Dedicated to Dr. Constantin VÂRSAN
on the occasion of his 70th birthday

CONTROL PROBLEMS WITH MIXED CONSTRAINTS AND APPLICATION TO AN OPTIMAL INVESTMENT PROBLEM

J.F. BONNANS and D. TIBA

We discuss two optimal control problems of parabolic equations, with mixed state and control constraints, for which the standard qualification condition does not hold. Our first example is a bottleneck problem, and the second one is an optimal investment problem where a utility type function is to be minimized. By an adapted penalization technique, we derive optimality conditions from which useful information on the solution can be derived. In the case of a control entering linearly in the state equation and cost function, we obtain generalized bang-bang properties.

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1. INTRODUCTION

Optimal control problems involving state constraints (and, in particular, mixed constraints) are well known for their intrinsic difficulty. There is a rich literature devoted to the optimality conditions and the regularity of the Lagrange multipliers for the case of parabolic control problems with mixed constraints: Arada and Raymond [1], Tröltzsch [17], De los Reyes and Tröltzsch [8], Rösch and Tröltzsch [15].

One special case very much discussed in the literature is the so-called "bottleneck problem", introduced by Bellman [3], further studied by Mirică [12], Bergounioux and Tiba [4], Bergounioux and Tröltzsch [5, 6]. We study in Section 2 a variant of the bottleneck problem. We fix a "polynomial" cost functional and a linear parabolic state system, and we investigate the situation when the state is "dominated" by the control.

Section 3 is devoted to an optimal investment problem that in some sense is the opposite case of the bottleneck problem. Given a distribution of capital over space, we assume that one cannot invest more than a fraction of the capital, and that some diffusion of the capital occurs. We study the case of a small aversion to risk.

Our approach is based on the adapted penalization of the state equation, while the constraints are kept explicit. Elements of our technique have been previously used by Lions [11], Barbu and Precupanu [2], Bergounioux and Tiba [4]. The form of the optimality conditions that we obtain has the advantage of a certain symmetry: the control and the state play a similar role which is a natural characteristic for mixed constraints. They are also accessible for further analysis in order to obtain supplementary information like bang-bang or regularity properties for the unknowns, which is the main aim of this paper.

2. CONTROL DOMINATED PROBLEMS

We analyze the model problem

$$\text{Min } \left\{ \frac{1}{2} \int_Q (y - y_d)^2 \mathrm{d}x\mathrm{d}t + \frac{N}{q} \int_{Q_\omega} |Bu|^q \, \mathrm{d}x\mathrm{d}t \right\}, \tag{2.1}$$

$$y_t - \Delta y = Bu \text{ in } Q = \Omega \times]0, T[, \tag{2.2}$$

$$y(x, 0) = 0 \text{ in } \Omega, \quad y(x, t) = 0 \text{ on } \Sigma = \partial\Omega \times [0, T], \tag{2.3}$$

$$[y, u] \in D \subset C(0, T; H_0^1(\Omega)) \times U. \tag{2.4}$$

Above, U is the reflexive Banach space of controls, Ω is a bounded smooth domain in $\mathbf{R}^d$, $d \geq 1$, ω is a measurable subset of Ω, and setting $Q_\omega = \omega \times]0, T[$, B is a linear continuous operator : $U \to L^q(Q_\omega)$, with extension by 0 to Q, D is a closed convex nonvoid subset of $C(0, T; H_0^1(\Omega)) \times U$, $q \geq 2$, $N \geq 0$, and $y_d \in L^2(Q)$ are given.

The cost functional is a direct generalization of the standard quadratic functional. More complex situations instead of (2.1)–(2.3) may be considered as well. Here, we concentrate on the treatment of the mixed constraint (2.4), which is formulated in a very general way.

Notice that the unique solution y of (2.2)–(2.3) belongs to $W^{2,1,q}(Q)$. If $q > \frac{1}{2}(n + 2)$ then $y \in C(\overline{Q})$ by the Sobolev embedding theorem.

If the set of admissible pairs $[y, u] \in D$ and satisfying (2.2)–(2.3) is nonvoid and if $N > 0$, then it is well known that problem (2.1)–(2.4) has a unique optimal pair $[y^*, u^*] \in W^{2,1,q}(Q) \times U$, by the coercivity and strict convexity of the cost functional, see, e.g., Neittaanmäki and Tiba [13]. Since U is a reflexive space, existence may be obtained as well for $N = 0$ if the set of controls satisfying (2.4) is bounded in U. In the sequel, we assume that

$$\text{problem (2.1)–(2.4) has at least one optimal pair } [y^*, u^*] \in D. \tag{2.5}$$

Examples for (2.4) that we have in mind are

$$\frac{1}{2}\int_{\Omega} y(x,t)^2 \mathrm{d}x \le C(u)(t), \quad t\in[0,T], \tag{2.6}$$

$$u\in U_{ad}, \text{ with } U_{ad} \text{ closed convex subset of } U. \tag{2.7}$$

Here, $C(\cdot)$ is a given operator: $U\to L^1(0,T)$. For instance, if $C:U\to R$ is a positive constant, then (2.6)–(2.7) is a standard example of separate state and control constraints. If $U=L^q(0,T)$ and $B:U\to L^q(Q_\omega)$, $(Bu)(x,t)=f(x)u(t)$, $f\in L^\infty(\omega)$ and $C(u)(t)=u(t)$, $t\in[0,T]$, we obtain a variant of the bottleneck problem. Inequality (2.6) justifies the title of this section. In (2.7), with the above notation, one may take

$$U_{ad}=\{u\in L^q(0,T);\ a(t)\le u(t)\le b(t) \text{ a.e. in } [0,T]\}$$

with a and b in $L^\infty(0,T)$. In this case, (2.5) holds even when $N=0$. The adapted penalization method applied to problem (2.1)–(2.4) is based on the approximation,

$$\operatorname*{Min}_{[y,u]}\left\{\frac{1}{2}\int_Q (y-y_d)^2\mathrm{d}x\mathrm{d}t+\frac{N}{q}\int_{Q_\omega}|Bu|^q\mathrm{d}x\mathrm{d}t+|u-u^*|_U^2 + \frac{1}{q\varepsilon}\int_Q |y_t-\Delta y-Bu|^q\mathrm{d}x\mathrm{d}t\right\},\quad \varepsilon>0, \tag{2.8}$$

subject to

$$y\in W^{2,1,q}(Q),\ y(x,0)=0 \text{ in } \Omega,\ y(x,t)=0 \text{ in } \Sigma,\ [y,u]\in D. \tag{2.9}$$

Due to the presence of the adapted term $|u-u^*|_U^2$, the minimization problem (2.8)–(2.9) has a unique minimal pair $[y_\varepsilon,u_\varepsilon]$. Moreover, since the $[y^*,u^*]$ satisfies (2.9), (2.2) and is feasible for (2.8), we have the inequality

$$\frac{1}{2}\int_Q (y_\varepsilon-y_d)^2\mathrm{d}x\mathrm{d}t+\frac{N}{q}\int_{Q_\omega}|Bu_\varepsilon|^q\,\mathrm{d}x\mathrm{d}t+|u_\varepsilon-u^*|_U^2 + \frac{1}{q\varepsilon}\int_Q |(y_\varepsilon)_t-\Delta y_\varepsilon-Bu_\varepsilon|^q\,\mathrm{d}x\mathrm{d}t \le \frac{1}{2}\int_Q (y^*-y_d)^2\mathrm{d}x\mathrm{d}t+\frac{N}{q}\int_{Q_\omega}|Bu^*|^q\mathrm{d}x\mathrm{d}t. \tag{2.10}$$

Therefore, $[y_\varepsilon,u_\varepsilon]$ is bounded in $W^{2,1,q}(Q)\times U$ and

$$(y_\varepsilon)_t-\Delta y_\varepsilon-Bu_\varepsilon\to 0 \text{ strongly in } L^q(Q). \tag{2.11}$$

Denote $r_\varepsilon=\varepsilon^{-1}\left[(y_\varepsilon)_t-\Delta y_\varepsilon-Bu_\varepsilon\right]\in L^q(Q)$. By (2.10), $\varepsilon^{(q-1)/q}r_\varepsilon$ is bounded in $L^q(Q)$. We may assume that (for some subsequence) $u_\varepsilon\to\widehat{u}$ weakly in U, we have $y_\varepsilon\to\widehat{y}$ weakly in $W^{2,1,q}(Q)$, and we get that $[\widehat{y},\widehat{u}]$ satisfies (2.9) since D is weakly closed. Letting $\varepsilon\to 0$ in (2.11), we obtain

$$\widehat{y}_t-\Delta\widehat{y}-B\widehat{u}=0 \quad \text{in Q},$$

i.e., the pair $[\widehat{y}, \widehat{u}]$ is feasible for problem (2.1)–(2.4). By (2.10) and the weak lower semicontinuity of the norm, we have

$$\frac{1}{2}\int_Q (\widehat{y} - y_d)^2 \mathrm{d}x\mathrm{d}t + \frac{N}{q}\int_{Q_\omega} |B\widehat{u}|^q \,\mathrm{d}x\mathrm{d}t + |\widehat{u} - u^*|_U^2 \leq$$
$$\leq \frac{1}{2}\int_Q (y^* - y_d)^2 \mathrm{d}x\mathrm{d}t + \frac{N}{q}\int_{Q_\omega} |Bu^*|^q \,\mathrm{d}x\mathrm{d}t.$$

Therefore, $[\widehat{y}, \widehat{u}]$ is optimal for problem (2.1)–(2.4) and $\widehat{u} = u^*$. Clearly, weak convergences are in fact strong since $|u_\varepsilon - u^*|_U \to 0$. We have thus proved

Proposition 2.1. *The assertions below hold.*

$$u_\varepsilon \to u^* \text{ strongly in } U, \tag{2.12}$$

$$y_\varepsilon \to y^* \text{ strongly in } W^{2,1,q}(Q), \tag{2.13}$$

$$\left\{\varepsilon^{\frac{q-1}{q}} r_\varepsilon\right\} \text{ is bounded in } L^q(Q). \tag{2.14}$$

For a given pair $[y, u]$ satisfying (2.9), let us consider convex variations denoted $[y_s, u_s]$, with $y_s = y_\varepsilon + s(y - y_\varepsilon)$, $u_s = u_\varepsilon + s(u - u_\varepsilon)$, for s in $[0, 1]$. Obviously, $[y_s, u_s]$ satisfies (2.9) and we can write the inequality

$$\begin{aligned}(2.15)\qquad &\frac{1}{2}\int_Q (y_\varepsilon - y_d)^2 \mathrm{d}x\mathrm{d}t + \frac{N}{q}\int_{Q_\omega} |Bu_\varepsilon|^q \,\mathrm{d}x\mathrm{d}t + |u_\varepsilon - u^*|_U^2 + \\ &+\frac{1}{q\varepsilon}\int_Q |(y_\varepsilon)_t - \Delta y_\varepsilon - Bu_\varepsilon|^q \,\mathrm{d}x\mathrm{d}t \leq \frac{1}{2}\int_Q (y_\varepsilon + s(y - y_\varepsilon) - y_d)^2 \,\mathrm{d}x\mathrm{d}t + \\ &+\frac{N}{q}\int_{Q_\omega} |Bu_\varepsilon + s(Bu - Bu_\varepsilon)|^q \,\mathrm{d}x\mathrm{d}t + |u_\varepsilon + s(u - u_\varepsilon) - u^*|_U^2 + \\ &+\frac{1}{q\varepsilon}\int_Q |(y_\varepsilon)_t + s(y - y_\varepsilon)_t - \Delta y_\varepsilon - s\Delta(y - y_\varepsilon) - Bu_\varepsilon - sB(u - u_\varepsilon)|^q \,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Let us denote by $\mathrm{sgn}(\cdot)$ the sign function, U^* the topological dual of U, and $F : U \to U^*$ the duality mapping. Standard computations in (2.15) allow us to obtain

Proposition 2.2. *The pair $[y_\varepsilon, u_\varepsilon]$ satisfies the following necessary and sufficient first order optimality condition: for any $[y, u]$ for which (2.9) holds, we have*

(2.16)

$$\begin{aligned}0 \leq &\int_Q (y_\varepsilon - y_d)(y - y_\varepsilon)\mathrm{d}x\mathrm{d}t + N\int_{Q_\omega} |Bu_\varepsilon|^{q-1} \mathrm{sgn}(Bu_\varepsilon)(Bu - Bu_\varepsilon)\mathrm{d}x\mathrm{d}t + \\ &+ \langle F(u_\varepsilon - u^*), u - u_\varepsilon\rangle_{U^*\times U} + \int_Q \varepsilon^{q-2} |r_\varepsilon|^{q-1} \mathrm{sgn}(r_\varepsilon)(y_t - \Delta y - Bu)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Proof. As already mentioned, the necessity follows from (2.15), by dividing each side by $s > 0$ and letting $s \to 0$. The sufficiency of (2.16) is a consequence of the definition of the subdifferential since the right-hand side in (2.16) may be upper bounded by

$$\frac{1}{2}\int_Q (y - y_d)^2 \mathrm{d}x\mathrm{d}t - \frac{1}{2}\int_Q (y_\varepsilon - y_d)^2 \mathrm{d}x\mathrm{d}t + \frac{N}{q}\int_{Q_\omega} |Bu|^q \,\mathrm{d}x\mathrm{d}t -$$
$$-\frac{N}{q}\int_{Q_\omega} |Bu_\varepsilon|^q \,\mathrm{d}x\mathrm{d}t + |u - u^*|_U^2 - |u_\varepsilon - u^*|_U^2 +$$
$$+\frac{1}{\varepsilon q}\int_Q |y_t - \Delta y - Bu|^2 \,\mathrm{d}x\mathrm{d}t - \frac{1}{q\varepsilon}\int_Q |(y_\varepsilon)_t - \Delta y_\varepsilon - Bu_\varepsilon|^q \,\mathrm{d}x\mathrm{d}t,$$

for any $[y, u]$ satisfying (2.9). The conclusion follows. □

We now consider the main example of this section, the operator B having the form $(Bu)(x,t) = f(x)u(t)$, $u \in U = L^q(0,T)$. Then (2.6)–(2.7) become

$$\frac{1}{2}\int_\Omega y(x,t)^2 \mathrm{d}x\mathrm{d}t \leq u(t), \quad \text{for a.a. } t \in [0,T], \tag{2.17}$$

$$u \in L^q(0,T); \quad a(t) \leq u(t) \leq b(t) \quad \text{a.e. } [0,T], \tag{2.18}$$

and there exist constants $\alpha_a < 0$ and $\alpha_b > 0$ such that

$$a(t) \leq \alpha_a < 0 < \alpha_b \leq b(t), \quad \text{for a.a. } t \in (0,T). \tag{2.19}$$

Proposition 2.3. *If (2.17)–(2.19) hold then $\{\varepsilon^{q-2}|r_\varepsilon|^{q-1}\}$ is bounded in $L^{q/(q-1)}(Q)$ (or equivalently, $\{\varepsilon^{(q-2)/(q-1)} r_\varepsilon\}$ is bounded in $L^q(Q)$).*

Proof. For $\lambda > 0$, let y^λ be the unique element of $W^{2,1,q}(Q)$ satisfying (2.3) and

$$y_t^\lambda - \Delta y^\lambda = f(x)\lambda \quad \text{in } Q.$$

That is, y^λ is the solution of (2.2)–(2.3) associated with $u^\lambda \equiv \lambda$. When $\lambda > 0$ is small enough, say $\lambda < \lambda_0$, with $\lambda_0 > 0$, u^λ satisfies (2.18) and $\frac{\lambda}{2}\int_\Omega y^1(x,t)^2 \mathrm{d}x \leq \frac{1}{2}$. Consequently,

$$0 \leq \frac{1}{2}\int_\Omega y^\lambda(x,t)^2 \mathrm{d}x = \frac{1}{2}\lambda^2 \int_\Omega y^1(x,t)^2 \mathrm{d}x \leq \frac{1}{2}\lambda. \tag{2.20}$$

Given $\rho \in L^q(Q)$ with $|\rho|_{L^q(Q)} \leq 1$, define y_ρ as the solution of (2.3) and

$$(y_\rho)_t - \Delta y_\rho = \rho \quad \text{in } Q. \tag{2.21}$$

Then, we have $|y_\rho|_{C(\overline{Q})} \leq K$ (some positive constant) if $|\rho|_{L^q(Q)} \leq 1$. By (2.20)–(2.21), for any $\delta \in \mathbb{R}$ we have

$$\tag{2.22} \begin{aligned} \frac{1}{2}\int_\Omega (y^\lambda + \delta y_\rho)^2 \mathrm{d}x &= \frac{1}{2}\int_\Omega (y^\lambda)^2 \mathrm{d}x + \frac{1}{2}\delta^2 \int_\Omega y_\rho^2 \mathrm{d}x + \delta \int_\Omega y^\lambda y_\rho \mathrm{d}x \\ &\leq \frac{1}{2}\lambda + \frac{1}{2}\delta^2 K^2 \operatorname{mes}(\Omega) + |\delta|\, \lambda K \operatorname{mes}(\Omega) |y^1|_{C(\overline{Q})}. \end{aligned}$$

Given $\lambda \in (0, \lambda_0)$, for small enough $\delta > 0$, (2.22) shows that the pair $(y^\lambda + \delta y_\rho, \lambda)$ belongs to D defined in (2.17)–(2.18). Using this pair in (2.16), we get the inequality

$$\begin{aligned} 0 \leq &\int_Q (y_\varepsilon - y_d)(y^\lambda + \delta y_\rho - y_\varepsilon)\mathrm{d}x\mathrm{d}t + \\ &+ N \int_{Q_\omega} |Bu_\varepsilon|^{q-1} \operatorname{sgn}(Bu_\varepsilon)(B\lambda - Bu_\varepsilon)\mathrm{d}x\mathrm{d}t + \\ &+ \langle F(u_\varepsilon - u^*), \lambda - u_\varepsilon \rangle_{U^* \times U} + \delta \int_Q \varepsilon^{q-2} |r_\varepsilon|^{q-1} \operatorname{sgn} r_\varepsilon \rho(x,t) \mathrm{d}x\mathrm{d}t. \end{aligned}$$

Since all terms except the last remain uniformly bounded over $\varepsilon > 0$ (remember that here $\lambda > 0$ and $\delta > 0$ are fixed), the last integral is uniformly lower bounded. Since ρ is an arbitrary element of the closed unit ball, and the spaces $L^q(Q)$ and $L^{(q-1)/q}(Q)$ are dual to each other, the infimum of this integral over the unit ball is $-\varepsilon^{q-2}\|r_\varepsilon\|_{L^{(q-1)/q}(Q)}$. The conclusion follows. □

Theorem 2.4. *Assume* (2.17)–(2.19) *hold. Then the pair* $[y^*, u^*] \in D$ *is optimal for problem* (2.1)–(2.4) *iff there exists* $r^* \in L^{\frac{q}{q-1}}(Q)$ *such that*

$$\tag{2.23} \begin{cases} 0 \leq \int_Q (y^* - y_d)(y - y^*)\mathrm{d}x\mathrm{d}t + N \int_{Q_\omega} |Bu^*|^{q-1} \operatorname{sgn}(Bu^*) \\ (Bu - Bu^*)\mathrm{d}x\mathrm{d}t + \int_Q r^*(y_t - \Delta y - Bu)\mathrm{d}x\mathrm{d}t. \end{cases}$$

for any (y, u) *satisfying* (2.9).

Proof. By Proposition 2.3, there exists a sequence $\varepsilon_k \downarrow 0$ such that $\{\varepsilon^{q-2} |r_\varepsilon|^{q-1}\}$ converges weakly in $L^{q/(q-1)}(Q)$ to r^*. Since the spaces $L^{q/(q-1)}(Q)$ and $L^q(Q)$ are dual to each other, we may let $\varepsilon \to 0$ in (2.16), proving (2.23). Conversely, the sufficiency is obvious since, for admissible pairs $[y, u]$ satisfying (2.1)–(2.4), inequality (2.23) becomes

$$0 \leq \int_Q (y^* - y_d)(y - y^*)\mathrm{d}x\mathrm{d}t + N \int_{Q_\omega} |Bu^*|^{q-1} \operatorname{sgn}(Bu^*)(Bu - Bu^*)\mathrm{d}x\mathrm{d}t,$$

which immediately gives the optimality of $[y^*, u^*]$ by the definition of the subdifferential. □

Remark 2.5. Notice the regularity (integrability) property of the Lagrange multiplier r^*.

Remark 2.6. Using (2.2), relation (2.23) may be rewritten as

$$0 \le \int_Q (y^* - y_d)(y - y^*)\mathrm{d}x\mathrm{d}t + N \int_{Q_\omega} |Bu^*|^{q-1} \operatorname{sgn}(Bu^*)(Bu - Bu^*)\mathrm{d}x\mathrm{d}t$$
$$+ \int_Q r^*(y_t - \Delta y - Bu - y_t^* + \Delta y^* + Bu^*)\mathrm{d}x\mathrm{d}t.$$

When r^* is sufficiently smooth and $r^*(x,T) = 0$ in Ω, one can integrate by parts in the last integral. If I_D denotes the indicator function of the convex set D in $L^2(0,T;\ H_0^1(\Omega)) \times U$, then (2.23) may be rewritten as

$$\left[y_d + y^* + r_t^* - \Delta r^*,\ -B^*r^* - NB^*\left(|Bu^*|^{q-1}\operatorname{sgn}(Bu^*)\right)\right] \in \partial I_D(y^*, u^*)$$

(here, B^* is the adjoint of B).

We denote by $\partial_1 I_D(y^*, u^*)$, $\partial_2 I_D(y^*, u^*)$ the two components of $\partial I_D(y^*, u^*)$ that occur above and can write

$$r_t^* + \Delta r^* \in y^* - y_D + \partial_1 I_D(y^*, u^*),$$
$$-B^*r^* \in NB^*\left(|Bu^*|^{q-1}\operatorname{sgn}(Bu^*)\right) + \partial_2 I_D(y^*, u^*).$$

This is the usual form of the optimality system, see Barbu and Precupanu [2]. This formal interpretation may be made rigorous since r^* is the transposition solution of the above adjoint equation, see Lions and Magenes [9, 10].

We next discuss the case when $N = 0$. In this case one typically expects that (a representative of) the optimal control u^* is piecewise continuous, i.e., continuous except for finitely many times $(t_1, \ldots, t_q)$ whose union is denoted by $\mathcal{T}^d$. Reminding that $y^* \in W^{2,1,q}(Q) \subset C(\overline{Q})$ by the Sobolev embedding theorem, denote the *set of interior times* by

$$\mathcal{T} := \left\{ t \in [0,T] \backslash \mathcal{T}^d;\ \frac{1}{2}\int_\Omega (y^*(x,t))^2 \mathrm{d}x < u^*(t) \right\}.$$

Then $Q_o := \Omega \times \mathcal{T}$ is an open subset. Since u^* is continuous over $\mathcal{T}$, for any $d \in \mathcal{D}(Q)$ with compact support in Q_o, and for $\delta \in R$ small enough, by the Weierstrass theorem, the pair $[y^* + \delta d, u^*]$ satisfies (2.9). By Theorem 2.4 we have

$$0 = \int_Q r^*(d_t - \Delta d)\mathrm{d}x\mathrm{d}t + \int_Q d(y^* - y_D)\mathrm{d}x\mathrm{d}t$$

and, consequently,

$$\text{(2.24)} \qquad \begin{cases} r_t^* + \Delta r^* + j = y^* - y_D \text{ in } \mathcal{D}'(Q), \\ j \in \mathcal{D}'(Q) \text{ distribution with support in } \overline{Q} \backslash Q_o. \end{cases}$$

Note that

$$\overline{Q}\backslash Q_o = \left\{(x,t) \in Q;\ t \in \mathcal{T}^d \ \text{ or } \ \frac{1}{2}\int_\Omega (y^*(x,t))^2 \mathrm{d}x = u^*(t)\right\}.$$

Relation (2.24) is another well known form of the adjoint equation in the case when state or mixed constraints are present. Raymond and Arada [1], Rösch and Tröltzsch [15], De Los Reyes and Tröltzsch [8], studied the regularity properties of the multiplier j associated with the mixed constraint (2.4) under various interiority hypotheses.

PROPOSITION 2.7. *Assume that u^* is piecewise continuous, the functions a and b are continuous,* (2.17)–(2.19) *hold, $N = 0$, and $y_d \in L^\infty(Q)$. Then, for all $t \in \mathcal{T}$ we have*

$$\begin{cases} u^*(t) = a(t) \ \textit{in} \ \left\{t \in \mathcal{T};\ \displaystyle\int_\Omega f(x) r^*(x,t)\mathrm{d}x < 0\right\}, \\ u^*(t) = b(t) \ \textit{in} \ \left\{t \in \mathcal{T};\ \displaystyle\int_\Omega f(x) r^*(x,t)\mathrm{d}x > 0\right\}. \end{cases} \tag{2.25}$$

Proof. Let $t_o \in \mathcal{T}$ be a Lebesgue point of the function $t \to \int_\Omega f(x) r^*(x,t)\mathrm{d}x$ such that $\int_\Omega f(x) r^*(x,t_o)\mathrm{d}x < 0$. If $u^*(t_o) > a(t_o)$, since a and u^* are continuous at time t_o, for $\eta > 0$ small enough define

$$v^\eta(t) = \begin{cases} a(t_o) - u^*(t_o) & \text{if } |t - t_o| \le \eta, \\ 0 & \text{otherwise.} \end{cases}$$

For small enough η, the pair $[y^*, u^* + v^\eta]$ belongs to D. By Theorem 2.4, we have

$$0 \le (u^*(t_o) - a(t_o)) \int_{t_o-\eta}^{t_o+\eta} \left(\int_\Omega f(x) r^*(x,t)\mathrm{d}x\right) \mathrm{d}t. \tag{2.26}$$

Dividing (2.26) by η, since t_o is a Lebesgue point of $\int_\Omega f(x) r^*(x,t)\mathrm{d}x$, we deduce that $0 \le \int_\Omega f(x) r^*(x,t_o)\mathrm{d}x$, which is the desired contradiction. The second relation is proved in the same way. □

3. OPTIMAL INVESTMENT AND STATE DOMINATED PROBLEMS

In this section, we discuss a variant of (2.1)–(2.4) corresponding in some sense to the "converse" of example (2.6), namely,

$$\text{Min } \left\{\int_Q F(x,t,y(x,t))\mathrm{d}x\mathrm{d}t + \int_Q (u + N u^q)\mathrm{d}x\mathrm{d}t\right\}, \tag{3.1}$$

$$y_t - \Delta y + ay = u \text{ in } Q, \tag{3.2}$$

$$y(x,0) = y_o(x) \text{ in } \Omega, \quad y(x,t) = 0 \text{ on } \Sigma, \tag{3.3}$$

$$0 \le u(x,t) \le cy(x,t) \quad \text{a.e. in } Q. \tag{3.4}$$

Here, a and c are positive constants and $y_o \in W_o^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega)$, $y_o \ge 0$ a.e., in Ω, and $y_o \not\equiv 0$ in Ω. For each (x,t) the measurable function F is convex and of class C^1 w.r.t. y, and such that $F(x,t,y(x,t))$ and $F_y(x,t,y(x,t))$ belong to $L^q(Q)$ for each continuous function y. A standard example is

$$F(x,t,y) = \mu(x,t)\pi(y), \tag{3.5}$$

where $\mu(x,t) > 0$ is an actualization coefficient, possibly depending on time, and $\pi : \mathbb{R}^+ \to \mathbb{R}$ is a *desutility function* (convex nonincreasing), and in that case the cost function can be interpreted as a compromise between the utility of y and the effort in resources u. The economic interpretation is as follows: $y(x,t) \ge 0$ is the capital at place x and time t. One cannot invest more than a fraction of the capital at every $(x,t) \in Q$. In addition, there is a depreciation of the capital with constant rate a. Finally, the evolution of the capital also depends of what happens at neighbouring points, and this justifies the diffusion term. The cost function takes into account the preference for a certain type of evolution of the capital, and N can be viewed as a risk aversion coefficient (the preference for constant investment). Obviously, there is a lot of freedom in the definition of the cost function. On the other hand, the problem has severe restrictions. If $u \in L^q(Q)$ then $y \in W^{2,1,q}(Q)$ and $y \in C(\overline{Q})$ if $q > \frac{1}{2}(n+2)$, $y \ge 0$ in Q if $u \in L^q(Q)_+$. The maximal state is obtained when taking $u = cy$, i.e., is a solution of

$$\bar{y}_t - \Delta\bar{y} + (a-c)\bar{y} = 0 \quad \text{in } Q.$$

Therefore, if $c > a$, state decreases exponentially to zero, uniformly over the controls.

Remark 3.1. By the boundary conditions in (3.3), constraint (3.4) excludes the standard "interiority" (Slater) assumptions used in the literature on control problems with state constraints. The interior of the set of feasible controls is also void, even in the $L^\infty(Q)$ topology. From this point of view, constraint (3.4) is more difficult than (2.6).

PROPOSITION 3.2. *The optimal control problem* (3.1)–(3.4) *has an optimal pair* $[y^*, u^*] \in W^{2,1,q}(Q) \times L^\infty(Q)$.

Proof. The control $u_o \equiv 0$ in Q, together with the corresponding solution of (3.1)–(3.3), is feasible. For any feasible pair $[y,u]$ in $W^{2,1,q}(Q) \times L^q(Q)$, we have $y \le \bar{y}$, hence $0 \le u \le \bar{u} := c\bar{y}$. So, we have a uniform bound on y and u in $W^{2,1,s}(Q)$ and $L^\infty(Q)$, respectively. The usual passage to the limit in a minimizing sequence (using the fact that $L^\infty(Q)$ is the dual of the separable space $L^1(Q)$, and that a bounded sequence in the dual of a separable Banach space has a weakly $*$ converging subsequence, and that the cost is l.s.c. for the weak $*$ topology) allows to prove the existence of a solution to problem (3.1)–(3.4). $\square$

The approximating problem is

$$(3.6)\quad \text{Min}\left\{\int_Q F(x,t,y(x,t))\mathrm{d}x\mathrm{d}t+\int_Q (u+Nu^q)\mathrm{d}x\mathrm{d}t+\frac{1}{q}\int_Q |u-u^*|^q\,\mathrm{d}x\mathrm{d}t+ +\frac{1}{q\varepsilon}\int_Q |y_t-\Delta y+ay-u|^q\,\mathrm{d}x\mathrm{d}t\right\} \text{ for all } [y,u]\in W^{2,1,q}(Q)\times L^q(Q),$$

subject to (3.3)–(3.4).

This strongly convex problem has a unique solution $[y_\varepsilon, u_\varepsilon]$. Let $r_\varepsilon \in L^q(Q)$ be defined by

$$r_\varepsilon = \varepsilon^{-1}((y_\varepsilon)_t - \Delta y_\varepsilon + a y_\varepsilon - u_\varepsilon).$$

In the same way as in Section 2, we infer

PROPOSITION 3.3. *The minimization problem* (3.6) *has a unique optimal pair* $[y_\varepsilon, u_\varepsilon] \in W^{2,1,q}(Q)\times L^q(Q)$, $[y_\varepsilon, u_\varepsilon] \to (y^*, u^*)$ *strongly in* $W^{2,1,q}(Q)\times L^q(Q)$, *and* $\{\varepsilon^{\frac{q-1}{q}} r_\varepsilon\}$ *is bounded in* $L^q(Q)$.

Moreover, $(y_\varepsilon, u_\varepsilon)$ *is characterized as follows: for any* $(y,u)\in W^{2,1,q}(Q)\times L^q(Q)$ *satisfying* (3.3)–(3.4), *we have*

$$(3.7)\quad 0 \le \int_Q F_y(x,t,y_\varepsilon(x,t)(y-y_\varepsilon)\mathrm{d}x\mathrm{d}t + \int_Q (1+qNu_\varepsilon^{q-1})(u-u_\varepsilon)\mathrm{d}x\mathrm{d}t+ +\int_Q |u_\varepsilon-u^*|^{q-1}\operatorname{sgn}(u_\varepsilon-u^*)(u-u_\varepsilon)\mathrm{d}x\mathrm{d}t+ +\int_Q \varepsilon^{q-2}\,|r_\varepsilon|^{q-1}\operatorname{sgn}(r_\varepsilon)(y_t-\Delta y+ay-u)\mathrm{d}x\mathrm{d}t.$$

Denote by $y^{oo}\in W^{2,1,q}(Q)\subset C(\overline{Q})$ the solution of (3.2)–(3.3) corresponding to $u_o\equiv 0$ in Q. We assume that y_o is no-zero and that Ω is connected. It follows (see Protter and Weinberger [14]) that

$$(3.8)\qquad y^{oo}(x,t) > 0,\quad \forall (x,t)\in Q.$$

THEOREM 3.4. *Assuming* (3.8), *the pair* $(y^*, u^*)\in W^{2,1,q}(Q)\times L^q(Q)$ *is optimal for problem* (3.1)–(3.4) *iff there exists* $r^*\in M_{\text{loc}}(\overline{Q})$ *such that*

$$(3.9)\quad 0\le \int_Q F_y(x,t,y^*(x,t))(y-y^*)\mathrm{d}x\mathrm{d}t + \int_Q (1+qN(u^*)^{q-1})(u-u^*)\mathrm{d}x\mathrm{d}t +\int_Q r^*(y_t-\Delta y+ay-u)\mathrm{d}x\mathrm{d}t,$$

for any $(y,u) \in W^{2,1,q}(Q)\times L^q(Q)$ *for which* (3.3)–(3.4) *hold,* $y_t-\Delta y+ay-u \in L^\infty(Q)$*, and there is a compact* $\mathcal{K}=\mathcal{K}_{y,u}\subset Q$ *such that*

$$y_t-\Delta y+ay-u=0 \quad \text{a.e. in } Q\backslash\mathcal{K}.$$

Remark 3.5. Here, $r^*\in M_{\text{loc}}(\overline{Q})$ means that for any compact $\mathcal{K}\subset Q$, $r^*\in M(\mathcal{K})$, the dual of $L^\infty(\mathcal{K})$, i.e., $M_{\text{loc}}(\overline{Q})=\bigcap\{L^\infty(\mathcal{K})^*,\ \mathcal{K}\subset Q,\ \text{compact}\}\subset\mathcal{D}'(Q)$. Obviously, any admissible pair $[y,u]$ for (3.1)–(3.4) satisfies all the conditions on the test pairs in (3.9) since $y_t-\Delta y+ay-u=0$ a.e. in Q by (3.2).

Proof. We show that $\varepsilon^{q-2}|r_\varepsilon|^{q-1}$ is bounded in $L^1_{\text{loc}}(Q)$. Let $\mathcal{K}$ be a compact subset of Q, and let $\chi_\mathcal{K}$ denote its characteristic function. Take in (3.7) $\tilde u=\delta[\operatorname{sgn} r_\varepsilon]_+\chi_\mathcal{K}$ and the associated state denoted $\tilde y$, for small $\delta>0$. The Weierstrass theorem and (3.8) yield $y^{oo}|_\mathcal{K}\geq\alpha_\mathcal{K}>0$. Then the pair $(y^{oo},\tilde u)$ satisfies (3.3)–(3.4) and may be used in (3.7), if $\delta>0$ is small enough.

By Proposition 3.3 all terms except the last one in (3.7) are bounded independently of $\varepsilon>0$ and we get

$$\delta\int_\mathcal{K}\varepsilon^{q-2}|r_\varepsilon|^{q-1}\operatorname{sgn} r_\varepsilon\,[\operatorname{sgn} r_\varepsilon]_+\,\mathrm{d}x\mathrm{d}t\leq O(1)\quad\text{for all }\varepsilon>0. \tag{3.10}$$

Take now $\widehat{y}\in W^{2,1,q}(Q)$ to be the solution of (3.3) and

$$\widehat{y}_t-\Delta\widehat{y}+a\widehat{y}=\chi_\mathcal{K}\quad\text{in } Q \tag{3.11}$$

and $\widehat{u}\equiv 0$ in Q. Using the pair $[\widehat{y},\widehat{u}]$ in (3.7), we obtain

$$-\int_\mathcal{K}\varepsilon^{q-2}|r_\varepsilon|^{q-1}\operatorname{sgn} r_\varepsilon\mathrm{d}x\mathrm{d}t\leq O(1)\quad\text{for all }\varepsilon>0. \tag{3.12}$$

Multiplying (3.12) by $\delta>0$ and adding (3.10) twice to it yield

$$\delta\int_\mathcal{K}\varepsilon^{q-2}|r_\varepsilon|^{q-1}|\operatorname{sgn} r_\varepsilon|\,\mathrm{d}x\mathrm{d}t\leq O(1),\quad\forall\varepsilon>0,$$

where the $O(1)$ depend on $\mathcal{K}$. This proves that $\left\{\varepsilon^{q-2}|r_\varepsilon|^{q-1}\right\}$ is bounded in $L^1_{\text{loc}}(Q)$. Next, for any compact subset $\mathcal{K}$ of Q we may define $r^*|_{L^\infty(\mathcal{K})}\in M(\mathcal{K})$ as the weak limit of $\varepsilon^{q-2}|r_\varepsilon|^{q-1}$ restricted to $\mathcal{K}$. Clearly, if the compact $\widehat{\mathcal{K}}\subset Q$ is such that $\mathcal{K}\subset\widehat{\mathcal{K}}$, then the limit obtained extends the previous one as any element in $L^\infty(\mathcal{K})$ may be extended to $L^\infty(\widehat{\mathcal{K}})$ by 0. In this way, we obtain $r^*\in M_{\text{loc}}(\overline{Q})$.

One can pass to the limit in (3.7) on any test pair $[y,u]$ satisfying the hypotheses of this theorem. This ends the proof of the necessity of (3.9). The sufficiency follows as in the previous section. □

COROLLARY 3.6. *Assume that $N = 0$. Let Q_o be the interior of the set of points where u^* is continuous. Then*

$$\begin{cases} u^*(x,t) = cy^*(x,t) & \text{if } r^*(x,t) > 1, \text{ a.e. in } Q_o, \\ u^*(x,t) = 0 & \text{if } r^*(x,t) < 1, \text{ a.e. in } Q_o. \end{cases} \tag{3.13}$$

In addition, a.e. on Q_o, one of the following three statements hold:

$$\begin{cases} u^*(x,t) = cy^*(x,t) \text{ and } r^*(x,t) > 1, \\ u^*(x,t) = 0 \text{ and } r^*(x,t) < 1, \\ -a = F_y(x,t,y^*(x,t)). \end{cases} \tag{3.14}$$

Proof. Let $d \in \mathcal{D}(Q)$ have compact support in the open set

$$Q^* = \{(x,t) \in Q_o;\ 0 \le u^*(x,t) < cy^*(x,t)\}.$$

Then for λ close enough to 0, the pair $[y^* \pm \lambda d, u^*]$ may be taken in (3.9), and it follows by standard arguments that

$$r_t^* + \Delta r^* - ar^* + j = F_y(x,t,y^*(x,t)) \quad \text{in } \mathcal{D}'(Q), \tag{3.15}$$

where $j \in \mathcal{D}'(Q)$ is a distribution with support in $\overline{Q} \backslash Q^*$. Since $F_y(x,t,y^*)$ belongs to $L^q(Q)$, we have $r^* \in W^{2,1,q}_{\text{loc}}(Q^*) \subset C(Q^*)$. Take now $[y,u] \in D$ with $y = y^*$; since $N = 0$, (3.9) implies

$$0 \le \int_Q (r^* - 1)(u^* - u)\mathrm{d}x\mathrm{d}t. \tag{3.16}$$

Let $(x_o,t_o) \in Q_o$ be such that $u^*(x_o,t_o) < cy^*(x_o,t_o)$. Then we may take $u = u^* + v$, with v nonnegative with small support near (x_o,t_o), over which r^* is positive, and it follows from (3.16) that $\int_Q (r^* - 1)v \le 0$, which gives the desired contradiction in the first relation of (3.13). The second one can be proved in the same way.

We next prove (3.14). By [7, p. 195] and (3.15), we have $F_y(x,t,y^*(x,t)) = -a$ a.e. over $\{(x,t) \in Q_o;\ r^* = 1\}$. Combining with (3.13), we obtain (3.14). □

Remark 3.7. Note that relations (3.13) and (3.14) contain information of different nature, and that neither of them implies the other one.

The previous result shows that although the properties of r^* in Theorem 3.3 are very weak, (3.9) allows to obtain useful information on the optimal pair $[y^*, u^*]$.

Consider for instance the case when $\mu(x,t)$ has the constant value 1 and the desutility function is exponential, i.e., when $F(x,t,y) = \mathrm{e}^{-y}$. Then $F_y(x,t,y^*) = -a$ iff $y^*(x,t) = -\log a$. Since $y^*(x,t)$ is positive over Q, this never occurs if $a \ge 1$. If $a \in (0,1)$, since $y^* \in W^{2,1,q}_{\text{loc}}(Q_o)$, by the state equation and [7, p. 195] we obtain the additional information that $u^* = -a \log a$ a.e., on the set

$$\{(x,t) \in Q_o;\ y^*(x,t) = -\log a\}. \tag{3.17}$$

Therefore, by (3.14), when $N = 0$, we have

$$u^*(x,t) \in \{0, -a\log a, cy^*(x,t)\} \quad \text{a.e. on } Q_o. \tag{3.18}$$

The same property holds on Q if u^* is continuous a.e.

Remark 3.8. Results such as that in Corollary 3.6 are called generalized bang-bang properties, see Tröltzsch [17], Bergounioux and Tiba [4]. Note that the sets where the constraints or relation $F_y(x,t,y^*(x,t)) = 0$ are active in Q need not be disjoint. In Rösch and Tröltzsch [15], Hölder continuity properties are obtained for the optimal control in problems with mixed constraints, in a different setting.

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INRIA-Saclay, Centre de Mathématiques Appliquées,
École Polytechnique &
Laboratoire de Finance des Marchés d'Énergie, France
Frederic.Bonnans@inria.fr

Romanian Academy
"Simion Stoilow" Institute of Mathematics
P.O. Box 1-764, 014700 Bucharest, Romania
dan.tiba@imar.ro