(B_i(x) + V_i) \cap X_i$ is u.s.c. by Lemma 2. Then reasoning as in the proof of Theorem 3, we can prove that $T_i^{V_i}$ is upper semicontinuous on $X$ and has closed convex values.

Define $T^V : X \rightarrow 2^X$ by $T^V(x) := \prod_{i \in I} T_i^{V_i}(x)$ for each $x \in X$. $T^V$ is an upper semicontinuous correspondence and also has non-empty convex closed values. Since $X$ is a compact convex set, by Fan’s fixed-point theorem [8], there exists $\varpi_V \in X$ such that $\varpi_V \in T^V(\varpi_V)$, i.e., for each $i \in I$, $(\varpi_V)_i \in T_i^{V_i}(\varpi_V)$. We claim that $\varpi_V \in X \setminus \bigcup_{i \in I} \text{int}_X K_i$. Indeed, if $\varpi_V \in \text{int}_X K_i$, $(\varpi_V)_i \in T_i^{V_i}(\varpi_V) = f_i(\varpi_V) \in ((A_i(\varpi_V) + V_i) \cap P_i)(\varpi_V) \subset P_i(\varpi_V)$, which contradicts assumption (5). Hence $(\varpi_V)_i \in \text{cl}(B_i(\varpi_V) + V_i) \cap X_i$ and $(A_i \cap P_i)(\varpi_V) = \emptyset$, i.e. $\varpi_V \in Q_V$ where

$$Q_V = \bigcap_{i \in I} \{x \in X : x_i \in \text{cl}(B_i(x) + V_i) \cap X_i \text{ and } (A_i \cap P_i)(x) = \emptyset\}.$$

Since $W_i$ is open, $Q_V$ is the intersection of non-empty closed sets, then it is non-empty, closed in $X$. Let us prove that the family $\{Q_V : V \in \prod_{i \in I} \beta_i\}$ has the finite intersection property. Let $\{V^{(1)}, V^{(2)}, \ldots, V^{(n)}\}$ be any finite set of
\[ \prod_{i \in I} \beta_i \text{ and let } V^{(k)} = (V_i^{(k)})_{i \in I}, \ k = 1, \ldots, n. \] 
For each \( i \in I, \) let \( V_i = \bigcap_{k=1}^n V_i^{(k)}. \) Then \( V_i \in \beta_i, \) thus, \( V = (V_i)_{i \in I} \in \prod_{i \in I} \beta_i. \) Clearly, \( Q_V \subset \bigcap_{k=1}^n Q_{V^{(k)}} \) so that 
\[ \bigcap_{k=1}^n Q_{V^{(k)}} \neq \emptyset. \]

Since \( X \) is compact and the family \( \left\{ Q_V : V \in \prod_{i \in I} \beta_i \right\} \) has the finite intersection property, we have 
\[ \bigcap \left\{ Q_V : V \in \prod_{i \in I} \beta_i \right\} \neq \emptyset. \] 
Take any \( \bar{x} \in \bigcap \{ Q_V : V \in \prod_{i \in I} \beta_i \}. \) Then \( \bar{x} \in \text{cl}(B_i(\bar{x}) + V_i) \cap X_i \) and \( (A_i \cap P_i)(\bar{x}) = \emptyset \) for each \( i \in I \) and each \( V_i \in \beta_i, \) but then \( \bar{x} \in \text{cl}(B_i(\bar{x})) \) by Lemma 3 and \( (A_i \cap P_i)(\bar{x}) = \emptyset \) for each \( i \in I, \) so that \( \bar{x} \) is an equilibrium point of \( \Gamma \) in \( X. \) \( \square \)

**Proof of Theorem 5.** Since \( \text{cl} B_i \) has the WCGS property on \( K_i, \) there exists a WCG correspondence \( F_i : X \to 2^{D_i} \) such that \( F_i(x) \subset \text{cl} B_i(x) \) and \( x_i \notin F_i(x) \) for each \( x \in K_i. \) Note that \( K_i \) is an \((n_i - 1)\)-dimensional simplex. By the selection theorem, there exists a continuous function \( f_i : K_i \to D_i \) such that \( f_i(x) \in F_i(x) \) for each \( x \in K_i. \) Because \( x_i \notin F_i(x) \) for each \( x \in K_i, \) we have \( x_i \neq f_i(x) \) for each \( x \in K_i. \)

Define the correspondence \( T_i : X \to 2^{D_i}, \) by
\[
T_i(x) := \begin{cases} 
\{f_i(x)\} & \text{if } x \in K_i, \\
\text{cl} B_i(x) & \text{if } x \notin K_i.
\end{cases}
\]

\( T_i \) is lower semicontinuous on \( X \) and has closed convex values.

Let \( U \) be a closed subset of \( X_i. \) Then
\[
U' := \{x \in X \mid T_i(x) \subset U\} = \{x \in K_i \mid T_i(x) \subset U\} \cup \{x \in X \setminus K_i \mid T_i(x) \subset U\} = \{x \in K_i \mid f_i(x, y) \in U\} \cup \{x \in X \mid \text{cl} B_i(x) \subset U\} = ((f_i)^{-1}(U \cap K_i) \cup \{x \in X \mid \text{cl} B_i(x) \subset U\}.
\]

\( U' \) is a closed set because \( K_i \) is closed, \( f_i \) is a continuous map on \( K_i \) and the set \( \{x \in X \mid \text{cl} B_i(x) \subset U\} \) is closed since \( \text{cl} B_i(x) \) is l.s.c. Then \( T_i \) is lower semicontinuous on \( X \) and has non-empty closed convex values.

By Theorem 2 (Wu’s fixed-point theorem) applied to the correspondences \( S_i = T_i \) and \( T_i : X \to 2^{D_i}, \) there exists \( \bar{x} \in D \) such that \( \bar{x}_i \in T_i(\bar{x}) \) for each \( i \in I. \) If \( \bar{x} \in W_i \) for some \( i \in I, \) then \( \bar{x}_i = f_i(\bar{x}), \) which is a contradiction.

Therefore, \( \bar{x} \notin W_i, \) hence \( (A_i \cap P_i)(\bar{x}) = \emptyset. \) Also, for each \( i \in I \) we have \( \bar{x}_i \in T_i(\bar{x}), \) and then \( \bar{x}_i \in \text{cl} B_i(\bar{x}). \) \( \square \)
Proof of Theorem 6. For each \( i \in I \), let \( \beta_i \) denote the family of all open convex neighborhoods of zero in \( E_i \). Let \( V = (V_i)_{i \in I} \subseteq \prod_{i \in I} \beta_i \). Since \( \text{cl} \, B_i \) has the e-WCGS property on \( K_i \), there exists a WCG correspondence \( F^V_i : X \to 2^{X_i} \) such that \( F^V_i(x) \subseteq \text{cl} \, B_i(x) + V_i \) and \( x_i \notin F^V_i(x) \) for each \( x \in K_i \).

\( K_i \) is an \((n_i - 1)\)-dimensional simplex. Then, by the selection theorem, there exists a continuous function \( f^V_i : K_i \to X_i \) such that \( f^V_i(x) \in F^V_i(x) \) for each \( x \in K_i \). Because \( x_i \notin F^V_i(x) \) for each \( x \in K_i \), we have \( x_i \notin f^V_i(x) \) for each \( x \in K_i \).

Define the correspondence \( T^V_i : X \to 2^{X_i} \) by
\[
T^V_i(x) := \begin{cases} \{f^V_i(x)\} & \text{if } x \in \text{int}_X K_i, \\ \text{cl}(B_i(x) + V_i) \cap X_i & \text{if } x \in X \setminus \text{int}_X K_i. \end{cases}
\]

Here, \( B_V : X \to 2^{X_i} \), \( B_V(x) = \text{cl}(B_i(x) + V_i) \cap X_i = (\text{cl} \, B_i(x) + \text{cl} \, V_i) \cap X_i \) is upper semicontinuous by Lemma 2.

Let \( U \) be an open subset of \( X_i \). Then
\[
U' := \{ x \in X \mid T^V_i(x) \subseteq U \}
\]
\[
= \{ x \in \text{int}_X K_i \mid T^V_i(x) \subseteq U \} \cup \{ x \in X \setminus \text{int}_X K_i \mid T^V_i(x) \subseteq U \}
\]
\[
= \{ x \in \text{int}_X K_i \mid f^V_i(x, y) \in U \} \cup \{ x \in X \mid (\text{cl} \, B_i(x) + V_i) \cap X_i \subseteq U \}
\]
\[
= ((f^V_i)^{-1}(U) \cap \text{int}_X K_i) \cup \{ x \in X \mid (\text{cl} \, B_i(x) + V_i) \cap X_i \subseteq U \}.
\]

\( U' \) is an open set because \( \text{int}_X K_i \) is open, \( f^V_i \) is a continuous map on \( K_i \) and the set \( \{ x \in X \mid (\text{cl} \, B_i(x) + \text{cl} \, V_i) \cap X_i \subseteq U \} \) is open since \( (\text{cl} \, B_i(x) + \text{cl} \, V_i) \cap X_i \) is u.s.c. Then \( T^V_i \) is upper semicontinuous on \( X \) and has closed convex values.

Define \( T^V : X \to 2^X \) by \( T^V(x) := \prod_{i \in I} T^V_i(x) \) for each \( x \in X \); \( T^V \) is an upper semicontinuous correspondence and has also non-empty convex closed values. Since \( X \) is a compact convex set, by Fan's fixed-point theorem [8], there exists \( \varphi_V \in X \) such that \( \varphi_V \in T^V(\varphi_V) \), i.e., \( (\varphi_V)_i \in T^V_i(\varphi_V) \) for each \( i \in I \). If \( \varphi_V \in \text{int}_X K_i \), \( (\varphi_V)_i = f^V_i(\varphi_V) \), which is a contradiction. Hence \( (\varphi_V)_i \in (\text{cl} \, B_i(\varphi_V) + V_i) \cap X_i \) and \( (A_i \cap P_i)(\varphi_V) = \emptyset \), i.e., \( \varphi_V \in Q_V \) where
\[
Q_V = \bigcap_{i \in I} \{ x \in X : x_i \in \text{cl}(B_i(x) + V_i) \cap X_i \text{ and } (A_i \cap P_i)(x) = \emptyset \}.
\]

Since \( W_i \) is open, \( Q_V \) is the intersection of non-empty closed sets, so it is non-empty, closed in \( X \).

We now prove that the family \( \{ Q_V : V \in \bigcap_{i \in I} \beta_i \} \) has the finite intersection property. Let \( \{ V^{(1)}, V^{(2)}, \ldots, V^{(n)} \} \) be any finite set of \( \prod_{i \in I} \beta_i \) and let
\[ V^{(k)} = (V_i^{(k)})_{i \in I}, \quad k = 1, \ldots, n. \] For each \( i \in I \), let \( V_i = \bigcap_{k=1}^{n} V_i^{(k)} \). Then \( V_i \in \beta_i \), thus \( V = (V_i)_{i \in I} \in \prod_{i \in I} \beta_i \). Clearly, \( Q_V \subset \bigcap_{k=1}^{n} Q_{V^{(k)}} \) so that \( \bigcap_{k=1}^{n} Q_{V^{(k)}} \neq \emptyset \). Since \( X \) is compact and the family \( \{Q_V : V \in \prod_{i \in I} \beta_i\} \) has the finite intersection property, we have \( \bigcap \{Q_V : V \in \prod_{i \in I} \beta_i\} \neq \emptyset \). Take any \( \pi \in \bigcap \{Q_V : V \in \prod_{i \in I} \beta_i\} \).

Then \( \pi_i \in \text{cl}(B_i(\pi) + V_i) \cap X_i \) and \( (A_i \cap P_i)(\pi) = \emptyset \) for each \( i \in I \) and each \( V_i \in \beta_i \), but then \( \pi_i \in \text{cl}(B_i(\pi)) \) by Lemma 3 and \( (A_i \cap P_i)(\pi) = \emptyset \) for each \( i \in I \), so that \( \pi \) is an equilibrium point of \( \Gamma \) in \( X \). \( \square \)

Example 5. This is an example of an abstract economy with finite number of agents where Theorem 3 is applicable. Let \( \Gamma = (X_i, A_i, P_i, B_i)_{i \in I} \) be an abstract economy, where \( I = \{1, 2, \ldots, n\} \) is a finite set of agents such that \( X_i = [0, 4] \) for each \( i \in I \) is a compact convex choice set, \( X := \prod_{i \in I} X_i \) and the correspondences \( A_i, P_i, B_i : X \to 2^{X_i} \) are defined as

\[
A_i(x) = \begin{cases}
[2, 3 - \frac{1}{2}x_i] & \text{if } x \in M, \\
[0, \frac{1}{2}] \cup [2, 4] & \text{if } x \in X - M,
\end{cases}
\]

\[
P_i(x) = \begin{cases}
[2, 3] & \text{if } x \in M, \\
[0, \frac{1}{2}] \cup [2, 4] & \text{if } x = (1, 1, \ldots, 1), \\
\{2\} & \text{if } x \in W \setminus (M \cup \{(1, 1, \ldots, 1)\}), \\
\emptyset & \text{otherwise},
\end{cases}
\]

\[
B_i(x) = X_i \quad \text{for each } x \in X.
\]

Here, \( W \) is the simplex with vertices \((0, 0, \ldots, 0), (3, 0, \ldots, 0), (0, 3, 0, \ldots, 0), \) and \((0, 0, \ldots, 0, 3)\) in \( X \) and \( M = \prod_{i \in I} \{0, 1\} \). Then we have

\[
(A_i \cap P_i)(x) = \begin{cases}
[2, 3 - \frac{1}{2}x_i] & \text{if } x \in M, \\
[0, \frac{1}{2}] \cup [2, 4] & \text{if } x = (1, 1, \ldots, 1), \\
\{2\} & \text{if } x \in W \setminus (M \cup \{(1, 1, \ldots, 1)\}), \\
\emptyset & \text{otherwise}.
\end{cases}
\]

It is easy to show that \( A_i \cap P_i \) is not lower semi-continuous or upper semi-continuously on \( X \), has not convex values but, since \( 2 \in \bigcap_{x \in X} S(x), A_i \cap P_i \) is a WCG correspondence on \( W \). Note that \( x_i \notin (A_i \cap P_i)(x) \) for each \( x \in X \). Clearly, \( \text{cl} B_i \) is lower semicontinuous, has non-empty convex values and
$A_i(x) \subset B_i(x)$ for each $x \in X$. Then the assumptions of Theorem 3 are satisfied, so we can obtain an equilibrium point $\pi = (\frac{7}{2}, \frac{7}{2}, \ldots, \frac{7}{2}) \in X$ for $\Gamma$ such that, $\pi \in \text{cl} B_i(\pi)$ and $A_i(\pi) \cap P_i(\pi) = \emptyset$ for each $i \in I$.

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