Dedicated to Professor Gheorghe Bucur
on the occasion of his 70th birthday

THE EQUIVALENCE OF CHEBYSHEV'S INEQUALITY TO THE HERMITE-HADAMARD INEQUALITY

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We prove that Chebyshev's inequality, Jensen's inequality and the Hermite-Hadamard inequality imply each other within the framework of probability measures. Some implications remain valid for certain classes of signed measures. By considering the case of semiconvex functions, we also prove a Kantorovich type variant of the Hermite-Hadamard inequality.

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1. INTRODUCTION

The Hermite-Hadamard inequality is a valuable tool in the theory of convex functions, providing a two-sided estimate for the mean value of a convex function with respect to a probability measure. Its formal statement is as follows:

THEOREM 1. *If $f : [a, b] \to \mathbb{R}$ is a continuous convex function, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}; \tag{1.1}$$

equality holds in either side only for the affine functions (i.e., for the functions of the form $mx + n$).

The middle point $(a + b)/2$ represents the barycenter of the probability measure $\frac{1}{b-a}\mathrm{d}x$ (viewed as a mass distribution over the interval $[a, b]$), while a and b represent the extreme points of $[a, b]$. Thus the Hermite-Hadamard inequality could be seen as a precursor of Choquet's theory. See [8] for details and further comments.

The optimal transport theory offers more insights into the mechanism of this inequality. In fact, from the point of view of that theory, the *barycenter* of a mass distribution on $[a, b]$, represented by a Borel probability measure μ, is the unique minimizer b_μ of the *transportation cost*,

$$C(y) = \frac{1}{2}\int_a^b |x-y|^2 \, \mathrm{d}\mu(x),$$

associated to the cost function $c(x, y) = \frac{1}{2}|x-y|^2$. See [11]. The transportation cost being uniformly convex, it attains its minimum at the unique root of its derivative, so that

$$b_\mu = \int_a^b x \mathrm{d}\mu(x).$$

It is useful to formulate the Hermite-Hadamard inequality (1.1) in the context of semiconvex functions and arbitrary mass distributions.

Definition 1. *A function $f : [a, b] \to \mathbb{R}$ is called semiconvex (of rate k) if the function $f + \frac{k}{2}|\cdot|^2$ is convex for some real constant k, that is,*

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y) + \frac{k}{2}\lambda(1-\lambda)|x-y|^2$$

for all $x, y \in [a, b]$ and all $\lambda \in [0, 1]$.

For example, every twice differentiable function f such that $f'' + k \geq 0$ is semiconvex of rate k. A semiconvex function of negative rate is usually known as an *uniformly convex* function.

A useful remark is that a function f is semiconvex of rate k if and only if for some point $x_0 \in [a, b]$ (equivalently, for any point x_0) the function

$$h(x) = f(x) + \frac{k}{2}|x - x_0|^2$$

is convex.

Based on Choquet's theory we can state the following generalization of the Hermite-Hadamard inequality in the context of semiconvex functions:

Theorem 2. *If μ is a Borel probability measure on an interval $[a, b]$, then for every semiconvex function $f : [a, b] \to \mathbb{R}$ of rate k we have*

$$f(b_\mu) \leq \int_a^b f(x)\mathrm{d}\mu(x) + \frac{k}{2}\int_a^b |x - b_\mu|^2 \, \mathrm{d}\mu(x) \tag{1.2}$$
$$\leq \frac{b - b_\mu}{b-a} \cdot f(a) + \frac{b_\mu - a}{b-a} \cdot f(b) + \frac{k}{2}(b_\mu - a)(b - b_\mu).$$

The term $(b_\mu - a)(b - b_\mu)$ represents the transportation cost of the mass δ_{b_μ} to $\frac{b-b_\mu}{b-a}\delta_a + \frac{b_\mu - a}{b-a}\delta_b$. This proves to be more expansive than the transportation cost of μ to δ_{b_μ} (a fact which follows from the right hand side inequality in

Theorem 2, when applied to the function $f(x) = |x - b_\mu|^2$ and $k = 0$). The difference of the two costs

$$D(\mu) = (b - b_\mu)(b_\mu - a) - \int_a^b |x - b_\mu|^2 \,\mathrm{d}\mu(x), \tag{1.3}$$

is thus nonnegative and it reflects both the geometry of the domain and the mass distribution on it.

The aim of this paper is to discuss the connection of the Hermite-Hadamard inequality to another classical result, Chebyshev's inequality.

THEOREM 3 (Chebyshev's inequality). *If $f, g : [a, b] \to \mathbb{R}$ are two monotonic functions of the same monotonicity, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \geq \left(\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right)\left(\frac{1}{b-a}\int_a^b g(x)\mathrm{d}x\right).$$

If f and g are of opposite monotonicity, then the above inequality works in the reverse way.

The proof of Theorem 3 is a direct consequence of the property of positivity of the integral. The basic remark is the inequality

$$(f(x) - f(y))(g(x) - g(y)) \geq 0 \quad \text{for all } x, y \in [a, b],$$

which is integrated with respect to x and y.

No argument of such simplicity is known for Theorem 1.

Remark 1. In the statement of Theorem 3, the interval $[a, b]$ can be replaced by any interval I, and the normalized Lebesgue measure $\frac{1}{b-a}\mathrm{d}x$ can be replaced by any Borel probability measure on the interval I. The argument remains the same!

Surprisingly, Theorem 1 can be derived from Theorem 3 (and vice-versa). This is shown in the next section. When the normalized Lebesgue measure is replaced by an arbitrary Borel probability measure (still defined on a compact interval), the left hand side inequality in Theorem 2 is equivalent to Jensen's inequality (for $k = 0$), which in turn is equivalent to Chebyshev's inequality.

Steffensen's extension of Jensen's inequality (see [8], p. 33), started the important subject of generalizing the classical inequalities to the framework of signed Borel measures.

Based on previous work done (independently) by T. Popoviciu and A.M. Fink, the first named author was able to provide a characterization of those signed Borel measures for which Jensen's inequality remains valid in full generality. See [8], Section 4.1, for details. A characterization of the signed Borel measures defined on a compact interval for which both sides of the Hermite-Hadamard inequality still work can be found in [4]. A tantalizing problem is the characterization of such measures in the case of several variables. A particular result in this direction is presented in [5].

The problem of characterizing the signed Borel measures for which Chebyshev's inequality remains valid was solved by Fink and Jodeit Jr. [3]. See also [6], Ch. IX. At about the same time Pečarić [9] has provided an elegant (partial) solution based on an identity describing the precision in Chebyshev's inequality. His approach is put here in full generality, being accompanied by an analogue of the Grüss inequality. See Theorem 5 below. As a consequence we are able to derive the Jensen-Steffensen inequality as well as an estimate of its precision.

The paper ends with a generalization of Kantorovich's inequality within the framework of signed measures.

2. PROOF OF THEOREM 2

We start by noticing that Theorem 3 is strong enough to yield the classical inequality of Jensen.

THEOREM 4 (Jensen's inequality). *If* (X, Σ, μ) *is a finite measure space,* $\varphi : X \to \mathbb{R}$ *is a* μ*-integrable function and* h *is a continuous convex function defined on an interval* I *containing the range of* φ*, then*

$$h\left(\frac{1}{\mu(X)}\int_X \varphi(x)\mathrm{d}\mu(x)\right) \leq \frac{1}{\mu(X)}\int_X h\left(\varphi(x)\right)\mathrm{d}\mu(x). \tag{2.1}$$

Proof. If $h : I \to \mathbb{R}$ is a convex function, and $c \in \operatorname{int} I$ is kept fixed, then the function

$$x \to \frac{h(x) - h(c)}{x - c}$$

is nondecreasing on $I\backslash\{c\}$. Let ν be a finite positive Borel measure on I with barycenter

$$b_\nu = \frac{1}{\nu(I)}\int_I x\mathrm{d}\nu(x).$$

Clearly, $b_\nu \in I$ (since otherwise $b_\nu - x$ or $x - b_\nu$ will provide an example of strictly positive function whose integral with respect to ν is 0). According to Chebyshev's inequality, if $b_\nu \in \operatorname{int} I$, then

$$\frac{1}{\nu(I)}\int_I \frac{h(x) - h(b_\nu)}{x - b_\nu}(x - b_\nu)\mathrm{d}\nu(x) \geq$$

$$\geq \frac{1}{\nu(I)}\int_I \frac{h(x) - h(b_\nu)}{x - b_\nu}\mathrm{d}\nu(x) \cdot \frac{1}{\nu(I)}\int_I (x - b_\nu)\,\mathrm{d}\nu(x) = 0,$$

which yields

$$h(b_\nu) \leq \frac{1}{\nu([a,b])} \int_a^b h(x)\mathrm{d}\nu(x). \tag{2.2}$$

Notice that the last inequality also works when b_ν is an endpoint of I (because in that case $\nu = \delta_{b_\nu}$).

Using the technique of pushing-forward measures, we can infer from (2.2) the general statement of the Jensen inequality. In fact, if μ is a positive Borel measure on X and $\varphi : X \to I$ is a μ-integrable map, then the push-forward measure $\nu = \varphi \# \mu$ is given by the formula $\nu(A) = \mu(\varphi^{-1}(A))$ and its barycenter is

$$\overline{\varphi} = \frac{1}{\mu(X)} \int_X \varphi(x)\mathrm{d}\mu(x).$$

According to (2.2),

$$h(\overline{\varphi}) \leq \frac{1}{\nu(I)} \int_I h(t)\mathrm{d}\nu(t) = \frac{1}{\mu(X)} \int_X h(\varphi(x))\mathrm{d}\mu(x)$$

for all continuous convex functions $h : I \to \mathbb{R}$, which ends the proof of Theorem 4. $\square$

It is clear that the inequality of Jensen implies in turn the positivity property of integral (and thus all are equivalent to the inequality of Chebyshev).

The inequality of Jensen and the inequality of Chebyshev are also dual each other. See [8], Section 1.8.

Coming back to the proof of Theorem 2, we will notice that the left hand side inequality in (2) is a consequence of the inequality of Jensen (2.1), applied to φ the identity of $[a,b]$, and to the convex function $h(x) = f(x) + \frac{k}{2}|x - b_\mu|^2$.

The right hand side inequality in (2) can be obtained in a similar manner, as a consequence of the following result:

LEMMA 1. *For every convex function $h : [a,b] \to \mathbb{R}$ and every Borel probability measure μ on $[a,b]$,*

$$\int_a^b h(x)\mathrm{d}\mu(x) \leq \frac{b - b_\mu}{b-a} \cdot h(a) + \frac{b_\mu - a}{b-a} \cdot h(b). \tag{2.3}$$

Proof. Formally, this is a special case of an important theorem due to Choquet. See [8], Section 4.4, for details. A more direct argument is in order.

If $\mu = \sum_{i=1}^n \lambda_i \delta_{x_i}$ is a discrete probability measure, then its barycenter is $b_\mu = \sum_{i=1}^n \lambda_i x_i$ and (2.3) takes the form

$$\sum_{i=1}^n \lambda_i h(x_i) \leq \frac{b - \sum\limits_{i=1}^n \lambda_i x_i}{b-a} h(a) + \frac{\sum\limits_{i=1}^n \lambda_i x_i - a}{b-a} h(b). \tag{2.4}$$

This inequality follows directly from the property of h of being convex. In fact,

$$x_i = \frac{b - x_i}{b - a} \cdot a + \frac{x_i - a}{b - a} \cdot b$$

which yields

$$h(x_i) \leq \frac{b - x_i}{b - a} \cdot h(a) + \frac{x_i - a}{b - a} \cdot h(b).$$

Multiplying both sides by λ_i and then summing over i we arrive at (2.4). The general case of (2.3) is now a consequence of the following approximation argument: every Borel probability measure λ on a compact Hausdorff space is the pointwise limit of a net of discrete probability measures, each having the same barycenter as λ (see [8], Lemma 4.1.10, p. 183). □

When $\mathrm{d}\mu = \frac{1}{b-a}\mathrm{d}x$ is the normalized Lebesgue measure on $[a, b]$, then $b_\mu = (a+b)/2$ and we can derive the inequality (2.3) directly from Chebyshev's inequality. In fact,

$$\frac{1}{b-a}\int_a^b \left(x - \frac{a+b}{2}\right) h'(x)\mathrm{d}x \geq$$

$$\geq \frac{1}{b-a}\int_a^b \left(x - \frac{a+b}{2}\right) \mathrm{d}x \cdot \frac{1}{b-a}\int_a^b h'(x)\mathrm{d}x = 0$$

and

$$\frac{1}{b-a}\int_a^b \left(x - \frac{a+b}{2}\right) h'(x)\mathrm{d}x =$$

$$= \frac{1}{b-a}\int_a^b \left(\left(x - \frac{a+b}{2}\right) h(x)\right)' \mathrm{d}x - \frac{1}{b-a}\int_a^b h(x)\mathrm{d}x =$$

$$= \frac{h(a)+h(b)}{2} - \frac{1}{b-a}\int_a^b h(x)\mathrm{d}x.$$

3. THE CASE OF SIGNED MEASURES

Is is well known that the integral inequalities with respect to signed measures are considerably more involving because the positivity property of the integral does not work in that context. Notably, both Chebyshev's inequality and Jensen's inequality admit extensions that work for certain signed measures.

The case of Chebyshev's inequality is discussed by J. Pečarić in [9], Theorem 1, in the case of Stieltjes measures associated to absolutely continuous functions. His basic argument is an identity whose validity can be established under less restrictive hypotheses:

Lemma 2. *Suppose that $p, f, g : [a,b] \to \mathbb{R}$ are functions of bounded variation and f and g are also continuous. Then*

$$(3.1)\qquad (p(b)-p(a))\int_a^b f(x)g(x)\mathrm{d}p(x) - \int_a^b f(x)\mathrm{d}p(x)\int_a^b g(x)\mathrm{d}p(x)$$

$$= \int_a^b p^*(x)\left(\int_a^x p_*(t)\mathrm{d}g(t)\right)\mathrm{d}f(x) + \int_a^b p_*(x)\left(\int_x^b p^*(t)\mathrm{d}g(t)\right)\mathrm{d}f(x),$$

where $p^*(x) = p(b) - p(x)$ and $p_*(x) = p(x) - p(a)$.

Proof. We start by noticing that the indefinite integral $\int_a^x h(t)\mathrm{d}p(t)$ of any continuous function h has bounded variation and verifies the formula

$$\int_a^b f(x)\mathrm{d}\left(\int_a^x h(t)\mathrm{d}p(t)\right) = \int_a^b f(x)h(x)\mathrm{d}p(x).$$

Thus for

$$h(t) = \int_a^b g(s)\mathrm{d}p(s) - (p(b)-p(a))\,g(t) = \int_a^b (g(s)-g(t))\,\mathrm{d}p(s),$$

the formula of integration by parts leads us to the identity

$$\int_a^b \left(\int_a^x h(t)\mathrm{d}p(t)\right)\mathrm{d}f(x) =$$

$$= (p(b)-p(a))\int_a^b f(x)g(x)\mathrm{d}p(x) - \int_a^b f(x)\mathrm{d}p(x)\int_a^b g(x)\mathrm{d}p(x).$$

On the other hand,

$$\int_a^x h(t)\mathrm{d}p(t) = (p(x)-p(a))\int_a^b g(t)\mathrm{d}p(t) - (p(b)-p(a))\int_a^x g(t)\mathrm{d}p(t) =$$

$$= (p(x)-p(a))\left(\int_a^x g(t)\mathrm{d}p(t) + \int_x^b g(s)\mathrm{d}p(s)\right) - (p(b)-p(a))\int_a^x g(t)\mathrm{d}p(t) =$$

$$= -p^*(x)\int_a^x g(t)\mathrm{d}p_*(t) - p_*(x)\int_x^b g(t)\mathrm{d}p^*(t) =$$

$$= p^*(x)\int_a^x p_*(t)\mathrm{d}g(t) + p_*(x)\left(\int_x^b p^*(t)\mathrm{d}g(t)\right),$$

so that

$$(p(b)-p(a))\int_a^b f(x)g(x)\mathrm{d}p(x) - \int_a^b f(x)\mathrm{d}p(x)\int_a^b g(x)\mathrm{d}p(x)$$

$$= \int_a^b p^*(x)\left(\int_a^x p_*(t)\mathrm{d}g(t)\right)\mathrm{d}f(x) + \int_a^b p_*(x)\left(\int_x^b p^*(t)\mathrm{d}g(t)\right)\mathrm{d}f(x). \qquad \square$$

An immediate consequence of Lemma 2 is the following stronger form of Chebyshev's inequality:

Theorem 5. *Suppose that $p : [a, b] \to \mathbb{R}$ is a function of bounded variation such that*

$$p(b) > p(a) \quad \text{and} \quad p(a) \le p(x) \le p(b) \quad \text{for all } x. \tag{3.2}$$

Then for every pair of continuous functions $f, g : [a, b] \to \mathbb{R}$ which are monotonic in the same sense, the Chebyshev functional

$$T(f, g; p) = \frac{1}{p(b) - p(a)} \int_a^b f(x)g(x)\mathrm{d}p(x) - \\ - \left(\frac{1}{p(b) - p(a)} \int_a^b f(x)\mathrm{d}p(x) \right) \left(\frac{1}{p(b) - p(a)} \int_a^b g(x)\mathrm{d}p(x) \right),$$

is bounded from above by

$$\frac{1}{p(b)-p(a)} \int_a^b \max\{p^*(x), p_*(x)\}\mathrm{d}f(x) \cdot \frac{1}{p(b)-p(a)} \int_a^b \max\{p^*(x), p_*(x)\}\mathrm{d}g(x) \\ \le (f(b) - f(a))(g(b) - g(a)),$$

and is bounded from below by

$$\frac{1}{p(b)-p(a)} \int_a^b \min\{p^*(x), p_*(x)\}\mathrm{d}f(x) \cdot \frac{1}{p(b)-p(a)} \int_a^b \min\{p^*(x), p_*(x)\}\mathrm{d}g(x) \ge 0.$$

Proof. In fact, we may assume that both functions f and g are nondecreasing (changing f and g by $-f$ and $-g$ if necessary). Since the integral of a nonnegative function with respect to a nondecreasing function is nonnegative, we have

$$\int_a^b p^*(x) \left(\int_a^x p_*(t)\mathrm{d}g(t) \right) \mathrm{d}f(x) + \int_a^b p_*(x) \left(\int_x^b p^*(t)\mathrm{d}g(t) \right) \mathrm{d}f(x) \ge \\ \ge \int_a^b \min\{p^*(x), p_*(x)\} \left(\int_a^x \min\{p^*(t), p_*(t)\} \mathrm{d}g(t) \right) \mathrm{d}f(x) + \\ + \int_a^b \min\{p^*(x), p_*(x)\} \left(\int_x^b \min\{p^*(t), p_*(t)\} \mathrm{d}g(t) \right) \mathrm{d}f(x) = \\ = \int_a^b \min\{p^*(x), p_*(x)\} \mathrm{d}f(x) \cdot \int_a^b \min\{p^*(x), p_*(x)\} \mathrm{d}g(x)$$

and

$$\int_a^b p^*(x)\left(\int_a^x p_*(t)\mathrm{d}g(t)\right)\mathrm{d}f(x)+\int_a^b p_*(x)\left(\int_x^b p^*(t)\mathrm{d}g(t)\right)\mathrm{d}f(x)\leq$$
$$\leq\int_a^b \max\{p^*(x),p_*(x)\}\,\mathrm{d}f(x)\cdot\int_a^b \max\{p^*(x),p_*(x)\}\,\mathrm{d}g(x).\quad\square$$

Remark 2. As was noticed by Pečarić [9], very efficient bounds can be indicated in the framework of C^1-differentiability of f and g. For example, if f and g are monotonic in the same sense, and $|f'|\leq\alpha$ and $|g'|\leq\beta$ (for some positive constants α and β), then

$$T(f,g;p)\leq\alpha\beta T(x-a,x-a;p).$$

The necessity of the condition (3.2) for the validity of Chebyshev's inequality is discussed in [3].

Theorem 5 yields the following improvement on the Jensen-Steffensen inequality:

THEOREM 6. *Suppose that $p:[a,b]\to\mathbb{R}$ is a function with bounded variation such that $p(b)>p(a)$ and $p(a)\leq p(x)\leq p(b)$ for all x. Then for every continuous and strictly monotonic function $\varphi:[a,b]\to\mathbb{R}$ and for every continuous convex function h defined on an interval I that contains the image of φ, the inequality*

$$0\leq\frac{1}{p(b)-p(a)}\int_a^b h\left(\varphi(x)\right)\mathrm{d}p(x)-h\left(b_p\right)\leq$$
$$\leq\left(\frac{h(\varphi(b))-h(b_p)}{\varphi(b)-b_p}-\frac{h(\varphi(a))-h(b_p)}{\varphi(a)-b_p}\right)\left(\varphi(b)-\varphi(a)\right).$$

holds. Here

$$b_p=\frac{1}{p(b)-p(a)}\int_a^b \varphi(x)\mathrm{d}p(x)$$

represents the barycenter of the push-forward measure $\varphi\#\mathrm{d}p$.

Proof. Using an approximation argument we may restrict ourselves to the case where h is differentiable. The functions $\varphi(x)$ and $\frac{h(\varphi(x))-h(b_p)}{\varphi(x)-b_p}$ being monotonic in the same sense, Theorem 5 applies and it gives us

$$\frac{1}{p(b)-p(a)}\int_a^b\frac{h(\varphi(x))-h(b_p)}{\varphi(x)-b_p}(\varphi(x)-b_p)\mathrm{d}p(x)\geq$$
$$\geq\frac{1}{p(b)-p(a)}\int_a^b\frac{h(\varphi(x))-h(b_p)}{\varphi(x)-b_p}\mathrm{d}p(x)\cdot\frac{1}{p(b)-p(a)}\int_a^b(\varphi(x)-b_p)\mathrm{d}p(x)=0.$$

Thus

$$h(b_p) \leq \frac{1}{p(b)-p(a)} \int_a^b h\left(\varphi(x)\right) \mathrm{d}p(x).$$

The proof ends by taking into account the discrepancy in Chebyshev's inequality (provided by Theorem 5). □

Remark 3. According to Lemma 2,

$$\frac{1}{p(b)-p(a)} \int_a^b h\left(\varphi(x)\right) \mathrm{d}p(x) - h\left(b_p\right) =$$
$$= \frac{1}{\left(p(b)-p(a)\right)^2} \int_a^b \left(p^*(x) \int_a^x p_*(t) \mathrm{d}\varphi(t)\right) \mathrm{d}\left(\frac{h(\varphi(x))-h(b_p)}{\varphi(x)-b_p}\right) +$$
$$+ \frac{1}{\left(p(b)-p(a)\right)^2} \int_a^b \left(p_*(x) \int_x^b p^*(t) \mathrm{d}\varphi(t)\right) \mathrm{d}\left(\frac{h(\varphi(x))-h(b_p)}{\varphi(x)-b_p}\right),$$

and this fact may be used to derive better bounds for the discrepancy in Chebyshev's inequality.

The fact that Theorem 1.2 works outside the framework of positive Borel measures is discussed in [4] (under the nonrestrictive assumption that $k = 0$).

Let us call a real Borel measure μ on $[a, b]$, with $\mu\left([a, b]\right) > 0$, a *Hermite-Hadamard measure* if for every convex function $f : [a, b] \to \mathbb{R}$ the following two inequalities hold true,

$$\text{(LHH)} \qquad f(b_\mu) \leq \frac{1}{\mu([a,b])} \int_a^b f(x) \mathrm{d}\mu(x)$$

and

$$\text{(RHH)} \qquad \frac{1}{\mu([a,b])} \int_a^b f(x) \mathrm{d}\mu(x) \leq \frac{b-b_\mu}{b-a} \cdot f(a) + \frac{b_\mu - a}{b-a} \cdot f(b),$$

where $b_\mu = \frac{1}{\mu([a,b])} \int_a^b x \mathrm{d}\mu(x)$.

According to [4], the measure $(x^2 + a)\mathrm{d}x$ verifies (LHH) (respectively (RHH)) for all convex functions defined on the interval $[-1, 1]$ if and and only if $a \geq -1/3$ (respectively $a \geq -1/6$).

As a consequence, we can exhibit examples of functions of bounded variation as in Theorem 5, which do not give rise to Hermite-Hadamard measures. So is the function $p(x) = (x^3 - \frac{3x}{4})$, for $x \in [-1, 1]$, which clearly verifies the condition $p(\cos x) = \frac{1}{4} \cos 3x \in [p(-1), p(1)]$. However, the associated measure $\mathrm{d}\mu = 3(x^2 - \frac{1}{4})\mathrm{d}x$ does not verify (RHH) for all convex functions defined on the interval $[-1, 1]$.

4. AN APPLICATION TO KANTOROVICH'S INEQUALITY

We start with the following variant of Theorem 2.

THEOREM 7. *Suppose that μ is a Hermite-Hadamard measure on $[a,b]$, with $b_\mu > 0$. Then for every k-convex function $f:[a,b]\to\mathbb{R}$ such that $f(a)>f(b)$,*

$$f(b_\mu)-\frac{k}{2}\int_a^b |x-b_\mu|^2\,\mathrm{d}\mu(x)\leq\int_a^b f(x)\mathrm{d}\mu(x)\leq$$
$$\leq\frac{\left[bf(a)-af(b)+\frac{1}{2}k(b-a)D(\mu)\right]^2}{4b_\mu\,(b-a)\,(f(a)-f(b))},$$

where $D(\mu)$ is given by the formula (1.3).

Proof. The left hand side inequality is equivalent to the left hand side inequality in Theorem 2. As concerns the right hand side inequality, it suffices to consider the case where $\int_a^b f(x)\mathrm{d}\mu(x)>0$. In this case, we make again an appeal to Theorem 2 in order to infer that

$$(b-a)\int_a^b f(x)\mathrm{d}\mu(x)+(b-a)\frac{k}{2}\int_a^b |x-b_\mu|^2\,\mathrm{d}\mu(x)\leq$$
$$\leq bf(a)-af(b)+(f(b)-f(a))b_\mu+\frac{k}{2}(b-a)\,(b-b_\mu)\,(b_\mu-a)\,.$$

This yields

$$bf(a)-af(b)+\frac{k}{2}(b-a)\left[(b-b_\mu)\,(b_\mu-a)-\int_a^b |x-b_\mu|^2\,\mathrm{d}\mu(x)\right]\geq$$
$$\geq(b-a)\int_a^b f(x)\mathrm{d}\mu(x)+(f(a)-f(b))b_\mu\geq$$
$$\geq 2\left[b_\mu\,(b-a)\,(f(a)-f(b))\int_a^b f(x)\mathrm{d}\mu(x)\right]^{1/2},$$

and the proof is done. □

COROLLARY 1. *If $f:[a,b]\to\mathbb{R}$ is a convex function such that $f(a)>f(b)$, and μ is a Hermite-Hadamard measure on $[a,b]$ with $b_\mu>0$, then*

$$\int_a^b f(x)\mathrm{d}\mu(x)\leq\frac{(bf(a)-af(b))^2}{4b_\mu\,(b-a)\,(f(a)-f(b))}.$$

In the discrete case, when μ is a Borel probability measure of the form $\mu=\sum_{i=1}^n\lambda_i\delta_{x_i}$, the result of Corollary 1 reads as

$$\left(\sum_{i=1}^n\lambda_i x_i\right)\left(\sum_{i=1}^n\lambda_i f(x_i)\right)\leq\frac{(bf(a)-af(b))^2}{4\,(b-a)\,(f(a)-f(b))}.$$

If we apply this remark to the convex function $f(x) = 1/x$ for $x \in [m, M]$ (where $0 < m < M$), we recover a classical inequality due to Kantorovich:

$$\left(\sum_{i=1}^{n} \lambda_i x_i\right)\left(\sum_{i=1}^{n} \frac{\lambda_i}{x_i}\right) \leq \frac{(M+m)^2}{4Mm},$$

for all $x_1, \ldots, x_n \in [m, M]$ and all $\lambda_1, \ldots, \lambda_n \in [0, 1]$ with $\sum_{i=1}^{n} \lambda_i = 1$. See [1].

However, our approach yields a bit more. Precisely, the Kantorovich inequality still works for those Hermite-Hadamard measures $\sum_{i=1}^{n} \lambda_i \delta_{x_i}$ (on $[m, M]$) such that $\sum_{i=1}^{n} \lambda_i x_i \in [m, M]$.

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