Dedicated to Dr. Constantin VÂRSAN
on the occasion of his 70th birthday

NONLINEAR AND TIME DELAY SYSTEMS FOR FLIGHT CONTROL

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Starting from the equations occurring in flight control, there are considered time delay systems with (possibly) discontinuous but bounded nonlinearities. A useful definition of the solution is based on the so-called extended nonlinearity which allows development of the standard absolute stability theory; within this theory a stability result is extended to a significant class of time delay systems and the so called minimal phase assumption is relaxed. The result is obtained without applying to sliding modes framework.

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1. THE STARTING POINTS OF THE PAPER

The interacting of the aircraft as a controlled system and of the human operator (pilot) dynamics results in a closed loop feedback system. As known [11], feedback may be the source of various instabilities, both in linear and nonlinear systems. The feedback system pilot-aircraft is a quite uncertain system since the human behavior – even of a trained operator, such as a pilot could be – may be quite unpredictable; at its turn the aircraft behavior is strongly dependent on the flight envelope which incorporates a description of the flight parameters (mainly the height and the Mach number – the speed). This cumulated uncertainty is the source of an instability phenomenon known as PIO – P(ilot) I(n-the-loop) O(scillations), which are in fact self sustained oscillations of the feedback system pilot-aircraft due, paradoxically, to the pilot's efforts to control the aircraft in limit situations. Since the first PIO event in 1959 there has been a tremendous effort to handle the phenomenon while new such events occurred. Even if much of the information is classified, there exists already published matter and the reader is referred to [9] for the basic notions and descriptions.

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A. From the mathematical point of view, this phenomenon can be viewed as loosing the stability via self sustained oscillations. The aircraft experts in PIO consider 3 categories of PIO called PIO I, PIO II, PIO III. The first category is considered to occur when both the pilot and the aircraft may be quite well described by their linear models. The second category is considered to be the effect of quite large control signals which activate the so-called position and rate limiters which are nonlinear elements containing a bounded nonlinearity – the saturation function. Category III PIO are considered fully nonlinear and non stationary, i.e., time varying. It is considered that if PIO I and PIO II are prevented, PIO III will not occur and up to now this assumption was verified practically.

It follows that if nonlinearity is to be taken into account, the theoretical research has to cope with the position and rate limiters. These are technical devices which should be modeled in the most adequate way. The analysis of the existing information, summarized in [10], shows that the position limiter is a simple saturated nonlinear function expressed, e.g.,

$$f(\sigma) = \begin{cases} V_L \operatorname{sgn} \sigma, & |\sigma| \geq \varepsilon_L, \\ \dfrac{V_L}{\varepsilon_L}\sigma, & |\sigma| \leq \varepsilon_L. \end{cases} \tag{1}$$

Here, the notation is that of the technological field-aircraft dynamics and control. The rate limiter is a dynamical structure with local feedback described by

$$\dot{\delta} = f(\delta_c - \delta) \tag{2}$$

with δ_c – the input signal and $f : \mathbb{R} \mapsto \mathbb{R}$ as in (1). It is not difficult to observe that $f(\sigma)$ thus defined is bounded on $\mathbb{R}$, non-decreasing and also sector restricted.

If static friction of the actuators is also taken into account, the nonlinearities of the limiters associated with the actuators may be also discontinuous with discontinuity of the first kind: if σ_0 is a discontinuity point, the limits $f(\sigma_0 - 0)$ and $f(\sigma_0 + 0)$ are both finite.

B. The pilot models used in the PIO analysis are usually linear and described by transfer functions of rational type multiplied by a pure delay accounting for a delayed reaction of the human operator. A transfer function of the form

$$H_p(s) = K_p \, \frac{1 + T_d s}{1 + T_i s} \, \mathrm{e}^{-\tau s} \tag{3}$$

is standard in the description of the human operator not only for aircraft but also for power and nuclear power engineering, possibly for chemical engineering, too. A system with the transfer function (3) may be described, for instance, by the delay-differential equation

$$T_i\,\dot{y}(t) + y(t) = K_p(T_d\,\dot{u}(t-\tau) + u(t-\tau)). \tag{4}$$

If such a model is integrated in a feedback structure, it will lead to a system described by delay-differential equations with delays occurring in state, input and output variables.

All these considerations show that it is of certain interest to discuss the absolute stability of time delay system containing bounded and (possibly) discontinuous nonlinear functions.

2. THE BASIC SYSTEM AND THE MAIN RESULT

We shall consider throughout this paper the system

$$\dot{x}(t) = Ax(t) + \sum_1^r bq_i^* x(t-\tau_i) - b\varphi(c^* x(t)) \tag{5}$$

which is obviously a special case of

$$\dot{x}(t) = Ax(t) + \sum_1^r B_i x(t-\tau_i) - b\varphi(c^* x(t)) \tag{6}$$

with $B_i = bq_i^*$ being special dyadic matrices. It is worth mentioning that for a single delay case ($r = 1$) such systems were introduced in [8] accompanied by the remark that higher order delay-differential equations of the form

$$y^{(n)}(t) + \sum_1^{n-1} a_i y^{(i)}(t) + \sum_1^{n-1} b_i y^{(i)}(t-\tau) = u(t) \tag{7}$$

can be given the above form with

$$x = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \quad q = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix}. \tag{8}$$

Using the Cauchy formula of constants variation an integral equation

$$\sigma(t) = \rho(t) - \int_0^t \kappa(t-\tau)\varphi(\sigma(\tau))\mathrm{d}\tau \tag{9}$$

can be associated with (6) hence to (5), where we denoted

$$\rho(t) = c^* \left(X(t)x_0(0) + \sum_1^r \int_{-\tau_i}^0 X(t-\tau_i-\theta)B_i x_0(\theta)\mathrm{d}\theta \right),$$

$$\kappa(t) = c^* X(t) b, \quad \sigma = c^* x,$$

$X(t)$ being the Cauchy matrix of

$$\dot{x}(t) = Ax(t) + \sum_1^r B_i x(t-\tau_i)$$

with the usual definition.

A. The nonlinear function $\varphi : \mathbb{R} \mapsto \mathbb{R}$ is supposed to satisfy the conditions of [5, 6, 7]: i) piecewise continuous with finite (first kind) discontinuities; ii) bounded, i.e., $|\varphi(\sigma)| \leq m$, $\forall \sigma \in \mathbb{R}$; iii) subject to the pseudo-sector condition

$$\varphi(\sigma)\sigma - \varepsilon\sigma^2 - \frac{\varphi^2(\sigma)}{k} > 0, \quad 0 < k \leq +\infty \tag{10}$$

for some $\varepsilon \in [0, k/4)$, $0 < |\sigma| \leq mc$, where $c > 0$ is defined by the conditions of the problem. Condition (10) is sector-like since we may consider polar coordinates in the plane (σ, φ), namely, $\sigma = r\cos\theta$, $\varphi = r\sin\theta$ to obtain from (10), $r \neq 0$,

$$\sin\theta\cos\theta - \varepsilon\cos^2\theta - \frac{\sin^2\theta}{k} > 0,$$

which becomes

$$\frac{\tan^2\theta}{k} - \tan\theta + \varepsilon < 0,$$

and this gives

$$\varepsilon < \frac{2\varepsilon}{1+\sqrt{1-4\varepsilon/k}} < \tan\theta < \frac{1}{2}(1+\sqrt{1-4\varepsilon/k})k < k.$$

Remark that (10) with finite k will require $\varphi(\sigma)$ being continuous at $\sigma = 0$. It is possible now to state the following basic result [6].

Theorem 1. *Consider a system described by the nonlinear integral equation* (9) *under the following assumptions:* a) $\rho, \kappa \in \mathcal{L}^1(0,\infty) \cap \mathcal{L}^2(0,\infty)$*;* b) $\dot{\rho}, \dot{\kappa} \in \mathcal{L}^1(0,\infty)$*;* c) $\int_t^\infty |\kappa(\lambda)|\mathrm{d}\lambda \in \mathcal{L}^2(0,\infty)$*;* d) $\varphi : \mathbb{R} \mapsto \mathbb{R}$ *is subject* *to conditions* i)–iii) *stated above, with c being the $\mathcal{L}^1$ norm of $\kappa(t)$. If there exists some real ϑ such that the frequency domain inequality*

$$\frac{1}{k} + \varepsilon|\chi(\mathrm{i}\omega)|^2 + \mathrm{Re}(1 + \mathrm{i}\omega\vartheta)\chi(\mathrm{i}\omega) \geq 0 \ , \ \forall\omega \in \mathbb{R}_+, \tag{11}$$

holds, then $\lim\limits_{t\to\infty} \sigma(t) = 0$. Here $\chi(s)$ is the Laplace transform of $\kappa(t)$.

While the result is quite old we shall not reproduce its proof but rather comment on it. Since $\varphi(\sigma)$ is discontinuous, we have to make clear the sense of defining the solution. As mentioned in [6], the solution of (2.5) is viewed in the sense of Azbelev [2, 3]. The time delay system (6) is a time delay system with discontinuous right hand side whose solution should be defined in one of the possible ways for such systems. With the same arguments as in [7], we choose the approach of the "extended nonlinearity" which may be called "partly differential inclusion": if σ_0 is some discontinuity point of $\varphi(\sigma)$, then the solution of (6) is defined as the solution of

$$\begin{cases} \dot{x}(t) = Ax(t) + \displaystyle\sum_1^r B_i x(t - \tau_i) + b\xi(t), \\ \xi(t) = -\varphi(c^* x(t)) \end{cases} \tag{12}$$

and if $c^*x(t) = \sigma_0$ then $\xi(t) \in [-\varphi(\sigma_0 + 0), -\varphi(\sigma_0 - 0)]$, where for convenience we took $\varphi(\sigma_0 - 0) < \varphi(\sigma_0 + 0)$. In this case, since the extended nonlinearity is obviously integrable, the solution of (9) may be considered as the solution of

$$\sigma(t) = \rho(t) + \int_0^t \kappa(t - \tau)\xi(\tau)\mathrm{d}\tau, \quad \xi(t) = -\varphi(\sigma(t)) \tag{13}$$

with the extended nonlinearity as previously.

The solution definition *via* extended nonlinearity allows an adequate definition of the sliding modes for time delay systems following the lines of [7] (a comparison with other reference is worth doing). Fortunately, the way of proving the results allows avoiding the problem of the sliding modes for (6). To end this section, if $\varphi(\sigma)$ is discontinuous at $\sigma = 0$, then $k = \infty$ in (10) and (11). However, the frequency domain condition is still improved with respect to the standard Popov inequality, due to the positive term $\varepsilon|\chi(\mathrm{i}\omega)|^2$; the same improvement may be obtained *via* a circle criterion but with the significant detail that (10) holds only on a finite interval.

B. We shall turn now to the case of system (5) having in mind the line of [6] where the system is without delays, i.e., $q_i = 0$, $i = \overline{1, r}$. The main problem here is to obtain $\lim\limits_{t\to\infty} x(t) = 0$. If $\varphi(\sigma)$ is continuous at 0, it is a standard way that gives not only the asymptotic behavior (attractiveness of 0) but also Liapunov stability, hence global asymptotic stability. If $\varphi(\sigma)$ is discontinuous at 0, then it is still possible to obtain for (5) a result which is analogous to Theorem 3 of (op. cit.)

THEOREM 2. *Consider system* (5) *under the following assumptions:*

a) *the characteristic equation*

$$\det\left(\lambda I - A - \sum_1^r bq_i^* \mathrm{e}^{-\lambda\tau_i}\right) = 0 \tag{14}$$

has all its roots in the left half plane of $\mathbb{C}$*, the pair* (c^*, A) *is observable, and there exists* k*,* $0 \le k \le n-1$*, such that* $c^* A^k b \neq 0$*;*

b) $\varphi : \mathbb{R} \mapsto \mathbb{R}$ *is such that* $|\varphi(\sigma)| \le m$*, has a discontinuity at* 0 *and* $|\varphi(\sigma)/\sigma| > \varepsilon \ge 0$ *for* $0 < |\sigma| \le mc$*, where* c *is the* $\mathcal{L}^1$ *norm of* $\kappa(t)$ *defined previously. Assume also that there exists some real* ϑ *such that*

$$\varepsilon|\chi(\mathrm{i}\omega)|^2 + \mathrm{Re}\,(1 + \mathrm{i}\omega\vartheta)\chi(\mathrm{i}\omega) \ge 0, \quad \forall\omega \in \mathbb{R}_+, \tag{15}$$

with $\chi(s) = c^*(sI - A - b\sum_1^r q_i^*\mathrm{e}^{-s\tau_i})^{-1}b$ *the transfer function of the linear part of* (5)*, being such that all its transmission* (*invariant*) *zeros are outside the imaginary axis* $\mathrm{i}\mathbb{R}$*. Then* $\lim_{t\to\infty} x(t) = 0$ *for any initial condition* $(x_0, \psi(\cdot))$*, where* ψ *is a* $\mathbb{R}^n$*-valued function defined on* $[-\tau, 0)$*,* $\tau = \max\{\tau_1, \ldots, \tau_r\}$*.*

The proof is sketched in the Appendix and makes largely use of a canonical change of coordinates used in [5, 6] but which goes back to [1] and [13].

3. CONCLUSIONS AND FURTHER DEVELOPMENT

The present paper represents an account of the extension of some results concerning systems with bounded and discontinuous nonlinearities, to a special class of time lag systems. In fact, the approach of [6] together with the idea of considering extended nonlinearities for systems with discontinuous right hand side [7], is a genuine programme that may be carried on for time delay systems with a single or several bounded and discontinuous nonlinear functions. Within this programme, it appears possible to discuss sliding modes for time delay systems in the simplest and most reasonable way. It is an interesting research for the future, with its applications going beyond the field that initiated the topics (flight control).

APPENDIX

We shall sketch here the proof of Theorem 2. If the Cauchy formula is used, we can associate with (5) the integral equation (9) with $X(t)$ defined by

$$\dot{X}(t) = AX(t) + \sum_1^r bq_i^* X(t - \tau_i), \quad X(t) \equiv 0,\ t < 0;\ X(0) = I.$$

Taking into account the assumption on the roots of the characteristic equation (14), we deduce the estimate $|X(t)| \leq \beta_0 \mathrm{e}^{-\alpha\tau}$ for some $\beta_0 > 0$, $\alpha > 0$. It follows that all assumptions of Theorem 1 on $\rho(t)$ and $\kappa(t)$ hold due to the exponential estimates. It is easily seen that the Laplace transform of the kernel $\kappa(t)$ is exactly the transfer function $\chi(s)$ in the statement of Theorem 2. Consequently, the fulfilment of (15) implies the fulfilment of (13) in Theorem 1. It follows that $\lim_{t\to\infty} \sigma(t) = \lim_{t\to\infty} c^* x(t) = 0$ along the solutions of (5), these solutions being defined using (12) and the idea of extended nonlinearity of [7] presented previously.

We shall now prove that $\lim_{t\to\infty} x(t) = 0$. Since $b \neq 0$ and (c^*, A) is an observable pair, there exists some k, $0 \leq k \leq n-1$, such that $\rho_{k+1} = c^* A^k b \neq 0$, where $\rho_i = c^* A^{i-1} b$, $i = 1, \ldots, n$. Perform now a change of the state variables in (5) in two steps: the first one takes into account the observability of the pair (c^*, A) and reads $z = Qx$, Q being the nonsingular observability matrix with the rows $c^* A^i$, $i = 0, \ldots, n-1$. If ζ_i are the entries of the vector z, system (5) becomes

$$(16)\quad \begin{cases} \dot{\zeta}_i(t) = \zeta_{i+1}(t),\ i = 1, \ldots, k, \\ \dot{\zeta}_i(t) = \zeta_{i+1}(t) - \rho_i(\varphi(\zeta_1(t)) - \sum_1^r q_j^* Q^{-1} z(t-\tau_j)),\ i = k+1, \ldots, n-1, \\ \dot{\zeta}_n(t) = \sum_1^n \gamma_{i-1}\zeta_i(t) - \rho_n(\varphi(\zeta_1(t)) - \sum_1^r q_j^* Q^{-1} z(t-\tau_j)), \end{cases}$$

where γ_i are the coefficients of the characteristic equation $\det(\lambda I - A) = 0$.

Next, in the last $n-k-1$ equations we shall eliminate the nonlinear and delay terms using the second change of variables that appears for the first time in [1] and is also used in [13], namely,

$$(17)\qquad \xi_i = \zeta_i,\ i = 1, \ldots, k+1, \quad \xi_i = \zeta_i - (\rho_i/\rho_{k+1})\zeta_{k+1},\ i = k+2, \ldots, n$$

which is obviously invertible. A straightforward computation leads to the system

$$\begin{aligned} \dot{\xi}_i(t) &= \xi_{i+1}(t),\ i = 1, \ldots, k, \\ \dot{\xi}_{k+1}(t) &= \frac{\rho_{k+2}}{\rho_{k+1}}\xi_{k+1}(t) + \xi_{k+2}(t) - \rho_{k+1}(\varphi(\xi_1(t)) - \sum_1^r q_j^*(TQ)^{-1}\widehat{x}(t-\tau_j)), \\ \dot{\xi}_i &= \left(\frac{\rho_{i+1}}{\rho_{k+1}} - \frac{\rho_i}{\rho_{k+1}} \cdot \frac{\rho_{k+2}}{\rho_{k+1}}\right)\xi_{k+1} - \frac{\rho_i}{\rho_{k+1}}\,\xi_{k+2} + \xi_{i+1},\ i = k+2, \ldots, n-1, \end{aligned}$$

$$\dot{\xi}_n = \sum_{i\neq k+1,k+2} \gamma_{i-1}\xi_i + \left(\sum_{k+1}^{n} \gamma_{i-1}\frac{\rho_i}{\rho_{k+1}} - \frac{\rho_n}{\rho_{k+1}} \cdot \frac{\rho_{k+2}}{\rho_{k+1}}\right)\xi_{k+1} + \\ + \left(\gamma_{k+1} - \frac{\rho_n}{\rho_{k+1}}\right)\xi_{k+2},$$

T being the nonsingular matrix of the second change of variables while $\widehat{x}$ is the state vector with entries ξ_i. Remark that only the $k+1$ equation contains nonlinear and delayed terms. Since $\xi_1 \equiv c^*x \equiv \sigma$, we deduce that $\lim_{t\to\infty} \xi_1(t) = 0$. But the state variables of (5), hence of the above system, are bounded due to the boundedness of φ and to the stability of the linear part of (5) – the assumption about the roots of the characteristic equation (14). Therefore, $\dot{\xi}_2$ is bounded, hence $\dot{\xi}_1$ is uniformly continuous; using the Barbălat lemma [12], we deduce that $\lim_{t\to\infty} \xi_2(t) = 0$. We can continue in this way to obtain $\lim_{t\to\infty} \xi_i(t) = 0$, $i = 3, \dots, k+1$.

Consider now the subsystem of the variables $\xi_{k+2}, \dots, \xi_n$, which is a linear system without delays, with input a linear combination of the previous state variables $\xi_i(t)$, $i = 1, \dots, k+1$, that approach asymptotically 0. But this system has as eigenvalues the invariant transmission zeros of (5), hence the zeros of the transfer function $\chi(s)$. Since the subsystem is finite dimensional, we show first that the numerator of $\chi(s)$ is a polynomial. Indeed, a rather straightforward manipulationwill give

$$\chi(s) = \frac{c^*(sI - A)^{-1}b}{1 - \sum_1^r \mathrm{e}^{-s\tau_i} q_i^*(sI - A)^{-1}b} \tag{18}$$

and the numerator of $\chi(s)$ is the numerator of $c^*(sI - A)^{-1}b$ – a polynomial. Next, the proof of the assertion concerning the eigenvalues of the subsystem of the variables $\xi_{k+2},\dots,\xi_n$ can be either performed directly or by applying a general result from e.g. [4] dealing with the properties of the linear systems subject to changes of the state variables defined by matrices of the form

$$T = \begin{pmatrix} c^* \\ \vdots \\ c^*A^k \\ W \end{pmatrix}, \tag{19}$$

where W is such that its rows are a maximal system of independent solutions of the linear homogeneous system

$$w^*b = 0,\ w^*Ab = 0,\ \dots,\ w^*A^kb = 0.$$

The eigenvalues of the subsystem of the variables $\xi_{k+2}, \dots, \xi_n$ are thus outside the imaginary axis, i.e., this subsystem is hyperbolic. On the other hand, its solutions are bounded (see above). We can thus apply Lemma 22.3 of [12] to find $\lim_{t\to\infty} \xi_i(t) = 0$, $i = k+2, \ldots, n$. This ends the proof. $\square$

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