Dedicated to Professor Gheorghe Bucur
on the occasion of his 70th birthday

ON A DENJOY-BOURBAKI TYPE INEQUALITY AND SOME APPLICATIONS

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Let $f : [a,b] \to \mathbb{R}$ be a continuous function on the compact interval $[a,b]$. We prove the inequality

$$f(x) - f(a) \le v*([a,x] \cap A) + \int^{*}_{[a,x]\cap A'} f(t)\mathrm{d}t, \quad \forall x \in [a,b],$$

where A is on arbitrary subset of $[a,b]$, $A' \cap A = \phi$, $[a,b]\backslash(A \cup A')$ is a countable set, $v*$ is the outer measure associated with f and $\int^{*}_{M} f\mathrm{d}t$ is the outer integral of the function f on the subset M. It turns out that there are a lot of consequences of this inequality.

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1. THE VARIATION OF A FUNCTION

In what follows an interval $[\alpha,\beta]$ is called proper (interval) if $\alpha < \beta$.

Let $f : [a,b] \to \mathbb{R}$ be a real continuous function on $[a,b]$. For any proper interval $[\alpha,\beta] \subset [a,b]$ we denote by $v_f[\alpha,\beta]$ or simply $v[\alpha,\beta]$ the variation of the function f on the interval $[\alpha,\beta]$, i.e.,

$$v[\alpha,\beta] = \sup\left\{\sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)|;\ \alpha = x_0 < x_1 < x_2 < \cdots < x_n = \beta\right\}.$$

It is known that if $a \le \alpha < \beta < \gamma \le b$ we have

$$0 \le v([\alpha,\gamma]) = v([\alpha,\beta]) + v([\beta,\gamma]),$$

$v[\alpha,\beta] = \sup v[\alpha_n,\beta_n]$ if $\alpha_n \downarrow \alpha$ and $\beta_n \uparrow \beta$.

Moreover, if $([\alpha_n,\beta_n])_n$ is a sequence of intervals such that the sets $[\alpha_n,\beta_n] \cap [\alpha_m,\beta_m]$ have no interior point, for $n \neq m$ and $\bigcup_n [\alpha_n,\beta_n] = (\alpha,\beta)$ then

$$v[\alpha,\beta] = \sum_n v[\alpha_n,\beta_n].$$

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Indeed, for any $m \in \mathbb{N}$ we have

$$(\alpha, \beta) = \bigcup_{n \le m} [\alpha_n, \beta_n] \cup \bigcup_{i=1}^{k} [\gamma_i, \delta_i],$$

where the sets $[\alpha_n, \beta_n] \cap [\alpha_k, \beta_k], [\gamma_i, \delta_i] \cap [\gamma_j, \delta_j],\ [\alpha_n, \beta_n] \cap [\gamma_i, \delta_i]$ have no interior point and therefore

$$v[\alpha, \beta] = \sum_{n \le m} v[\alpha_n, \beta_n] + \sum_{i=1}^{k} v[\gamma_i, \delta_i] \ge \sum_{n \le m} v[\alpha_n, \beta_n],$$

$$v[\alpha, \beta] \ge \sup_{m \in \mathbb{N}} \sum_{n \le m} v[\alpha_n, \beta_n] = \sum_{n \in \mathbb{N}} v[\alpha_n, \beta_n].$$

Using a compacity argument, for any $\varepsilon, \varepsilon' > 0$ we find a finite covering of the compact interval $[\alpha + \varepsilon,\ \beta - \varepsilon]$ with intervals of the type $\left(\alpha_n - \frac{\varepsilon}{2^n}, \beta_n + \frac{\varepsilon}{2^n}\right)$, i.e.,

$$[\alpha + \varepsilon, \beta - \varepsilon] \subset \bigcup_{n \le m} \left[\alpha_n - \frac{\varepsilon'}{2^n}, \beta_n + \frac{\varepsilon'}{2^n}\right]$$

and therefore

$$v[\alpha + \varepsilon, \beta - \varepsilon] \le \sum_{n \le m} v \left[\alpha_n - \frac{\varepsilon'}{2^n}, \beta_n + \frac{\varepsilon'}{2^n}\right] \le \sum_{n \in \mathbb{N}} v \left[\alpha_n - \frac{\varepsilon'}{2^n}, \beta_n + \frac{\varepsilon'}{2^n}\right].$$

Hence, letting $\varepsilon' \to 0$, we get

$$v[\alpha + \varepsilon, \beta - \varepsilon] \le \sum_{n \in \mathbb{N}} v[\alpha_n, \beta_n].$$

If ε tends to zero we obtain

$$v[\alpha, \beta] \le \sum_{n \in \mathbb{N}} v[\alpha_n, \beta_n], \quad v[\alpha, \beta] = \sum_{n \in \mathbb{N}} v[\alpha_n, \beta_n].$$

For any subset A of $[a, b]$ we define

$$v^*(A) = \begin{cases} \inf \left\{ \sum_n v[\alpha_n, \beta_n] \mid \alpha_n < \beta_n,\ A \subset \bigcup_n (\alpha_n, \beta_n) \right\} & \text{if } A \ne \phi, \\ 0 & \text{if } A = \phi. \end{cases}$$

It is not difficult to show that v^* is an outer measure on $\mathrm{P}([a, b])$, i.e., $v^*(\phi) = 0$, $v^*(A) \le v^*(B)$ if $A \subset B$ and

$$v^* \left(\bigcup_{n=1}^{\infty} A_n \right) \le \sum_{n=1}^{\infty} v^*(A_n)$$

for any sequence $(A_n)_n$ of subsets of the interval $[a, b]$.

Moreover, if $A, B \in \mathrm{P}([a,b])$ are such that $\widehat{A} \cap \widehat{B} = \emptyset$ then $v^*(A \cup B) = v^*(A) + v^*(B)$. In this case, the family B_f of all subsets M of $[a,b]$ which are v^*-measurable is a σ-algebra on $[a,b]$, the restruction to B_f of v^* is a positive measure (called the variation measure associated with f) and B_f contains any open subsect of $[a,b]$. Moreover, for any $\alpha, \beta \in [a,b]$, $\alpha < \beta$, we have

$$v^*(\alpha, \beta) = v[\alpha, \beta].$$

It is interesting to remark that if $v[a,b] < \infty$ then

$$v^*[\alpha,\beta] = v^*(\alpha,\beta) = v[\alpha,\beta], \quad \forall \alpha, \beta \in [a,b],$$
$$v^*(\{\alpha\}) = 0, \quad \forall \alpha \in (a,b].$$

In the general case we have $v^*(\{\alpha\}) \in \{0, \infty\}$ and for any sequence $(A_n)_n$ from B_f such that $A_n \cap A_m = \phi$ if $n \neq m$ we have

$$v^*\left(A \cap \left(\bigcup_n A_n\right)\right) = \sum_n v^*(A \cap A_n).$$

Moreover, for any increasing sequence $(A_n)_n$, $A_n \subset [a,b]$ we have $v^*(\bigcup_n A_n) = \sup_n v^*(A_n)$

2. OUTER LEBESGUE INTEGRAL

For any function $g : [a,b] \to \overline{\mathbb{R}}$ we associate an element of $\overline{\mathbb{R}}$ denoted by $\int^*_{[a,b]} g \mathrm{d}\lambda$ or $\int^*_{[a,b]} g \mathrm{d}t$ given by

$$\int^*_{[a,b]} g\mathrm{d}\lambda = \inf \left\{ \int_{[a,b]} \varphi \mathrm{d}\lambda | \varphi : [a,b] \to (-\infty, \infty],\ \varphi \text{ lower semicontinuous}, \ \varphi \geq g \right\}.$$

Usually, this element is called the *outer Lebesgue integral* of the function g with respect to the Lebesgue measure λ on $[a,b]$.

In the same way it is defined the *interior Lebesgue integral* of the function g on the interval $[a,b]$ with respect to the Lebesgue measure λ, namely,

$$\int_{*[a,b]} g\mathrm{d}\lambda =$$
$$= \sup \left\{ \int_{[a,b]} \Psi \mathrm{d}\lambda,\ \Psi : [a,b] \to [-\infty, \infty),\ \Psi \leq g,\ \Psi \text{ upper semicontinuous} \right\}.$$

Obviously, we have

$$\int_{*[a,b]} g\mathrm{d}\lambda = -\int^*_{[a,b]} (-g)\mathrm{d}\lambda \leq \int^*_{[a,b]} g\mathrm{d}\lambda.$$

It is known that the function g is integrable with respect to the Lebesgue measure λ on $[a,b]$ iff we have

$$\int_{[a,b]}^{*} g\mathrm{d}\lambda = \int_{*[a,b]} g\mathrm{d}\lambda \in \mathbb{R}.$$

In this case we have

$$\int_{[a,b]} g\mathrm{d}\lambda = \int_{[a,b]}^{*} g\mathrm{d}\lambda = \int_{*[a,b]} g\mathrm{d}\lambda.$$

3. THE KEY LEMMA

As above, we consider a continuous function $f : [a,b] \to \mathbb{R}$ and for any point $x_0 \in [a,b]$ we denote

$$Df(x_0) = \lim_{x\to x_0} \sup \frac{f(x) - f(x_0)}{x - x_0}.$$

Let $A_1 = \{x_1, x_2, x_3, \ldots\}$ be a countable subset of the interval $[a,b]$, A_2 an arbitrary subset of $[a,b]$ and G an open subset of $[a,b]$ (endowed with the topology associated with the usual distance $d(x,y) = |x-y|$) such that

$$[a,b] = A_1 \cup A_2 \cup G.$$

Let further φ be a lower semicontinuous function

$$\varphi : [a,b] \to (-\infty, \infty]$$

such that

$$Df(x)1_{A_2}(x) \le \varphi(x), \quad \forall x \in [a,b]$$

and let us suppose $Df(x) < \infty$ for all $x \in A_2$ and $\varepsilon, \varepsilon_{x_n}$ be strictly positive real numbers for all $n \in \mathbb{N}$ such that $\sum_n \varepsilon_{x_n} \le \varepsilon$.

LEMMA 1. *If f, A_1, A_2, G, φ, ε, εx_n are as above, we have*

$$f(x) - f(a) \le \sum_{x_i<x} \varepsilon x_i + \int_{[a,x]} \varphi(t)\mathrm{d}t + v([a,x]\cap G) + \varepsilon(x-a), \quad \forall x \in [a,b],$$

where v is the variation measure associated with f.

Proof. Since φ is lower semicontinuous and $\varphi(x) > -\infty$ for all $x \in [a,b]$ we deduce that φ is lower bounded and the integral $\int_{[a,x]} \varphi(t)\mathrm{d}t > -\infty$ for any $x \in [a,b]$. If there exists $x_0 \in [a,b]$ such that $\int_{[a,x_0]} \varphi(t)\mathrm{d}t = +\infty$, then we have $\int_{[a,x]} \varphi(t)\mathrm{d}t = +\infty$ for all $x \in [x_0, b]$. The function

$$x \to \sum_{x_i<x} \varepsilon x_i, x \to v([a,x]\cap G)$$

being increasing, the stated inequality holds for any $x \in [x_0, b]$ if there exists a point $x_0 \in [a, b]$ such that $v([a, x_0] \cap G) = +\infty$ or $\int_{[a,x_0]} \varphi(t)\mathrm{d}t = +\infty$.

So, we may consider a point $y_0 \in [a, b]$ such that $v([a, x] \cap G) < \infty$ and $\int_{[a,x]} \varphi(t)\mathrm{d}t < \infty$, $\forall x \in [a, y_0)$.

For our purpose we may suppose that $y_0 = b$. Let now M be the subset of $[a, b]$ given by

$$M = \left\{ y \in [a, b] \mid f(x) - f(a) \leq \sum_{x_i < x} \varepsilon x_i + \int_{[a,x]} \varphi(t)\mathrm{d}t + \varepsilon(x - a),\ \forall x \in [a, y] \right\}.$$

Obviously, $a \in M$ and we shall denote $z_0 = \sup M$. We have to show that $z_0 = b$. We suppose the contrary, i.e., $z_0 < b$.

The following situations arise.

Case 1. $z_0 \in A_1$, i.e., there exists $n_0 \in \mathbb{N}$ such that $z_0 = x_{n_0}$. Since $z_0 = x_{n_0}$, $\varepsilon_{x_{n_0}} > 0$ and

$$f(z_0) - f(a) \leq \sum_{x_i < z_0} \varepsilon x_i + \int_{[a,z_0]} \varphi(t)\mathrm{d}t + v([a, z_0] \cap G) + \varepsilon(z_0 - a) < +\infty$$

we deduce

$$f(z_0) - f(a) < \sum_{x_i \leq z_0} \varepsilon x_i + \int_{[a,z_0]} \varphi(t)\mathrm{d}t + v([a, z_0] \cap G) + \varepsilon(z_0 - a) < \infty$$

and therefore, taking into account that the functions

$$x \to f(x), \quad x \to \int_{[a,x]} \varphi(t)\mathrm{d}t, \quad x \to v([a, x] \cap G), \quad x \to \varepsilon(x - a)$$

are continuous (they are finite) we get

$$f(x) - f(a) < \sum_{x_i < x} \varepsilon x_i + \int_{[a,x]} \varphi(t)\mathrm{d}t + v([a, x] \cap G) + \varepsilon(x - a),$$

for any $x \in [z_0, z_0 + \eta]$ where $\eta \in \mathbb{R}$, $\eta > 0$ is sufficiently small. Hence $z_0 + \eta \in M$ and this fact contradicts the choice of z_0.

Case 2. $z_0 \in A_2$. Since $Df(z_0) < \infty$ and $\varphi(z_0) \geq Df(z_0)$ there exists $\alpha \in \mathbb{R}$ such that

$$Df(z_0) < \alpha < \varphi(z_0) + \varepsilon.$$

Using the fact that φ is a lover semicontinuous function we have $\alpha < \varphi(x) + \varepsilon$ for any x belonging to a neighbourhood of z_0. On the other hand, from the definition of $Df(z_0)$ we deduce that we have

$$\frac{f(x) - f(z_0)}{x - z_0} < \alpha$$

for any $x(x \neq z_0)$ from a neighbourhood of z_0. Hence there exists $z > z_0$ such that

$$\frac{f(x)-f(z_0)}{x-z_0} < \alpha < \varphi(t)+\varepsilon, \quad \forall t, x \in (z_0, z]$$

and therefore, by integration on the interval $(z_0, x]$,

$$f(x)-f(z_0) < \int_{[z_0,x]} \varphi(t)\mathrm{d}t + \varepsilon(x-z_0).$$

Since by the hypothesis we have

$$f(z_0)-f(a) \leq \sum_{x_i<z_0} \varepsilon x_i + \int_{[a,z_0]} \varphi(t)\mathrm{d}t + v([a,z_0]\cap G) + \varepsilon(z_0-a),$$

we deduce

$$f(x)-f(a) \leq \sum_{x_i<z_0} \varepsilon x_i + \int_{[a,x]} \varphi(t)\mathrm{d}t + v([a,z_0]\cap G) + \varepsilon(x-a), \quad \forall x \in (z_0, z]$$

and therefore

$$f(x)-f(a) \leq \sum_{x_i<x} \varepsilon x_i + \int_{[a,x]} \varphi(t)\mathrm{d}t + v([a,x]\cap G) + \varepsilon(x-a), \quad \forall x \in (z_0, z].$$

Since by the hypotheses the above inequality holds also for any $x \in [a, z_0]$, we deduce that $z \in M$ and again the contradictory relation $z > z_0$.

Case 3. $z_0 \in G \setminus A_2$. Since G is a countable union of pairwise disjoint open interval of the topological space $[a, b]$ and $z_0 < b$ we deduce that there exists $c \in (z_0, b)$ such that $[z_0, c] \subset G$. On the other hand, $z_0 \notin A_2$ and therefore $\varphi(z_0) \geq 0$. Using again the fact that φ is lower semicontinuous we may choose $c' \in (z_0, c)$ such that $\varphi(t) > -\varepsilon$ for any $t \in (z_0, c')$ and therefore we have

$$\int_{[z_0,x]} \varphi(t)\mathrm{d}t > -\varepsilon(x-z_0), \quad -\int \varphi(t)\mathrm{d}t < \varepsilon(x-z_0), \quad \forall x \in [z_0, c'].$$

From the previous considerations we have also

$$f(x)-f(z_0) \leq v([z_0,x]) = v([z_0,x]\cap G), \quad \forall x \in [z_0, c']$$

and therefore

$$f(x) - f(a) = f(z_0) - f(a) + f(x) - f(z_0) \le$$
$$\le \sum_{x_i < z_0} \varepsilon x_i + \int_{[a,z_0]} \varphi(t)\mathrm{d}t + v([a,z_0] \cap G) + \varepsilon(z_0 - a) + v([z_0,x] \cap G) \le$$
$$\le \sum_{x_i < x} \varepsilon x_i + \int_{[a,x]} \varphi(t)\mathrm{d}t - \int_{[z_0,x]} \varphi(t)\mathrm{d}t + v([a,x] \cap G) + \varepsilon(z_0 - a) \le$$
$$\le \sum_{x_i < x} \varepsilon x_i + \int_{[a,x]} \varphi(t)\mathrm{d}t + v([a,x] \cap G) + \varepsilon(x - a)$$

for all $x \in [z_0, c']$. Hence we arrive again to a contradiction and we get $z_0 = b$. □

4. THE MAIN RESULT

As before, $f : [a,b] \to \mathbb{R}$ will be a continuous function, v, respectively v^*, denote the variation, respectively, outer variation of the function f defined as above.

THEOREM 2. *Let A and B be two subset of the interval $[a,b]$ such that the set $[a,b] \backslash (A \cup B)$ is at most countable and $Df(x) < \infty$ for any $x \in A$. We have*

$$f(b) - f(a) \le \int_{[a,b]}^{*} Df \cdot 1_A \mathrm{d}x + v^*(B)$$

whenever the sum from the right hand side makes sense.

Proof. We suppose that $v^*(B) < \infty$ or $\int_{[a,b]}^{*} Df \cdot 1_A \mathrm{d}x > -\infty$. In the first case $v^*(B) < \infty$ we consider an arbitrary real number $\varepsilon > 0$ such that $B \subset G$ and $v(G) < v^*(B) + \varepsilon$. The set $[a,b] \backslash (A \cup G) = A_1$ is at most countable. If $x_1, x_2, x_3, \dots$ are the points of A_1 we consider the real numbers $\varepsilon_{x_i} > 0$, $i = 1, 2, 3, \dots$ such that $\sum_i \varepsilon_{x_i} \le \varepsilon$.

Using now Lemma we obtain

$$f(b) - f(a) \le \sum_i \varepsilon_i + \int_{[a,b]}^{*} Df \cdot 1_A \mathrm{d}x + v(G) + \varepsilon(b - a),$$
$$f(b) - f(a) \le \varepsilon + \varepsilon(b - a) + \varepsilon + v^*(B) + \int_{[a,b]}^{*} Df \cdot 1_A \mathrm{d}x.$$

The number ε being arbitrary, we get the stated assertion. □

Remark 3. In this situation, as corollary, we get

$$\int_{[a,b]}^{*} Df \cdot 1_A \mathrm{d}x > -\infty.$$

We suppose now that $-\infty < \int^*_{[a,b]} Df \cdot 1_A \mathrm{d}x < \infty$. Then the assertion of the above theorem is obvious if $v^*(B) = +\infty$. If $v^*(B) < \infty$ the assertion was already proved.

Corollary 4. *If $f : [a,b] \to \mathbb{R}$ is a continuous function which is derivable at any point $x \in A$ where $[a,b]\backslash A$ is at most countable and the function f' is integrable with respect to the Lebesgue measure λ we have*

$$f(b) - f(a) = \int_{[a,b]} f'(x)\mathrm{d}x.$$

Proof. Taking in the above theorem $B = \phi$, we have

$$f(b) - f(a) \leq \int_{[a,b]} f'(x)\mathrm{d}x.$$

On the other hand, applying the same results to the function $-f$ we get

$$-f(b) + f(a) \leq \int_{[a,b]} -f'(x)\mathrm{d}x = -\int_{[a,b]} f'(x)\mathrm{d}x$$

and therefore

$$f(b) - f(a) = \int_{[a,b]} f'(x)\mathrm{d}x. \quad \square$$

Corollary 5. *If $f : [a,b] \to \mathbb{R}$ is a continuous function which is derivable at any point $x \in A_1$ where $A \in \mathrm{B}_f$, if the function $Df \cdot 1_A$ is integrable with respect to the Lebesgue measure on $[a,b]$ and $v([a,b]\backslash A) = 0$, then we have*

$$f(b) - f(a) \leq \int_{[a,b]} Df \cdot 1_A \mathrm{d}x = \int_A f'\mathrm{d}x.$$

Proof. We apply again the above theorem for the functions f and $-f$. $\square$

Corollary 6 (Denjoy–Bourbaki). *Let E be Banach space and let $f : [a,b] \to E$, $\varphi : [a,b] \to \mathbb{R}$ two continuous functions witch are derivable outside of a countable subset A of $[a,b]$ and such that*

$$\|f'(x)\| \leq \varphi'(x), \quad \forall x \in [a,b]\backslash A.$$

We have

$$\|f(b) - f(a)\| \leq \varphi(b) - \varphi(a).$$

Proof. Let $L : E \to \mathbb{R}$ be a continuous linear functional such that $\|L\| = 1$ and $L(f(b) - f(a)) = \|f(b) - f(a)\|$. The function $g : [a,b] \to \mathbb{R}$ given by $g(x) = L(f(x))$ is continuous on $[a,b]$ and derivable at any point $[a,b]\backslash A$. We have

$$g'(x) = L(f'(x)), \quad |g'(x)| \leq \|L\| \cdot \|f'(x)\| \leq \varphi'(x), \quad \forall x \in [a,b]\backslash A.$$

Using the above theorem we obtain

$$\|f(b)-f(a)\|=\|g(b)-g(a)\| \leq \int_{[a,b]}^{*} g'\mathrm{d}x \leq \int_{[a,b]} \varphi'(x)\mathrm{d}x=\varphi(b)-\varphi(a). \quad \square$$

COROLLARY 7 (Denjoy–Bourbaki). *Let E be a Banach space and let $f : [a, b] \to E$ be a continuous function which is also derivable outside of a countable subset A of $[a, b]$. We have*

$$\|f(b) - f(a)\| \leq (b - a) \sup_{t\in[a,b]\setminus A} \|f'(t)\|.$$

Proof. We apply the previons corollary taking as φ the function $x \to Mx \in [a, b]$, where $M = \sup_{t\in[a,b]\setminus A} \|f'(t)\|$. $\square$

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