HAMiLTON-JACOBI EQUATIONS WITh JUMPS:
ASYMPTOTIC STABILITY

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The asymptotic stability of a global solution satisfying Hamilton-Jacobi equations with jumps is analyzed in dependence on the strong dissipativity of the jump control function and using orbits of the differentiable flows to describe the corresponding characteristic system.

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Asymptotic behavior of Hamilton-Jacobi (H-J) equations with jumps will be concentrated on revealing the significance of dissipativity property acting in the Hamiltonian $H(x, u, p) = \langle p, g(x, u) \rangle + L(x, u)$ and the jump functions $h(x, u) \in \mathbb{R}^m$, $(x, u) \in \mathbb{R}^n \times [-1, 1]$. We analyze (H-J) equations of the form

$$\partial_t u + H(x, u, \partial_x u) = 0, \quad t \in [t_j, t_{j+1}], \quad x \in B(0, 1) \subset \mathbb{R}^n,$$

where the jumps

$$u(t_j, x) = u(t_{j-}, x) + \langle h(x, u(t_{j-}, x)), \Delta y(t_j) \rangle, \quad j = 0, 1, 2, \ldots$$

are defined by a sequence $\{t_j\}_{j \geq 0} \uparrow \infty$ and a piecewise constant process $\{y(t) = y(t_j) \in \mathbb{R}^m : t \in [t_j, t_{j+1}], j \geq 0, \ y(0) = 0\}$. The Cauchy method of characteristic systems allows to construct a global solution $\{\hat{x}(t, \lambda), \hat{u}(t, \lambda)\} \in \mathbb{R}^n \times [-1, 1] : t \in [t_j, t_{j+1}], \lambda \in \mathbb{R}^n, j \geq 0$ provided some weak dissipativity with respect to $u \in [-1, 1]$ of $L$ and $h$ is assumed. On the other hand, a global bounded solution for the H-J equation (1) with jumps is constructed combining $\hat{u}(t, \lambda) \in [-1, 1]$ with a diffeomorphism $\{\lambda = \psi(t, \cdot) \in \mathcal{C}^1(D \subset \mathbb{R}^n; \mathbb{R}^n) : t \geq 0\}$ which is piecewise smooth for $t \in [t_j, t_{j+1}], j \geq 0$. In doing this we need to impose additional conditions which are strictly related with asymptotic stability for both components of a solution $\{u(t, x) = \hat{u}(t; \psi(t, x)) : t \geq 0\}$ and

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\{p(t,x) = \partial_x u(t,x) : t \geq 0\}. Here, \(L(u) \in \mathbb{R}, h(u) \in \mathbb{R}^m\) and \(u \in [-1, 1]\) only depend on the unknown solution. When \(g(x,u) = \alpha(u)\hat{g}(x)\) and \(\hat{g} \in C^1(\mathbb{R}^n, \mathbb{R}^n)\) agrees with a nonlinear growth condition, the analysis of the asymptotic stability is given in Theorem 2.1, but the conclusion is only locally valid with respect to \(x \in D \subset \mathbb{R}^n\). In Section 1 we give some auxiliary results regarding the global existence of the characteristic system solution. They imply that both components \(\{\hat{u}(t,\lambda), \partial_\lambda \hat{u}(t,\lambda)\} \in \mathbb{R}^{n+1} : t \geq 0, \lambda \in \mathbb{R}^n\) are bounded. Also included are two results (Lemmas 1.5 and 1.7) where the construction of the smooth mapping \(\{\lambda = \psi(t,x) : t \geq 0, x \in D \subseteq \mathbb{R}^n\}\) is given. Usually, the construction of such a smooth mapping involves a backward integral equation combined with a contractive mapping theorem when the original equation \(\hat{\chi}(t;\lambda) = x\) can be rewritten as \(\hat{G}(\tau(t;\lambda))|\lambda| = x, \tau(t;\lambda) = \int_0^t \alpha(\hat{\psi}(s;\lambda)) \, ds\), where \(\{\hat{G}(\sigma|\lambda : \sigma \in \mathbb{R}, \lambda \in \mathbb{R}^n\}\) is the global flow generated by \(\hat{\psi} \in C^1(\mathbb{R}^n; \mathbb{R}_n)\). As far as \(\hat{x}(t;\lambda) = x\) and \(\psi(t,\hat{x}(t;\lambda)) = \lambda\) for any \(t \geq 0, x \in \mathbb{R}^n, \lambda \in \mathbb{R}^n\), we get a direct verification that \(\{u(t,x) \overset{\text{def}}{=} \hat{u}(t;\psi(t,x)) : t \geq 0, x \in \mathbb{R}^n\}\) is the solution of the H-J equation with jumps. The situation changes when the vector field \(\hat{\psi} \in C^1(\mathbb{R}^n, \mathbb{R}_n)\) is a nonlinear and unbounded one. It is analyzed in Theorem 2.1 pointing out that the contractive mapping theorem can be only locally applied (see \(x \in D = \text{int} B(x_+, \gamma) \subseteq \mathbb{R}^n\)). It implies that a nonstandard method must be used in order to get that \(\{u(t,x) \overset{\text{def}}{=} \hat{u}(t;\psi(t,x)) : t \geq 0, x \in D \subseteq \mathbb{R}^n\}\) is the solution of the H-J equation with jumps. The last theorem in Section 2 (see Theorem 2.2) tells us that in the case when \(g(x,u) = \sum_{k=1}^l \alpha_k(u)\hat{g}_k(x)\) is a summation of several vector fields of the type occurring in Theorem 2.1, we need to impose a commutativity hypothesis.

1. SOME AUXILIARY RESULTS

We are given an increasing sequence \{\(t_j\)\}_{j\geq0}, and a piecewise constant process

\(\{y(t) = y(t_0) \in \mathbb{R}^m : t \in [t_j, t_{j+1}), j \geq 0, y(0) = 0\}\)

for which \(\Delta y(t_j) = y(t_j) - y(t_{j-}), j \geq 1\), can be taken as a bounded value from \(\mathbb{R}_m^+\) and suitable for a fixed goal. Consider the H-J equation with jumps

\[
\begin{cases}
  \partial_t u(t,x) + H(x,u(t,x),\partial_x u(t,x)) = 0, & t \in [t_j,t_{j+1}), x \in \mathbb{R}^n, u \in \mathbb{R}, \\
  u(t_j, x) = u(t_{j-}, x) + (h(x,u(t_{j-}, x)), \Delta y(t_j)), & j \geq 0, \\
  u(0,x) = u_0(x), \ u_0 \in \hat{C}_b^1(\mathbb{R}^{n+1}),
\end{cases}
\]

where \(H(x,u,p) = \langle p, g(x,u) \rangle + L(x,u), \ g \in \hat{C}_b^1(\mathbb{R}^{n+1}; \mathbb{R}^n), \ L \in C^1(\mathbb{R}^{n+1})\) and \(h \in \hat{C}_b^1(\mathbb{R}^{n+1})\). By \(\hat{C}_b^1\) we denote the space consisting of all continuously
differentiable functions $f(x, u) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^k$, $k = n, 1, m$, for which the partial derivatives $\partial_i f(x, u) = \partial_{x_i} f(x, u)$ and $\partial_u f(x, u)$, $i \in \{1, \ldots, n\}$, are bounded. To construct a global bounded solution for (2), we need to convince ourselves that the global solution
\[ \{ (\hat{x}(t, \lambda), \hat{u}(t, \lambda)) \} \in \mathbb{R}^{n+1} : t \in [t_j, t_{j+1}], \lambda \in \mathbb{R}^{n+1}, j \geq 0 \]
eq \text{exists and satisfies the corresponding characteristic system}
\begin{align*}
\frac{d\hat{x}}{dt} &= g(\hat{x}, \hat{u}), \quad \hat{x}(0, \lambda) = \lambda \in \mathbb{R}^n, \quad t \geq 0, \\
\frac{d\hat{u}}{dt} &= -L(\hat{x}, \hat{u}), \quad t \in [t_j, t_{j+1}], \\
\hat{u}(t_j; \lambda) &= \hat{u}(t_j; \lambda) + \langle h(\hat{x}(t_j, \lambda), \hat{u}(t_j; \lambda)), \Delta y(t_j) \rangle, \quad j \geq 0, \\
\hat{u}(0; \lambda) &= u_0(\lambda), \quad u_0 \in C^1_b(\mathbb{R}^n).
\end{align*}

Without any dissipativity conditions on $L$ and $h$, the standard analysis assume $L \in C^1_b(\mathbb{R}^{n+1})$ and $g \in C^1_b(\mathbb{R}^{n+1}, \mathbb{R}^n)$, and yields a unique global solution
\[ \{ z(t, \lambda) = (\hat{x}(t, \lambda), \hat{u}(t, \lambda)) \} \in \mathbb{R}^{n+1} : t \geq 0, \lambda \in \mathbb{R}^n \]
of (3). On each interval $t \in [t_j, t_{j+1})$ we obtain a smooth function $\{ z(t; \lambda) \in \mathbb{R}^{n+1}, t \in [t_j, t_{j+1}], \lambda \in \mathbb{R}^n \}$ satisfying
\[ \frac{dz}{dt} = Z(z), \quad t \in [t_j, t_{j+1}), \quad z(t; \lambda) = z(t_j; \lambda) + b(z(t_j; \lambda), \Delta y(t_j)), \quad j \geq 0, \]
where $z(0; \lambda) = (\lambda, u_0(\lambda)) = z_0(\lambda) \in \mathbb{R}^{n+1}$ and
\[ Z(z) \overset{\text{def}}{=} \begin{pmatrix} g(\hat{x}, \hat{u}) \\ -L(\hat{x}, \hat{u}) \end{pmatrix} \in \mathbb{R}^{n+1}, \quad z = (\hat{x}, \hat{u}) \in \mathbb{R}^{n+1}, \]
\[ b(z, \nu) = \begin{pmatrix} 0 \\ \langle h(z), \nu \rangle \end{pmatrix}, \quad \text{for } \nu \in \mathbb{R}^n. \]

Using the piecewise smooth solution $z(t, \lambda) = (\hat{x}(t, \lambda), \hat{u}(t, \lambda)), t \in [t_j, t_{j+1}), \lambda \in \mathbb{R}^n, j \geq 0$, of (3), we get the solution of the H-J equation (2) as a composition
\begin{align*}
(4) \quad u(t, x) &\overset{\text{def}}{=} \hat{u}(t, \psi(t, x)), \quad t \geq 0, \quad x \in \mathbb{R}^n, \\
\intertext{where the smooth mapping $\{ \lambda = \psi(t, x) \in \mathbb{R}^n : t \geq 0, x \in \mathbb{R}^n \}$ is found as a unique solution fulfilling $\hat{x}(t; \lambda) = x \in \mathbb{R}^n, t \geq 0$, and}
(5) \quad \psi(0, x) = x, \quad \hat{x}(t, \psi(t, x)) = x, \quad \forall t \geq 0.
\end{align*}
As far as $\{ \lambda = \psi(t, x) \in \mathbb{R}^n : t \geq 0, x \in \mathbb{R}^n \}$ is the unique solution of equation (5), we notice that
\begin{align*}
(6) \quad \psi(t, \hat{x}(t; \lambda)) = \lambda, \quad \lambda \in \mathbb{R}^n, t \geq 0,
\end{align*}
and this will be the main ingredient supporting the idea that \( \{u(t, x) : t \in [t_j, t_{j+1}), x \in \mathbb{R}^n, j \geq 0\} \) defined in (4) is the solution of the H-J equation (2). The construction of the mapping \( \lambda = \psi(t, x) \in \mathbb{R}^n : t \geq 0, x \in \mathbb{R}^n \) is given using a bounded solution \( \{\tilde{u}(t; \lambda) : t \geq 0, \lambda \in \mathbb{R}^n\} \) for which the gradient \( \partial \lambda \tilde{u}(t, x) : t \geq 0, \lambda \in \mathbb{R}^n \) is a bounded mapping. In this respect we recall that \( \{\tilde{u}(t, x) \in \mathbb{R} : y \in [t_j, t_{j+1}), j \geq 0, \lambda \in \mathbb{R}^n\} \) is a piecewise smooth scalar function and \( \{\tilde{x}(t, \lambda) \in \mathbb{R}^n : t \geq 0, \lambda \in \mathbb{R}^n\} \) is a smooth mapping satisfying

\[
\hat{x}(t, \lambda) = \tilde{x}(t, \lambda) + \int_{t_{j}}^{t} g(\tilde{x}(s, \lambda), \tilde{u}(s, \lambda))ds, \quad t \in [t_j, t_{j+1}), \ j \geq 1,
\]

\[
\hat{x}(t, \lambda) = \lambda + \int_{0}^{t} g(\tilde{x}(s, \lambda), \tilde{u}(s, \lambda))ds, \quad t \in [0, t_1],
\]

\[
(7) \begin{cases} 
\frac{d\tilde{u}}{dt} = -L(\hat{x}(t; \lambda), \tilde{u}), \ t \in [t_j, t_{j+1}), \\
\tilde{u}(t_j; \lambda) = \tilde{u}(t_{j-1}; \lambda) + (h(\tilde{x}(t_j; \lambda), \tilde{u}(t_{j-1}; \lambda)), \Delta y(t_j)), \ j \geq 1, \\
\tilde{u}(0; \lambda) = u_0(\lambda), \ u_0 \in C^1_b(\mathbb{R}^n), \ \sup \|u_0(x)\| = k_0 < 1.
\end{cases}
\]

The weak dissipativity conditions

\[
(8) \begin{cases} 
L(x, 0) = 0, \ \partial_u L(x, u) \geq 0, \ (x, u) \in \mathbb{R}^n \times [-1, 1], \\
h_i(x, 0) = 0, \ \partial_u h_i(x, u) \leq 0, \ i \in \{1, \ldots, m\}, \ (x, u) \in \mathbb{R}^n \times [-1, 1],
\end{cases}
\]

will be assumed.

**Lemma 1.1.** Consider \( g \in C^1_b(\mathbb{R}^{n+1}; \mathbb{R}^n) \) and let \( L \in C^1(\mathbb{R}^{n+1}) \) and \( h \in C^1_b(\mathbb{R}^n \times [-1, 1]; \mathbb{R}^m) \) satisfy condition (8). Let \( \Delta y(t_j) \in \mathbb{R}^n_m \) be such that

\[-1 \leq \langle \partial_u h(x, u), \Delta y(t_j) \rangle \leq 0, \quad (x, u) \in \mathbb{R}^n \times [-1, 1], \ j \geq 1.
\]

Then the unique global solution \( \{\tilde{x}(t; \lambda), \tilde{u}(t; \lambda) : t \geq 0, \lambda \in \mathbb{R}^n\} \) of the characteristic system (7) satisfies

\[
\begin{cases} 
|\tilde{u}(t_j; \lambda)| \leq |\tilde{u}(t_{j-1}; \lambda)|, \ |\tilde{u}(t; \lambda)| \leq |\tilde{u}(t_{j-1}; \lambda)|, \ t \in [t_{j-1}, t_j), \ j \geq 1, \\
|\tilde{u}(t; \lambda)| \leq |u_0(\lambda)| \leq 1, \ \forall t \geq 0, \ \lambda \in \mathbb{R}^n.
\end{cases}
\]

**Remark 1.2.** Taking \( u_0 \in C^1_b(\mathbb{R}^n) \) such that \( \sup \|u_0(x)\| = K_0 < 1 \), under the hypothesis of Lemma 1.1, we get a solution \( \{\tilde{u}(t; \lambda) : t \geq 0, \lambda \in \mathbb{R}^n\} \) of (7) fulfilling \( |\tilde{u}(t; \lambda)| \leq 1, \ t \geq 0, \lambda \in \mathbb{R}^n \). Now, we are interested to obtain a bounded solution for which \( \{\partial \lambda \tilde{u}(t, \lambda) : t \geq 0, \lambda \in \mathbb{R}^n\} \) is bounded and
\[|\partial_x u(t; \lambda)| \leq |\partial_\lambda u_0(\lambda)|, \ t \geq 0, \ \lambda \in \mathbb{R}^n.\]

In this respect, we assume that

\[
\begin{align*}
\text{(a)} \ L \in C^1(\mathbb{R}), \ L(0) = 0, \ \partial_\gamma L(u) \geq 0, \ u \in [-1, 1]; \\
\text{(b)} \ h_i \in C^1([-1, 1]), \ h_i(0) = 0, \ \partial_u h_i(u) \leq 0, \ u \in [-1, 1], \ i \in \{1, \ldots, m\}; \\
\text{(c)} \ g \in \tilde{C}^1_b(\mathbb{R}^n \times [-1, 1]; \mathbb{R}^n) \text{ and } u_0 \in C^1_b(\mathbb{R}^n) \text{ satisfies}
\end{align*}
\]

\[\sup |u_0(x)| = K_0 < 1.\]

**Lemma 1.3.** Consider \(H(x, u, p) = \langle p, g(x, u) \rangle + L(u)\), where \(g\) and \(L\) fulfil (9). Let \(h(u) \in \mathbb{R}^m\) and \(u_0 \in C^1_b(\mathbb{R})\) be such that (9) are satisfied. Take \(\Delta y(t_j) \in \mathbb{R}^m_+\) such that

\[0 \geq \langle \partial_u h(u), \Delta y(t_j) \rangle \geq -1, \quad u \in [-1, 1], \ j \geq 1,
\]

and consider the global solution \(\{\hat{u}(t; \lambda) : t \geq 0, \ \lambda \in \mathbb{R}^n\}\)

satisfying (3). Then,

\[
\begin{cases}
|\hat{u}(t; \lambda)| \leq |u_0(\lambda)| \leq 1, & t \geq 0, \ \lambda \in \mathbb{R}^n, \\
|\partial_\lambda \hat{u}(t; \lambda)| \leq |\partial_\lambda u_0(\lambda)|, & t \geq 0, \ \lambda \in \mathbb{R}^n.
\end{cases}
\]

**Remark 1.4.** Lemmas 1.2 and 1.3 help us to conclude that looking for a bounded global solution \(\{u(t;x) : t \geq 0, x \in \mathbb{R}^n\}\) for which the gradient \(p(t,x) = \partial_x u(t;x) : t \geq 0, x \in \mathbb{R}^n\) also is a bounded function, we need to assume some weak dissipativity condition for both \(L \in C^1(\mathbb{R})\) and \(h \in C^1([-1, 1]; \mathbb{R}^m)\). Unfortunately, the same dissipativity conditions are not sufficient for obtaining the smooth mapping \(\{\lambda = \psi(t,x) \in \mathbb{R}^n : t \geq 0, x \in \mathbb{R}^n\}\) satisfying equations (5) and (6). We shall present two types of necessary conditions which lead us to the smooth global mapping \(\{\lambda = \psi(t,x) : t \geq 0, \ x \in \mathbb{R}^n\}\). For \(L\) and \(g\) we assume that

\[
\begin{align*}
\text{(a)} \ L \in C^1(\mathbb{R}), \ L(0) = 0, \ 0 < \gamma \leq \partial_\gamma L(u), \ u \in [-1, 1], \\
\text{(b)} \ g(x, u) = \alpha(u) \hat{g}(x), \ \alpha \in C^1([-1, 1]), \ \hat{g} \in \tilde{C}^1_b(\mathbb{R}^n, \mathbb{R}^m), \ \alpha(0) = 0.
\end{align*}
\]

Denote \(C_1 = \max\{|\partial_u \alpha(u)| : u \in [-1, 1]\}, \ C_2 = \sup\{||\hat{g}(x)|| : x \in \mathbb{R}^n\}\) and assume that \(u_0\) and \(h\) fulfil

\[
\begin{align*}
\text{(a)} \ u_0 \in C^1_b(\mathbb{R}^n), \ \sup |u_0(x)| = K_0 < 1, \ \sup |\partial_x u_0(x)| = K_1; \\
\text{(b)} \ \frac{1}{2} C_1 C_2 K_1 \leq \rho, \ \text{where } \rho \in (0, 1) \ \text{is fixed}; \\
\text{(c)} \ h_i \in C^1([-1, 1]), \ h_i(0) = 0, \ \partial_u h_i(u) \leq 0, \ u \in [-1, 1], \ i \in \{1, \ldots, m\}.
\end{align*}
\]

**Lemma 1.5.** Consider \(H(x, u, p) = \langle p, g(x, u) \rangle + L(u)\) and the corresponding characteristic system (3) when \(g, L, u_0\) and \(h_i, i \in \{1, \ldots, m\}\), fulfil (10) and (11). Take \(\Delta y(t_j) \in \mathbb{R}^m_+\) sufficiently small such that

\[0 \geq \langle \partial_u h(u), \Delta y(t_j) \rangle \geq -1, \quad u \in [-1, 1], \ j \geq 1.
\]
Then there exist a unique global solution \( \{( \mathcal{H}(t, \lambda), \hat{u}(t, \lambda) ) : t \geq 0, \lambda \in \mathbb{R}^n \} \) of (3) and a smooth mapping \( \{ \lambda = \psi(t, x) : t \geq 0, x \in \mathbb{R}^n \} \) such that

\[
\mathcal{H}(t; \psi(t, x)) = x, \quad \psi(t, \mathcal{H}(t, \lambda)) = \lambda, \quad \psi(0, x) = x, \quad t \geq 0, \quad x, \lambda \in \mathbb{R}^n,
\]

where \( |\hat{u}(t, \lambda)| \leq K_0 < 1 \) and \( |\partial_\lambda \hat{u}(t, \lambda)| \leq K_1, \quad t \geq 0, \quad \lambda \in \mathbb{R}^n \).

Remark 1.6. The unique global solution \( \{ \lambda = \psi(t, x) : t \geq 0, x \in \mathbb{R}^n \} \) given in Lemma 1.5 fulfills equation (12), \( \psi_i(t, \mathcal{H}(t, \lambda)) = \lambda_i, \quad t \geq 0, \quad i \in \{1, \ldots, n\} \), where \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \). By a direct computation we get

\[
\frac{d}{dt}[\psi_i(t, \mathcal{H}(t; \lambda))] = \partial_t \psi_i(t, \mathcal{H}(t; \lambda)) + (\partial_x \psi(t, \mathcal{H}(t; \lambda)), g(\mathcal{H}(t; \lambda), \hat{u}(t; \lambda))) = 0
\]

for any \( t \geq 0, \quad i \in \{1, \ldots, n\} \) and \( \lambda \in \mathbb{R}^n \). In addition, for \( \lambda = \psi(t, x) \) we obtain \( \mathcal{H}(t, \psi(t, x)) = x, \quad \hat{u}(t, \psi(t, x)) = u(t, x) \) and equation (13) becomes \( \psi_i(0, x) = x_i \) and

\[
\partial_t \psi_i(t, x) + (\partial_x \psi_i(t, x), g(x, u(t, x))) = 0, \quad \forall t \geq 0, \quad x \in \mathbb{R}^n, \quad i \in \{1, \ldots, n\}, \quad \text{where } g(x, u) = \alpha(u)\tilde{g}(x).
\]

We are going to analyze a second type of conditions which leads us to a smooth mapping \( \{ \lambda = \psi(t, x) : t \geq 0, x \in \mathbb{R}^n \} \) satisfying (14) and a solution \( u(t, x) = \mathcal{H}(t, \psi(t, x)), \quad t \geq 0, \quad x \in \mathbb{R}^n \), of H-J equation (2) which is only asymptotically stable. In this respect, suppose \( L \) and \( g \) fulfil

\[
\begin{align*}
(a) & L \in C^1(\mathbb{R}), \quad L(0) = 0, \quad \partial_u L(u) \geq 0, \quad u \in [-1, 1], \\
(b) & g(x, u) = \alpha(u)\tilde{g}(x), \quad \alpha \in C^1([-1, 1]), \quad \tilde{g} \in C_b^1(\mathbb{R}^n, \mathbb{R}^n), \quad \alpha(0) = 0.
\end{align*}
\]

Denote \( C_1 = \max\{|\partial_u \alpha(u)| : u \in [-1, 1]\}, \quad C_2 = \sup\{|\tilde{g}(x)| : x \in \mathbb{R}^n\} \) and take \( u_0 \) and \( h \) such that

\[
\begin{align*}
(a) u_0 \in C_b^1(\mathbb{R}^n), \quad \sup |u_0(x)| = K_0, \quad \sup |\partial_u u_0(x)| = K_1; \\
(b) h_i \in C^1([-1, 1]), \quad h_i(0) = 0, \quad \partial_u h_i(u) \leq 0, \quad i \in \{1, \ldots, m\}
\end{align*}
\]

and

\[
\sum_{i=1}^m \partial_u h_i(u) \leq -\delta < 0, \quad u \in [-1, 1];
\]

\[
(c) 2dC_1C_2K_1 \leq \rho, \quad \text{where } \rho \in (0, 1) \text{ is fixed and } d = \max_{j \geq 0}(t_{j+1} - t_j).
\]

Lemma 1.7. Consider \( H(x, u, p) = \langle p, g(x, u) \rangle + L(u), \quad \text{where } g \text{ and } L \)

fulfil (15). Let \( u_0 \) and \( h_i, \quad i \in \{1, \ldots, m\}, \) be such that conditions (16) are satisfied. Take \( \Delta y(t_j) \in \mathbb{R}^n_+ \) satisfying the inequalities

\[
-1 \leq \langle \partial_u h(u), \Delta y(t_j) \rangle \leq -\frac{1}{2}, \quad u \in [-1, 1], \quad j \geq 1.
\]

Then a smooth mapping \( \{ \lambda = \psi(t, x) : t \geq 0, x \in \mathbb{R}^n \} \) satisfying the equations

\[
\hat{\mathcal{H}}(t, \psi(t, x)) = x, \quad \psi(0, x) = x, \quad \psi(t, \hat{\mathcal{H}}(t, \lambda)) = \lambda, \quad t \geq 0, \quad x, \lambda \in \mathbb{R}^n,
\]

does exist where \( \{( \hat{\mathcal{H}}(t; \lambda), \hat{u}(t; \lambda)) : t \geq 0, \lambda \in \mathbb{R}^n \} \) is the global solution of (3) and \( |\hat{u}(t, \lambda)| \leq 1, \quad t \geq 0, \quad \lambda \in \mathbb{R}^n \).
2. MAIN RESULTS

The first theorem concerns the local asymptotic stability when the vector field \( g(x, u) = \alpha(u)\hat{g}(x) \), and \( \hat{g} \in C^1(\mathbb{R}^n; \mathbb{R}^n) \) agrees with a nonlinear growth condition. The second theorem given here analyzes the asymptotic stability property when the vector field \( g(x, u) = \alpha_1(u)\hat{g}_1(x) + \alpha_2(u)\hat{g}_2(x) \) contains two commuting vector fields \( \hat{g}_1, \hat{g}_2 \in C^1(\mathbb{R}^n; \mathbb{R}^n) \). Asymptotic stability of the H-J equation \( \partial_t u + H(x, u, \partial_x u) = 0 \) is concerned. It will be the goal of the next two theorems. We are going to construct a local asymptotically stable solution for the H-J equation (2). In this respect, for some \( u_* \in \mathbb{R} \), \( x_* \in \mathbb{R}^n \) fixed we should assume

\[
\begin{align*}
(a) \quad & L \in C^1(\mathbb{R}), \quad L(u_*) = 0, \quad \partial_t L(u) \geq 0, \quad u \in [a, b]; \\
(b) \quad & g(x, u) = \alpha(u)\hat{g}(x), \quad \alpha \in C^1([a, b]), \quad \alpha(u_*) = 0; \\
(c) \quad & \hat{g} \in C^1(\mathbb{R}^n; \mathbb{R}^n), \quad \hat{g}(x_*) = 0,
\end{align*}
\]

where \( a = u_* - 1 \), \( b = u_* + 1 \). Denote

\[
c_1 = \max \{|\partial_u \alpha(u)| : u \in [a, b]\}, \quad c_2 = \max \{\hat{g}(x) : x \in B(x_*, \gamma)\},
\]

where \( \gamma > 0 \) is fixed. Take \( u_0 \) and \( h \) such that

\[
\begin{align*}
(a) \quad & u_0 \in C^1(\mathbb{R}^n), \quad \sup |u_0(x)| = K_0 < 1, \quad \sup |\partial_x u_0(x)| = K_1; \\
(b) \quad & h_i \in C^1([a, b]), \quad h_i(u_*) = 0, \quad \partial h_i(u) \leq 0 \\
\text{and} \quad & \sum_{i=1}^m \partial_u h_i(u) \leq \delta < 0, \quad i \in \{1, \ldots, m\}, \quad u \in [a, b]; \\
(c) \quad & 2dc_1c_2K_1 \leq \rho, \quad \text{where} \quad \rho \in (0, 1) \quad \text{and} \quad \rho = \max_{j \geq 0} (t_{j+1} - t_j),
\end{align*}
\]

satisfies \( 2dc_1c_2 \leq \gamma \).

**Theorem 2.1.** Consider \( H(x, u, p) = \langle p, g(x, u) \rangle + L(u) \), where \( g \) and \( L \) satisfy (17) and let \( u_0, h \) be such that (18) are verified. Take \( \Delta y(t_j) \in \mathbb{R}^n_+ \) for which

\[-1 \leq \langle \partial_u h(u), \Delta y(t_j) \rangle \leq -\frac{1}{2}, \quad u \in [a, b], \quad j \geq 0.\]

Then there exists an asymptotic stable solution

\[
\{u(t, x) \in [a, b] : t \in [t_j, t_{j+1}), \quad x \in B(x_*, \gamma), \quad j \geq 0\}
\]

of the H-J equation with jumps (2) such that

\[
u(0, x) = u_* + u_0(x),
\]

\[
|u(t, x) - u_*| \leq \left(\frac{1}{2}\right)^j, \quad t \in [t_j, t_{j+1}), \quad x \in B(x_*, \gamma), \quad j \geq 0,
\]

\[
|\partial_t u(t, x)| \leq \frac{L_i K_1}{1 - \rho} \left(\frac{1}{2}\right)^j, \quad t \in [t_j, t_{j+1}), \quad x \in B(x_*, \gamma), \quad j \geq 0, \quad i \in \{1, \ldots, n\},
\]

for some constant \( L_i > 0 \).
Proof. By hypothesis, the conclusion of Lemma 1.3 holds. Using (18) and Lemma 1.2, by a direct computation we obtain the estimates
\[
|\hat{u}(t; \lambda) - u_s| \leq |u_0(\lambda)| \leq K_0 < 1, \quad |\partial_\lambda \hat{u}(t, \lambda)| \leq |\partial_\lambda u_0(\lambda)|, \quad t \geq 0, \lambda \in \mathbb{R}^n,
\]
(19) \[|\hat{u}(t; \lambda) - u_s| \leq |\hat{u}(t_j; \lambda) - u_s| \leq \left(\frac{1}{2}\right)^j K_0, \quad t \in [t_j, t_{j+1}), \lambda \in \mathbb{R}^n, j \geq 0,
\]
\[|\partial_\lambda \hat{u}(t; \lambda)| \leq |\partial_\lambda \hat{u}(t_j; \lambda)| \leq \left(\frac{1}{2}\right)^j K_1, \quad t \in [t_j, t_{j+1}), \lambda \in \mathbb{R}^n, j \geq 0.
\]
This time, the smooth mapping \(\lambda = \psi(t, x)\) will be found as a unique solution of the integral equation
\[
\lambda = \hat{G}(-\tau(t; \lambda))[x], \quad \tau(t; \lambda) = \int_0^t \alpha(\hat{u}(s; \lambda))ds, \quad t \geq 0, x \in B(x_s, \gamma) \subseteq \mathbb{R}^n.
\]
The vector field \(\hat{g} \in C^1(\mathbb{R}^n; \mathbb{R}^n)\) is a nonlinear unbounded one and the corresponding local flow \(\{\hat{G}(\sigma)[x] : \sigma \in [-\beta, \beta], x \in B(x_s, \gamma)\}\) is constructed for \(\beta = 2dC_1\) satisfying
\[
\beta c_2 \leq \gamma \quad (\text{see (18, c)}).
\]
A direct computation shows that
\[
|\tau(t, \lambda)| \leq \sum_{j=0}^{\infty} \int_{t_j}^{t_{j+1}} \left[\int_0^1 |\partial_\alpha u_s + \theta(\hat{u}(t; \lambda) - u_s)|d\theta\right] |\hat{u}(t, \lambda) - u_s|dt
\]
\[
\leq dC_1 \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j = 2dC_1 = \beta, \quad \forall t \geq 0, \lambda \in \mathbb{R}^n.
\]
Using (21) we easily get that the right hand side
\[
V(t, x; \lambda) \overset{\text{def}}{=} \hat{G}(-\tau(t; \lambda))[x] = x - \left[\int_0^1 \hat{g}(\hat{G}(-\theta \tau(t; \lambda)))d\theta\right] \tau(t; \lambda)
\]
in (20) satisfies
\[
|V(t, x; \lambda) - x_s| \leq |x| + 2dC_1 C_2 = |x - x_s| + \beta C_2 \leq 2\gamma
\]
for any \(t \geq 0, x \in B(x_s, \gamma)\) and \(\lambda \in \mathbb{R}^n\). Here, we used the fact that the local flow \(\{\hat{G}(\sigma)[x] : \sigma \in [-\beta, \beta], x \in B(x_s, \gamma)\}\), is bounded and \(\hat{G}(\sigma)[x] \in B(x_s, 2\gamma)\) for any \(\sigma \in [-\beta, \beta], x \in B(x_s, \gamma)\). It can be easily seen, noticing that \(\{y_k(\sigma, x) : \sigma \in [-\beta, \beta], x \in B(x_s, \gamma)\}_{k \geq 0}\) defining the local flow is uniformly bounded and verifies
\[
y_k(\sigma, x) \in B(x_s, 2\gamma), \quad \sigma \in [-\beta, \beta], \quad x \in B(x_s, \gamma), \quad k \geq 0.
\]
With this notation, the integral equation (20) can be written as \(\lambda = V(t, x; \lambda)\), where the smooth mapping \(V(t, x; \lambda)\) is a contractive application with respect
to \( \lambda \in \mathbb{R}^n \). We then compute \( M(t, x; \lambda) = \partial_\lambda V(t, x; \lambda) = \tilde{g}(V(t, x; \lambda)) \partial_\lambda \tau(t; \lambda) \), where (see (20))

\[
\partial_\lambda \tau(t, \lambda) = \int_0^t \partial_\alpha \tilde{u}(s; \lambda)) \partial_\lambda \tilde{u}(s; \lambda) ds.
\]

Using (19) we get

\[
|\partial_\lambda \tau(t, \lambda)| \leq C_1 K_1 \sum_{j=0}^{\infty} \left( \frac{1}{2} \right)^j = 2dC_1 K_1
\]

and from (23) and (24) we obtain that \( |M(t, x; \lambda)| \leq 2dC_1 C_2 K_1 = \rho \in (0, 1) \) for any \( t \geq 0, x \in B(x_*, \gamma) \) and \( \lambda \in \mathbb{R}^n \). Applying the contractive mapping theorem we obtain a smooth mapping \( \{ \lambda = \psi(t, x) \in B(x_*, 2\gamma) : t \geq 0, x \in B(x_*, \gamma) \} \) as the unique solution of the integral equation

\[
\psi(t, x) = V(t, x; \psi(t, x)), \quad t \geq 0, x \in B(x_*, \gamma).
\]

Define

\[
u(t, x) = \tilde{u}(t, \psi(t, x)), \quad t \geq 0, x \in B(x_*, \gamma).
\]

To prove that \( \{ u(t, x) : t \geq 0, x \in B(x_*, \gamma) \} \) defined in (26) is the solution of the H-J equation (2) we need to show that the smooth function satisfying (25) is a solution of the H-J equations

\[
\begin{align*}
\partial_t \psi(t, x) + \partial_x \psi(t, x) g(x, u(t, x)) &= 0, \quad t \geq 0, x \in \text{int} B(x_*, \gamma), \\
\psi(0, x) &= x.
\end{align*}
\]

In this respect, for \( \bar{t} \in [t_j, t_{j+1}] \) and \( \bar{x} \in \text{int} B(x_*, \gamma) \) fixed define \( \{ \tilde{x}(s; \bar{x}), \tilde{u}(s; \bar{x}) : s \in [\bar{t}, \bar{t} + \varepsilon] \subseteq [t_j, t_{j+1}] \} \) as the solution of the characteristic system

\[
\begin{align*}
\frac{d\tilde{x}}{ds} &= g(\tilde{x}, \tilde{u}), \quad \tilde{x}(\bar{t}; \bar{x}) = \bar{x}, \\
\frac{d\tilde{u}}{ds} &= -L(\tilde{u}), \quad \tilde{u}(\bar{t}; \psi(\bar{t}, \bar{x})) = u(\bar{t}, \bar{x})
\end{align*}
\]

satisfying \( \tilde{x}(s, \bar{x}) \in B(x_*, \gamma), \quad s \in [\bar{t}, \bar{t} + \varepsilon] \). Notice that the solution of (28) can be obtained as the restriction to \( [\bar{t}, \bar{t} + \varepsilon] \) of the global solution \( \{ \tilde{x}(s; \lambda), \tilde{u}(s; \lambda) : s \geq 0 \} \) satisfying the original characteristic system

\[
\begin{align*}
\frac{d\tilde{x}}{ds} &= g(\tilde{x}, \tilde{u}), \quad \tilde{x}(0; \lambda) = \lambda, \\
\frac{d\tilde{u}}{ds} &= -L(\tilde{u}), \quad \tilde{u}(t_j, \lambda) = \tilde{u}(t_j^-; \lambda) + \left( h(\tilde{u}(t_j^-; \lambda)), \Delta \gamma(t_j) \right), \\
s \in [t_j, t_{j+1}], \quad j \geq 0,
\end{align*}
\]
with \( \tilde{u}(0; \lambda) = u_0(\lambda) + u_* \) for \( \lambda = \psi(\tilde{t}, \tilde{x}) \) fixed. We now use the representation
\[
\tilde{x}(s, \tilde{x}) = \tilde{G}(\tau(s; \psi(\tilde{t}, \tilde{x}))) [\psi(\tilde{t}, \tilde{x})], \quad s \in [\tilde{t}, \tilde{t} + \varepsilon],
\]
and replacing \( \hat{u} \) with \( \lambda \) in (30) we get
\[
V(s, \tilde{x}(s, \tilde{x}); \psi(\tilde{t}, \tilde{x})) = \tilde{G}( - \tau(s; \psi(\tilde{t}, \tilde{x})) ) \circ \tilde{G}(\tau(s; \psi(\tilde{t}, \tilde{x}))) [\psi(\tilde{t}, \tilde{x})]
= \psi(\tilde{t}, \tilde{x}) = \text{const., } s \in [\tilde{t}, \tilde{t} + \varepsilon].
\]
Differentiating with respect to \( s \), from (29) we obtain the H-J equations
\[
(30) \quad \partial_s V(\tilde{t}, \tilde{x}; \psi(\tilde{t}, \tilde{x})) + \partial_x V(\tilde{t}, \tilde{x}; \psi(\tilde{t}, \tilde{x})) g(\tilde{x}, u(\tilde{t}, \tilde{x})) = 0
\]
at \( s = \tilde{t} \), where \( (\tilde{t}, \tilde{x}) \in [t_j, t_{j+1}) \times \text{int } B(x_*, \gamma) \) was arbitrarily fixed. On the other hand, differentiating with respect to \( t \) and \( x \), from (25) we obtain
\[
(31) \quad \left\{ \begin{array}{l}
\partial_t \psi(t, x) = [I_n - M(t, x; \psi(t, x))]^{-1} [\partial_t V(t, x, \lambda)](\lambda = \psi(t, x)), \\
\partial_t \psi(t, x) = [I_n - M(t, x; \psi(t, x))]^{-1} [\partial_t V(t, x, \lambda)](\lambda = \psi(t, x)), \quad i \in \{1, \ldots, n\}.
\end{array} \right.
\]
Combining (30) and (31) we conclude that the H-J equations (27) are valid. Therefore, the piecewise smooth scalar function
\[
\{ u(t, x) = \tilde{u}(t; \psi(t, x)) : t \in [t_j, t_{j+1}), \quad x \in \text{int } B(x_*, \gamma), \quad j \geq 0 \}
\]
fulfils the original H-J equations with jumps (2). In addition, we get
\[
u(t, x) \in [a, b] \quad \text{and} \quad |u(t, x) - u_*| \leq \left( \frac{1}{2} \right)^j, \quad t \in [t_j, t_{j+1}), \quad j \geq 0
\]
for any \( x \in B(x_*, \gamma) \) and as far as \( \partial_t u(t, x) = (\partial_\lambda \tilde{u}(t; \psi(t, x)), \partial_t \psi(t, x)) \) we easily see (see (19) and (31)) that
\[
|\partial_t u(t, x)| \leq L_1 \frac{K_1}{1 - \rho} \left( \frac{1}{2} \right)^j, \quad t \in [t_j, t_{j+1}), \quad x \in B(x_*, \gamma), \quad j \geq 0,
\]
where \( L_1 = \max(|\partial_x \tilde{G}(\sigma)| |x| : \sigma \in [-\beta, \beta], \ x \in B(x_*, \gamma) \} \) for \( i \in \{1, \ldots, n\} \). The proof is complete. \( \square \)

**Comment.** The result given in Theorem 2.1 relies essentially on the special structure we have assumed for the vector field \( g(x, u) = \alpha(u) \tilde{g}(x), \tilde{g} \in C^1(\mathbb{R}^n, \mathbb{R}^n), \alpha \in C^1([a, b]), \) where \( a = u_* - 1, b = u_* + 1 \) and \( \alpha(u_* = 0). A relaxation of this hypothesis allows vector fields
\[
g(x, u) = \sum_{k=1}^l \alpha_k(u) \tilde{g}_k(x), \quad \alpha_k \in C^1([a, b]), \tilde{g}_k \in C^1(\mathbb{R}^n, \mathbb{R}^n),
\]
where \( \alpha_k(u_* = 0, k \in \{1, \ldots, l\} \), for some \( u_* \in \mathbb{R} \). To get an asymptotic stable solution, we must assume, in addition, that \( \{\tilde{g}_1, \ldots, \tilde{g}_l\} \subseteq C^1(\mathbb{R}^n; \mathbb{R}^n) \) commute using the standard Lie product. The second theorem of this paper is
encompassing the details we need in order to get an asymptotic stable solution when the vector field \( g(x, u) = \alpha_1(u)\hat{g}_1(x) + \alpha_2(u)\hat{g}_2(x) \) is defined by \( \alpha_k \in C^1([a, b]) \), \( \hat{g}_k \in C^1(\mathbb{R}^n; \mathbb{R}^n) \), \( k = 1, 2 \), where \( \alpha_k(u) = 0 \), \( k \in \{1, 2\} \), for some \( u_0 \in \mathbb{R} \) fixed. Everywhere in what follows, we take a domain \( D = B(x_0, \gamma) \subset \mathbb{R}^n \), where \( x_0 \in \mathbb{R}^n \) and \( \gamma > 0 \) are fixed. Denote the standard Lie product \( [\hat{g}_1, \hat{g}_2] = \{\{\partial_x\hat{g}_1\}\hat{g}_2 - \{\partial_x\hat{g}_2\}\hat{g}_1\} \), \( x \in \mathbb{R}^n \) and let \( a = u_0 - 1, b = u_0 + 1 \), where \( u_0 \in \mathbb{R} \) is fixed. Using the Hamiltonian function

\[
H(x, u, p) = \langle p, g(x, u) \rangle + L(u), \quad g(x, u) = \alpha_1(u)\hat{g}_1(x) + \alpha_2(u)\hat{g}_2(x),
\]

we assume that \( \{\hat{g}_1, \hat{g}_2\} \subseteq C^1(\mathbb{R}^n; \mathbb{R}^n) \) and \( L \in C^1(\mathbb{R}) \) satisfy

\[
\begin{align*}
(a) & \quad L(u_0) = 0, \quad \partial_u L(u) \geq 0, \quad u \in [a, b]; \\
(b) & \quad \alpha_k(u_0) = 0, \quad k = 1, 2; \\
(c) & \quad [\hat{g}_1, \hat{g}_2](x) = 0, \quad \text{for any } x \in B(x_0, 3\gamma).
\end{align*}
\]

The corresponding (H-J) equations with jumps are

\[
\begin{align*}
\partial_t u + H(x, u, \partial_x u) = 0, & \quad t \in [t_j, t_{j+1}), \quad x \in D = B(x_0, \gamma), \\
 u(t_j, x) = u(t_j-, x) + h(u(t_j-, x))\Delta y(t_j), & \quad j \geq 0, \quad x \in D, \\
u(0, x) = u_0(x), & \quad x \in D,
\end{align*}
\]

where the scalar jump function \( h \in C^1([a, b]) \) and \( \Delta y(t_j) = y(t_j) - y(t_{j-}) \geq 0 \), \( j \geq 1 \), are associated with the scalar piecewise constant process \( \{y(t) \geq 0 : y(0) = 0\} \). Define

\[
\begin{align*}
C_1 &= \max\{|\partial_u \alpha_1(u)| + |\partial_u \alpha_2(u)| : u \in [a, b]\}, \\
C_2 &= \max\{|\hat{g}_1(x)| + |\hat{g}_2(x)| : x \in B(x_0, 3\gamma)\}.
\end{align*}
\]

Take \( u_0 \) and \( h \) such that

\[
\begin{align*}
(a) & \quad u_0 \in C^1(\mathbb{R}^n), \quad \sup |u_0(x)| = K_0 < 1 \text{ and } \sup |\partial_x u_0(x)| = K_1; \\
(b) & \quad h \in C^1([a, b]), \quad h(u_0) = 0, \quad \partial_u h(u) \leq \delta < 0, \quad u \in [a, b]; \\
(c) & \quad 2dC_1C_2K_1 = \rho \in (0, \frac{1}{2}), \quad \text{where } d = \max_{j \geq 0} (t_{j+1} - t_j) \text{ and } \beta = 2dC_1 \text{ satisfies } \beta C_2 \leq \frac{\gamma}{2}.
\end{align*}
\]

**Theorem 2.2.** Consider \( H(x, u, p) = \langle p, g(x, u) \rangle + L(u) \), where \( g(x, u) = \alpha_1(u)\hat{g}_1(x) + \alpha_2(u)\hat{g}_2(x) \) and \( L(u) \) fulfill (32). Let \( u_0 \) and \( h \) be such that (33) are satisfied. Take \( \Delta y(t_j) \geq 0 \) such that

\[
-1 \leq \partial_u h(u)\Delta y(t_j) \leq -\frac{1}{2}, \quad u \in [a, b], \quad j \geq 1.
\]
Then there exists a global solution \( \{ u(t,x) : t \geq 0, x \in D \} \) of the H-J equations with jumps (2) which is asymptotically stable such that

\[
\begin{cases}
\left| u(t,x) - u_* \right| \leq \left( \frac{1}{2} \right)^j, & t \in [t_j, t_{j+1}), x \in D, j \geq 0, \\
|\partial_t u(t,x)| \leq L_i \frac{K_i}{R} \left( \frac{1}{2} \right)^j, & t \in [t_j, t_{j+1}), x \in D, j \geq 0, i \in \{1, \ldots, n\},
\end{cases}
\]

for some constant \( L_i > 0 \), where \( K_1 > 0 \) and \( \rho \in (0,1) \) are given in (33).

Proof. We use the same arguments as in the proof of Theorem 2.1. By hypothesis, the conclusions of Lemma 1.5 regarding the global solution \( \{ \hat{u}(t;\lambda) : t \geq 0, \lambda \in \mathbb{R}^n \} \) hold. The equations

\[
\begin{align*}
\frac{d\hat{u}}{dt} &= -L(\hat{u}), \quad \hat{u}(t_-;\lambda) + h(\hat{u}(t_-;\lambda))\Delta y(t_j), \ t \in [t_j, t_{j+1}), j \geq 0, \\
\hat{u}(0;\lambda) &= u_* + u_0(\lambda), \ \lambda \in \mathbb{R}^n,
\end{align*}
\]

are satisfied. In addition, using (33 (a), (b)) and (34), by a direct computation we get

\[
\begin{cases}
|\partial_t \hat{u}(t;\lambda)| \leq \left( \frac{1}{2} \right)^j K_1, & t \in [t_j, t_{j+1}), j \geq 0, \lambda \in \mathbb{R}^n, \\
|\hat{u}(t;\lambda) - u_*| \leq |u_0(\lambda)| \leq K_0 < 1, & t \geq 0, \lambda \in \mathbb{R}^n, \\
|\partial_t \hat{u}(t;\lambda)| \leq |\partial_\lambda u_0(\lambda)|, & t \geq 0, \lambda \in \mathbb{R}^n.
\end{cases}
\]

This time, \( \{ \hat{x}(t;\lambda) : t \geq 0, \lambda \in B(x_*,2\gamma) \subseteq \mathbb{R}^n \} \) is the global solution of the characteristic system

\[
\begin{align*}
\frac{d\hat{x}}{dt}(t;\lambda) &= \alpha_1(\hat{u}(t;\lambda)) \hat{g}_1(\hat{x}(t;\lambda)) + \alpha_2(\hat{u}(t;\lambda)) \hat{g}_2(\hat{x}(t;\lambda)), \ t \geq 0, \\
\hat{x}(0;\lambda) &= \lambda \in B(x_*,2\gamma).
\end{align*}
\]

As far as \( [\hat{g}_1, \hat{g}_2](x) = 0, \forall x \in B(x_*,3\gamma) \), we may and do write \( \hat{x}(t;\lambda) \) as

\[
\hat{x}(t;\lambda) = \hat{G}_1(\tau_1(t,\lambda)) \circ \hat{G}_2(\tau_2(t,\lambda)) |\lambda|, \ t \geq 0, \lambda \in B(x_*,2\gamma)
\]

provided \( \{ \hat{x}(t;\lambda) \in B(x_*,3\gamma) : t \geq 0, \lambda \in B(x_*,2\gamma) \} \). Here \( \{ \hat{G}_k(t_k)[\lambda] : t_k \in [-\beta, \beta], \lambda \in B(x_*,2\gamma + \frac{\gamma}{2}) \} \) is the local flow generated by \( \hat{g}_k \in C^1(\mathbb{R}^n,\mathbb{R}^n) \). Since

\[
\tau_k(t,\lambda) = \int_0^t \alpha_k(\hat{u}(s;\lambda))ds = \int_0^t \left[ \int_0^1 \partial_s \alpha_k(u_* + \theta(\hat{u}(s;\lambda) - u_*))d\theta \right] (\hat{u}(s;\lambda) - u_*)ds, \ k \in \{1,2\},
\]

we get (see (2) and (33, c))

\[
|\tau_k(t,\lambda)| \leq C_1 \int_0^1 |\hat{u}(s;\lambda) - u_*|ds \leq C_1 \sum_{j=0}^\infty \int_{t_j}^{t_{j+1}} \left( \frac{1}{2} \right)^j ds \leq 2dC_1 = \beta
\]
for any \( t \geq 0, \lambda \in \mathbb{R}^n, k \in \{1, 2\}, \) where \( \beta C_2 \leq \frac{\gamma}{2}. \) This shows that the flow 
\( \{ \tilde{x}(t; \lambda) : t \geq 0, \lambda \in B(x, 2\gamma) \} \) defined in (36) fulfills \( \tilde{x}(t, \lambda) \in B(x, 3\gamma) \) and the equations

\[
\begin{align*}
    x &= \tilde{x}(t; \lambda) = \tilde{G}_1(-\tau_1(t; \lambda)) \circ \tilde{G}_2(-\tau_2(t; \lambda))[x], \\
    x &\in B(x, \gamma), \quad \lambda \in B(x, 2\gamma),
\end{align*}
\]

are equivalent to

\[
\lambda = \tilde{G}_2(-\tau_2(t; \lambda)) \circ \tilde{G}_1(-\tau_1(t; \lambda))[x], \quad \lambda \in B(x, 2\gamma), \quad x \in D.
\]

Denote

\[
V(t, x; \lambda) = \tilde{G}_2(-\tau_2(t; \lambda)) \circ \tilde{G}_1(-\tau_1(t; \lambda))[x] = \tilde{G}_1(-\tau_1(t; \lambda)) \circ \tilde{G}_2(-\tau_2(t; \lambda))[x]
\]

for \( t \geq 0, x \in B(x, \gamma), \lambda \in B(x, 2\gamma). \) To prove that the smooth mapping

\[
M(t, x; \lambda) = \partial_\lambda V(t, x; \lambda) = -[\tilde{g}_1(V(t, x; \lambda)) \partial_\lambda \tau_1(t; \lambda) + \tilde{g}_2(V(t, x; \lambda)) \partial_\lambda \tau_2(t; \lambda)]
\]

Using (2) and (33, c) we obtain

\[
|M(t, x; \lambda)| \leq \rho \in (0, \frac{1}{2}) \quad \text{for any } t \geq 0, x \in B(x, \gamma), \lambda \in B(x, 2\gamma).
\]

This allows to use the contractive mapping theorem for the integral equations (37) and to get the smooth solution \( \lambda = \psi(t, x) = \lim_{k \to \infty} \lambda_k(t, x) \in B(x, 2\gamma) \)

satisfying

\[
\begin{align*}
    \lambda_0(t, x) &= x, \quad |\lambda_k(t, x) - x| \leq \frac{1}{1-\rho} \gamma \leq 2\gamma, \quad \forall k \geq 0, \\
    \psi(t, x) &= V(t, x; \psi(t, x)), \quad t \geq 0, \quad x \in B(x, \gamma).
\end{align*}
\]

Define \( \{ u(t, x) = \hat{u}(t, \psi(t, x)) : t \in [t_j, t_{j+1}), \quad x \in B(x, \gamma), \quad j \geq 0 \}. \) Using the same argument as in the proof of Theorem 2.1, we get (35) and the proof is complete.

\[\square\]

REFERENCES


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