The notion of differential superordination was introduced by Miller and Mocanu [3] as a dual concept of differential subordination [2] and was developed in [4]. The notion of strong differential subordination was introduced by Antonino and Romaguera [1]. The notion was developed in [8], [9], [10]. In [5] the author introduced the dual concept of strong differential superordinations. In this paper, a Briot-Bouquet strong differential superordination is studied.

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1. INTRODUCTION AND PRELIMINARIES

Let the unit disc of the complex plane
\[ U = \{ z \in \mathbb{C} : |z| < 1 \} \quad \text{and} \quad \overline{U} = \{ z \in \mathbb{C} : |z| \leq 1 \}. \]

Let \( \mathcal{H}(U \times \overline{U}) \) denote the space of holomorphic functions in \( U \times \overline{U} \). For \( n \) a positive integer and \( a \in \mathbb{C} \), in [7] the authors introduced the classes
\[ \mathcal{H}^*[a,n,\xi] = \{ f \in \mathcal{H}(U \times \overline{U}) : f(z,\xi) = a + a_n(\xi)z^n + a_{n+1}(\xi)z^{n+1} + \ldots, \ z \in U, \ \xi \in \overline{U} \}, \]

with \( a_k(\xi) \) holomorphic functions in \( \overline{U} \), \( k \geq n \), and
\[ \mathcal{H}_u(U) = \{ f \in \mathcal{H}^*[a,n,\xi] : f(\cdot,\xi) \text{ univalent in } U \text{ for all } \xi \in \overline{U} \}, \]
\[ K = \left\{ f \in \mathcal{H}^*[a,n,\xi] : \text{Re} \frac{zf''(z,\xi)}{f'(z,\xi)} + 1 > 0, \ z \in U \text{ for all } \xi \in \overline{U} \right\} \]
the class of convex functions.

Definition 1 ([6]). We denote by \( Q \) the set of function \( f(\cdot,\xi) \) that are analytic and injective on the set \( \overline{U} \setminus E(f) \), where
\[ E(f) = \left\{ \xi \in \partial U : \lim_{z \to \xi} f(z,\xi) = \infty, \ z \in U, \ \xi \in \overline{U} \right\} \]

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and are such that \( f'(z, \xi) \neq 0 \) for \( \zeta \in \partial U \setminus E(f), \xi \in \bar{U} \).

The subclass of \( Q \) for which \( f(0, \xi) \equiv a \) is denoted by \( Q(a) \).

**Definition 2 ([7]).** Let \( f(z, \xi) \) and \( H(z, \xi) \) be analytic in \( U \times \bar{U} \). The function \( f(z, \xi) \) is said to be strongly **subordinate** to \( H(z, \xi) \), or \( H(z, \xi) \) is said to be strongly **superordinate** to \( f(z, \xi) \), if there exists a function \( w \) analytic in \( U \), with \( w(0) = 0 \) and \( |w'(z)| < 1 \), and such that \( f(z, \xi) = H(w(z), \xi) \) for all \( \xi \in \bar{U} \). In such a case we write

\[
f(z, \xi) \prec \prec H(z, \xi), \quad z \in U, \quad \xi \in U.
\]

If \( H(z, \xi) \) is univalent in \( U \), for all \( \xi \in \bar{U} \), then \( f(z, \xi) \prec \prec F(z, \xi) \) if and only if \( f(0, \xi) = F(0, \xi) \) and \( f(U \times \bar{U}) \subset F(U \times \bar{U}) \).

**Remark 1.** If \( H(z, \xi) \equiv H(z) \), and \( f(z, \xi) \equiv f(z) \), then the strong superordination becomes the usual notion of superordination.

Let \( \beta \) and \( \gamma \) be complex numbers, let \( \Omega_{\xi} \) and \( \Delta_{\xi} \) be sets in the complex plane, and \( p(\cdot, \xi) \) analytic in \( U \times \bar{U} \).

In [6] the authors have determined conditions such that

\[
(1) \quad \left\{ p(z, \xi) + \frac{zp'(z, \xi)}{\beta p(z, \xi) + \gamma} \right\} \subset \Omega_{\xi} \Rightarrow p(U \times \bar{U}) \subset \Delta_{\xi}, \quad z \in U, \quad \xi \in \bar{U}.
\]

In this article we consider the dual problem of determining conditions such that

\[
(2) \quad \Omega_{\xi} \subset \left\{ p(z, \xi) + \frac{zp'(z, \xi)}{\beta p(z, \xi) + \gamma} \right\} \Rightarrow \Delta_{\xi} \subset p(U \times \bar{U}), \quad z \in U, \quad \xi \in \bar{U}.
\]

In particular, we are interested in determining the largest set \( \Delta_{\xi} \) in \( \mathbb{C} \) for which (2) holds.

If the sets \( \Omega_{\xi} \) and \( \Delta_{\xi} \) in (1) and (2) are simply connected domains not equal to \( \mathbb{C} \), then it is possible to rephrase these expressions very neatly in terms of strong subordination and to obtain

\[
(1') \quad p(z, \xi) + \frac{zp'(z, \xi)}{\beta p(z, \xi) + \gamma} \prec \prec h_2(z, \xi) \Rightarrow p(z, \xi) \prec \prec q_2(z, \xi),
\]

\[
(2') \quad h_1(z, \xi) \prec \prec p(z, \xi) + \frac{zp'(z, \xi)}{\beta p(z, \xi) + \gamma} \Rightarrow g_1(z, \xi) \prec \prec p(z, \xi), \quad z \in U, \quad \xi \in \bar{U}.
\]

The left side of (1') is called a Briot-Bouquet strong differential subordination, and the function \( q_2 \) is called a **dominant** of the differential subordination. The best dominant, which provides a sharp result, is the dominant that is subordinate to all other dominant.

The left side of (2') is called a Briot-Bouquet strong differential superordination, and the function \( q_1(\cdot, \xi) \) is called a **subordinant** of the strong
differential subordination. The **best subordinant**, which provides a sharp
result is the subordinant which is superordinate to all other subordinants.

**Definition 3 ([5])**. Let \( \Omega_\xi \) be a set in \( \mathbb{C} \) and \( q(\cdot, \xi) \in \mathcal{H}^*[a, n, \xi] \) with
\( q'(z, \xi) \neq 0, z \in U, \xi \in \overline{U} \). The class of admissible functions \( \phi_n[\Omega_\xi, q(\cdot, \xi)] \)
consists of those functions \( \varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C} \) that satisfy the admissibility
condition:

\[
(A) \quad \varphi(r, s; t; \xi, \xi) \in \Omega_\xi
\]

whenever

\[
r = q(z, \xi), \quad s = \frac{zq'(z, \xi)}{m}, \quad \text{Re} \frac{t}{s} + 1 \leq \frac{1}{m} \text{Re} \left[ \frac{zq''(z, \xi)}{q'(z, \xi)} + 1 \right],
\]

where \( \xi \in \partial U, z \in U, \xi \in \overline{U} \) and \( m \geq n \geq 1 \). When \( n = 1 \) we write
\( \phi_1[\Omega_\xi, q(\cdot, \xi)] \) as \( \phi[\Omega_\xi, q(\cdot, \xi)] \).

In the special case when \( h(\cdot, \xi) \) is an analytic mapping of \( U \times \overline{U} \) onto
\( \Omega_\xi \neq \mathbb{C} \) we denote this class \( \phi_n[h(U \times \overline{U}), q(\cdot, \xi)] \) by \( \phi_n[h(\cdot, \xi), q(\cdot, \xi)] \).

If \( \varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C} \) and \( q(\cdot, \xi) \in \mathcal{H}^*[a, n, \xi] \), then the admissibility
condition \( (A) \) reduces to

\[
(A') \quad \varphi \left( q(z, \xi), \frac{zq'(z, \xi)}{m}; \xi, \xi \right) \in \Omega_\xi
\]

whenever \( r = q(z, \xi), s = \frac{zq'(z, \xi)}{m}, \) where \( z \in U, \xi \in \overline{U}, \xi \in \partial U \) and \( m \geq n \geq 1 \).

**Lemma A ([6])**. Let \( p(\cdot, \xi) \in Q(a) \), and let

\[
q(z, \xi) = a + a_n(\xi)z^n + a_{n+1}(\xi)z^{n+1} + \ldots
\]

be analytic in \( U \times \overline{U} \) with \( q(z, \xi) \neq a \) and \( a \geq 1 \). If \( q(\cdot, \xi) \) is not subordinate
to \( p(\cdot, \xi) \), then there exist points \( z_0 = r_0e^{ib_0} \in U \) and \( \zeta_0 \in \partial U \setminus E(p), \) and \( m \geq n \geq 1 \) for which
\( q(U_{r_0} \times \overline{U}_{r_0}) \subset p(U \times \overline{U}) \),

(i) \( q(z_0, \xi) = p(\xi, \xi) \),
(ii) \( z_0q'(z_0, \xi) = m\zeta_0 p'(\xi, \xi) \) and
(iii) \( \text{Re} \frac{z_0q''(z_0, \xi)}{q'(z_0, \xi)} + 1 \geq m \text{Re} \left[ \frac{z_0 p''(\xi, \xi)}{p'(\xi, \xi)} + 1 \right] \).

**2. MAIN RESULTS**

**Theorem 1**. Let \( \Omega_\xi \subset \mathbb{C}, q(\cdot, \xi) \in \mathcal{H}^*[a, n, \xi], \varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}, \) and
suppose that

\[
(3) \quad \varphi(q(z, \xi), tzq'(z, \xi); \xi, \xi) \in \Omega_\xi
\]
for $z \in U$, $\xi \in \partial U$, $\xi \in U$ and $0 < t \leq \frac{1}{n} \leq 1$. If $p(\cdot , \xi) \in Q(a)$ and $\varphi(p(z, \xi), zp'(z, \xi); z, \xi)$ is univalent in $U$, then

$$\Omega_{\xi} \subset \{ \varphi(p(z, \xi), zp'(z, \xi); z, \xi) \}$$

implies

$$q(z, \xi) \prec \prec p(z, \xi), \quad z \in U, \ \xi \in \overline{U}.$$  


Proof. Assume $q(z, \xi) \not\prec p(z, \xi)$. By Lemma A, there exist points $z_0 = r_0e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \setminus E(p)$, and an $m \geq n \geq 1$ that satisfy conditions (i)–(iii) of Lemma A. Using these conditions with $r = p(\zeta_0, \xi)$, $s = \zeta_0p'(\zeta_0, \xi)$ and $\zeta = \zeta_0$ in Definition 3 we obtain

$$\varphi(p(\zeta_0, \xi), \zeta_0p'(\zeta_0, \xi); \zeta_0, \xi) \in \Omega_{\xi}.$$  

Since this contradicts (4) we must have $q(z, \xi) \prec \prec p(z, \xi), z \in U, \ \xi \in \overline{U}$.  

We next consider the special situation when $h(z, \xi)$ is analytic on $U \times \overline{U}$ and $h(U \times \overline{U}) = \Omega_{\xi} \neq C$. Then Theorem 1 becomes

**Theorem 2.** Let $h(\cdot, \xi)$ be analytic in $U \times \overline{U}$, $q(\cdot, \xi) \in H^*[a,n,\xi]$, $\varphi : C^2 \times U \times \overline{U} \to C$, and suppose that

$$\varphi(q(z, \xi), tzq'(z, \xi); z, \xi) \in H(U \times \overline{U}),$$

for $z \in U$, $\zeta \in \partial U$ and $0 < t \leq \frac{1}{n} \leq 1$. If $p(\cdot, \xi) \in Q(a)$ and $\varphi(p(z, \xi), zp'(z, \xi); z, \xi)$ is univalent in $U$ for all $\xi \in \overline{U}$, then

$$h(z, \xi) \prec \prec \varphi(p(z, \xi), zp'(z, \xi); z, \xi), \quad z \in U, \ \xi \in \overline{U}$$

implies

$$q(z, \xi) \prec \prec p(z, \xi), \quad z \in U, \ \xi \in \overline{U}.$$  

Furthermore, if

$$\varphi(q(z, \xi), zq'(z, \xi); z, \xi) = h(z, \xi), \quad z \in U, \ \xi \in \overline{U}$$

has a univalent solution $q(\cdot, \xi) \in Q(a)$, then $q(\cdot, \xi)$ is the best subordinant.

**Theorem 3.** Let $h(\cdot, \xi)$ be convex in $U$, for all $\xi \in \overline{U}$ with $h(0, \xi) = a$, and let $\theta$ and $\psi$ be analytic in a domain $D \subset C$. Let $p(\cdot, \xi) \in H^*[a,1,\xi] \cap Q$ and suppose that $\theta[p(z, \xi)] + zp'(z, \xi)\psi[p(z, \xi)]$ is univalent in $U$ for all $\xi \in \overline{U}$.

If the differential equation

$$\theta[q(z, \xi)] + zq'(z, \xi)\psi[q(z, \xi)] = h(z, \xi), \quad z \in U, \ \xi \in \overline{U}$$

has a univalent solution $q(\cdot, \xi)$ that satisfies $q(0, \xi) = a$, $q(U \times \overline{U}) \subset D$, and

$$\theta[q(z, \xi)] \prec \prec h(z, \xi), \quad z \in U, \ \xi \in \overline{U},$$

then

$$h(z, \xi) \prec \prec \theta[p(z, \xi)] + zp'(z, \xi)\psi[p(z, \xi)]$$
implies
\[ q(z, \xi) \prec\prec p(z, \xi), \quad z \in U, \ \xi \in \overline{U}. \]

Function \( q \) is the best subordinant.

**Proof.** We can assume that \( h(\cdot, \xi), p(\cdot, \xi) \) and \( q(\cdot, \xi) \) satisfy the conditions of this theorem on the closed \( \overline{U} \times \overline{U} \), and that \( q'(\xi, \xi) \neq 0 \) for \( |\xi| = 1 \). If not, then we can replace \( h(\cdot, \xi), p(\cdot, \xi) \) and \( q(\cdot, \xi) \) with \( h(\rho z, \xi), p(\rho z, \xi) \) and \( q(\rho z, \xi) \), where \( 0 < \rho < 1 \). These new functions have the desired properties on \( \overline{U} \times \overline{U} \), and we can use them in the proof of the theorem. Theorem 3 would then follow by letting \( \rho \to 1 \). We will use Lemma A to prove this result. If we let \( \varphi(r, s) = \theta[r] + s\psi[r] \), \( r = q(z, \xi) \), \( s = zq'(z, \xi) \), then (8) becomes
\[ \varphi(q(z, \xi), tzq'(z, \xi)) = h(z, \xi), \quad 0 < t \leq 1. \]

From (9) and the convexity of \( h(U \times \overline{U}) \) we conclude that
\[ \varphi(q(z, \xi), tzp'(z, \xi)) \in h(U \times \overline{U}) \quad \text{for} \quad 0 < t \leq 1. \]
Hence condition (5) of Theorem 2 is satisfied and the conclusions of this theorem follow. □

In the special case when \( \theta[q(z, \xi)] = q(z, \xi) \) and
\[ \theta[q(z, \xi)] = \frac{1}{\beta q(z, \xi) + \gamma}, \quad \beta, \gamma \in \mathbb{C}, \]
we obtain the following result for the Briot-Bouquet strong differential superordination.

**Corollary 1.** Let \( \beta, \gamma \in \mathbb{C} \), and let \( h(\cdot, \xi) \) be convex in \( U \) for all \( \xi \in \overline{U} \), with \( h(0, \xi) = a \). Suppose that the differential equation
\[ q(z, \xi) + \frac{zq'(z, \xi)}{\beta q(z, \xi) + \gamma} = h(z, \xi), \quad z \in U, \ \xi \in \overline{U} \quad \text{(11)} \]
has a univalent solution \( q(\cdot, \xi) \) that satisfies \( q(0, \xi) = a \) and \( q(z, \xi) \prec\prec h(z, \xi), \quad z \in U, \ \xi \in \overline{U} \).

If \( p(\cdot, \xi) \in \mathcal{H}[a, 1] \cap Q \) and \( p(z, \xi) + \frac{zp'(z, \xi)}{bp(z, \xi) + \gamma} \) is univalent in \( U \) for all \( \xi \in \overline{U} \), then
\[ h(z, \xi) \prec\prec p(z, \xi) + \frac{zp'(z, \xi)}{bp(z, \xi) + \gamma} \quad \text{(12)} \]
implies
\[ q(z, \xi) \prec\prec p(z, \xi), \quad z \in U, \ \xi \in \overline{U}. \]
The function $q(\cdot, \xi)$ is the best subordinant.

We can combine that result with Theorem 3 and we obtain the following sandwich theorem.

**Theorem 4.** Let $h_1(\cdot, \xi)$ and $h_2(\cdot, \xi)$ be convex in $U \times \overline{U}$, for all $\xi \in \bar{U}$ with $h_1(0, \xi) = h_2(0, \xi) = a$, and let $\theta$ and $\psi$ be analytic in a domain $D \subset \mathbb{C}$. Let $p(\cdot, \xi) \in \mathcal{H}[a, 1, \xi] \cap Q$ and suppose that $\theta[p(z, \xi)] + z p'[z, \xi] \psi[p(z, \xi)]$ is univalent in $U$, for all $\xi \in \bar{U}$. If the differential equations

$$
\theta[q_i(z, \xi)] + z q_i'(z, \xi) \psi[q_i(z, \xi)] = h_i(z, \xi),
$$

have univalent solutions $q_i$ that satisfy $q_i(0, \xi) = a$, $q_i(U \times \overline{U}) \subset D$, and

$$
\theta[q_i(z, \xi)] \ll h_i(z, \xi),
$$

for $i = 1, 2$, then

$$
h_1(z, \xi) \ll \theta[p(z, \xi)] + z p'[z, \xi] \psi[p(z, \xi)] \ll h_2(z, \xi)
$$

implies

$$
q_1(z, \xi) \ll p(z, \xi) \ll q_2(z, \xi), \quad z \in U, \quad \xi \in \overline{U}.
$$

In the special case when $\theta[p(z, \xi)] = p(z, \xi)$ and

$$
\psi[p(z, \xi)] = \frac{1}{\beta p(z, \xi) + \gamma},
$$

we obtain the following Briot-Bouquet sandwich result.

**Corollary 2.** Let $\beta, \gamma \in \mathbb{C}$ and let $h_i(\cdot, \xi)$ be convex in $U$, for all $\xi \in \overline{U}$, with $h_i(0, \xi) = a$, for $i = 1, 2$. Suppose that the differential equations

$$
q_i(z, \xi) + \frac{z q_i'(z, \xi)}{\beta q_i(z, \xi) + \gamma} = h_i(z, \xi)
$$

have univalent solutions $q_i(\cdot, \xi)$ that satisfy $q_i(0, \xi) = a$ and $q_i(z, \xi) \ll h_i(z, \xi)$, for $i = 1, 2$, $z \in U$, $\xi \in \overline{U}$. If

$$
p(\cdot, \xi) \in \mathcal{H}[a, 1, \xi] \cap Q
$$

and

$$
p(z, \xi) + \frac{z p'(z, \xi)}{\beta p(z, \xi) + \gamma} \in \mathcal{H}_a(U) \quad \text{for all } \xi \in \overline{U},
$$

then

$$
h_1(z, \xi) \ll p(z, \xi) + \frac{z p'(z, \xi)}{\beta p(z, \xi) + \gamma} \ll h_2(z, \xi)
$$

implies

$$
q_1(z, \xi) \ll p(z, \xi) \ll q_2(z, \xi), \quad z \in U, \quad \xi \in \overline{U}.
$$

The functions $q_1(\cdot, \xi)$ and $q_2(\cdot, \xi)$ are the best subordinant and best dominant respectively.
If $\beta = 0$ and $\gamma \neq 0$ with $\text{Re} \gamma \geq 0$, then (13) has univalent (convex) solutions given by

$$q_i(z, \xi) = \frac{\gamma}{z^\gamma} \int_0^z h_i(t, \xi)t^{\gamma-1}dt,$$

for $i = 1, 2$. In this case we obtain the following sandwich corollary.

**Corollary 3.** Let $h_1(\cdot, \xi)$ and $h_2(\cdot, \xi)$ be convex in $U$, for all $\xi \in \overline{U}$, with $h_1(0, \xi) = h_2(0, \xi) = a$. Let $\gamma \neq 0$ with $\text{Re} \gamma \geq 0$, and let the functions $q_i(\cdot, \xi)$ be defined by (14) for $i = 1, 2$. If $p(\cdot, \xi) \in \mathcal{H}^c[a, 1, \xi] \cap Q$ and $p(z, \xi) + \frac{zp'(z, \xi)}{\gamma}$ is univalent in $U$ for all $\xi \in \overline{U}$, then

$$h_1(z, \xi) \preceq \prec p(z, \xi) + \frac{zp'(z, \xi)}{\gamma} \preceq \prec h_2(z, \xi), \quad z \in U, \ \xi \in \overline{U}$$

implies

$$q_1(z, \xi) \preceq \prec p(z, \xi) \preceq \prec q_2(z, \xi), \quad z \in U, \ \xi \in \overline{U}.$$ 

The functions $q_1(z, \xi)$ and $q_2(z, \xi)$ are the best subordinant and best dominant respectively.

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