SOME CHARACTERIZATIONS OF b-WEAKLY COMPACT OPERATORS

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We establish some properties of the class of b-weakly compact operators on Banach lattices and we give some consequences.

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1. INTRODUCTION

In their paper [2], Alpay-Altin-Tonyali introduced the class of b-weakly compact operators. After that, a series of papers which gave different characterizations of this class of operators were published [3, 4, 5, 6, 7]. Let us recall that an operator T from a Banach lattice E into a Banach space X is said to be b-weakly compact if it carries each b-order bounded subset of E into a relatively weakly compact subset of X.

The most beautiful property of the class of b-weakly compact operators is that it satisfies the domination property as proved in [2, Corollary 2.9]. But, unfortunately, one of shortcomings of this class is that it does not satisfy the duality property, i.e., there exist b-weakly compact operators whose adjoints are not b-weakly compact, and conversely, there exist operators which are not b-weakly compact but their adjoints are b-weakly compact. And this problem is studied in [8]. Finally, note that each weakly compact operator is b-weakly compact but the converse is false in general. And in [9], we characterized Banach lattices on which each b-weakly compact operator is weakly compact.

The objective of this paper is to give other interesting properties of this class of operators. But to state our results, we need to fix some notation and recall some definitions. Let E be a vector lattice, for each $x, y \in E$ with $x \leq y$, the set $[x, y] = \{z \in E : x \leq z \leq y\}$ is called an order interval. A subset of E is said to be order bounded if it is included in some order interval. A nonzero element x of a vector lattice E is discrete if the order ideal generated by x equals the subspace generated by x. The vector lattice E is discrete, if it admits a complete disjoint system of discrete elements. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm and the dual order, is also a Banach lattice. Recall that a norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, (x_α) converges to 0 for the norm $\|\cdot\|$, where the notation $x_\alpha \downarrow 0$ means that (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. A Banach lattice E is said to be an AM-space if for each $x, y \in E$ such that $\inf(x, y) = 0$ we have $\|x + y\| = \max\{\|x\|, \|y\|\}$. The Banach lattice E is an AL-space if its topological dual E' is an AM-space.

We use the term operator $T : E \to F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. Note that each positive linear mapping on a Banach lattice is continuous. If an operator $T : E \to F$ between two Banach lattices is positive, then its adjoint operator $T' : F' \to E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$. For terminologies concerning Banach lattice theory and positive operators, we refer the reader to [1].

2. MAIN RESULTS

Recall from [2] that a subset A of a vector lattice E is called b-order bounded in E if it is order bounded in the order bidual $(E^\sim)^\sim$ where $E^\sim$ denotes the order dual of E. Note that every order bounded subset of E is b-order bounded. However, the converse is not true in general.

A vector lattice E is said to have property (b) if $A \subset E$ is order bounded whenever A is order bounded in $(E^\sim)^\sim$, i.e., if every b-order bounded subset of E is order bounded in E. Equivalently, if for each net (x_α) in E with $0 \leq x_\alpha \uparrow \leq \hat{x}$ for some $\hat{x}$ in $(E^\sim)^\sim$, (x_α) is order bounded in E. For an example, every AM-space with unit has the property (b), but the Banach lattice c_0 does not have the property (b). For other examples, see [5].

Remark. It is easy to see that each b-order bounded subset of a Banach lattice E is norm bounded (in fact, if $A \subset E$ is b-order bounded, then A is order bounded in the topological bidual E'', hence A is norm bounded in E'' and so A is norm bounded in E).

In general, the converse is not true. In fact, the closed unit ball of the Banach lattice l^1 is not b-order bounded in l^1. However, if E is an AM-space then a subset $A \subset E$ is b-order bounded in E if and only if it is norm bounded.

A linear map T between two vector lattices E and F is called b-order bounded if it maps b-order bounded subsets of E into b-order bounded subsets of F. It is clear that every order bounded linear map between two vector lattices is b-order bounded. Recall from [2, 4] that an operator T from a Banach lattice E into a Banach space X is said to be b-weakly compact whenever T carries each b-order bounded subset of E into a relatively weakly compact subset of X.

Also, recall from [10] that an operator T from a Banach lattice E into a Banach space F is said to be order weakly compact if for each $x \in E^+$, the subset $T([0,x])$ is relatively weakly compact in F. Note that each b-weakly compact operator is order weakly compact, but the converse is not true in general. In fact, the identity operator $\mathrm{Id}_{c_0} : c_0 \to c_0$ is order weakly compact but it is not b-weakly compact.

The first property gives a sufficient condition under which each order weakly compact operator is b-weakly compact.

Proposition 2.1. *Let E be a Banach lattice. If E has property* (b), *then each order weakly compact operator from E into X is b-weakly compact for every Banach space X.*

In particular, each order weakly compact operator from E' into X is b-weakly compact for each Banach lattice E and each Banach space X.

Proof. Assume that E has property (b) and let X be a Banach space, $T : E \to X$ an order weakly compact operator and let A be a b-order bounded subset of E. Since E has the property (b), A is order bounded in E. And so, by using that T is order weakly compact, we deduce that $T(A)$ is a relatively weakly compact subset of X. Hence, $T : E \to X$ is b-weakly compact, and this shows the result.

The particular case follows immediately by observing that the topological dual of each Banach lattice has property (b). □

As a consequence of Proposition 2.1 and Examples of [5, p. 576], we obtain

Corollary 2.2. *Let E be a Banach lattice. Then each order weakly compact operator from E into X is b-weakly compact for every Banach space X if one of the following assertions is valid:*

(1) *E is a KB-space.*
(2) *E is reflexive.*
(3) *E is an AM-space with unit.*
(4) *E is a perfect space.*

An operator T from a Banach space E into another F is said to be Dunford-Pettis if it carries weakly compact subsets of E onto compact subsets of F. In [7, Proposition 5], it was established that each Dunford-Pettis operator is b-weakly compact. To give the converse, we need to recall that an operator T from a Banach lattice E into a Banach space F is said to be M-weakly compact if for each disjoint bounded sequence (x_n) of E, we have $\lim_n \|T(x_n)\| = 0$. Note that a M-weakly compact operator is not necessary Dunford-Pettis. In fact, the inclusion map $i : L^2[0,1] \to L^1[0,1]$ is M-weakly compact but it is not Dunford-Pettis. The converse is not always true. In fact, the identity operator of the Banach lattice l^1 is Dunford-Pettis, but it is not M-weakly compact. If not, it would be weakly compact and this is impossible.

The following result gives a sufficient condition under which these two classes of operators coincide with the class of b-weakly compact operators.

Proposition 2.3. *Let E be an AM-space and X a Banach space. Then for an operator $T : E \to X$ the following assertions are equivalent:*

(1) *T is M-weakly compact.*

(2) *T is weakly compact.*

(3) *T is Dunford-Pettis.*

(4) *T is b-weakly compact.*

(5) *$(T(x_n))$ is norm convergent for every positive increasing sequence (x_n) of the closed unit ball B_E of E.*

In addition, if E has an order continuous norm or E' is discrete, then we may add

(6) *T is compact.*

Proof. (1) ⇒ (2) ⇒ (3) Obvious.

(3) ⇒ (1) follows from Theorem 3.7.10 of [11] if we observe that the norm of E' is order continuous.

(2) ⇒ (4) follows immediately if we observe that each b-order bounded subset of E is norm bounded.

(4) ⇒ (2) follows immediately if we observe that the closed unit ball B_E of E is b-order bounded in E.

(2) ⇒ (5). Let (x_n) be a positive increasing sequence of B_E. If T is weakly compact, then its adjoint $T' : X' \to E'$ is also weakly compact, i.e., the subset $B = T'(B_{X'})$ is relatively compact in E' for $\sigma(E', E'')$. Since E' is an AL-space, by Theorem 2.5.5 of [11] (x_n) is ρ_B-Cauchy, where ρ_B is the lattice seminorm on E defined by

$$\rho_B(x) = \{|f|(|x|) : f \in B\}, \quad x \in E.$$

Since $\rho_B(x) \geq \|T(x)\|$ for all $x \in E$, $(T(x_n))$ is a norm Cauchy sequence of X. Since X is a Banach space, $(T(x_n))$ is norm convergent and we are done.

(5) ⇒ (1). Let (x_n) be a positive disjoint sequence of B_E. Pick $u_n = x_1 + \cdots + x_n = \vee_{k=1}^n x_k$ for each n. It is clear that (u_n) is a positive increasing sequence of E and since E is an AM-space,

$$\|u_n\| = \|\vee_{k=1}^n x_k\| \leq \vee_{k=1}^n \|x_k\| \leq 1 \text{ for each } n.$$

By (5) the sequence $(T(u_n))$ is norm convergent. Now, since $T(x_n) = T(u_n) - T(u_{n-1})$ for each n, $\|T(x_n)\| \to 0$. And then T is M-weakly compact.

Finally, assume, in addition, that E has an order continuous norm or E' is discrete. It suffices to establish that (2) $\Rightarrow$ (6). Note first that if E is an AM-space with an order continuous norm, then E is discrete (see the proof of Theorem 1.4 of [12]). Hence, Theorem 6.1 of [12], E' is also discrete. Then we can assume that E' is discrete. If T is weakly compact, then its adjoint $T' : X' \to E'$ is also weakly compact, i.e., $T'(B_{X'})$ is relatively compact for $\sigma(E', E'')$ in E'. Since E' is a discrete AL-space, E' has the Schur property (i.e., every weakly convergent sequence to 0 in E is norm convergent to zero), and this implies that $T'(B_{X'})$ is relatively compact for the norm of E'. Then T' is compact and hence T is also compact. This ends the proof. □

As the norm of the Banach lattice c_0 is order continuous and the topological dual of the Banach lattice of all convergent sequences c is discrete, we obtain from Proposition 2.3 the following characterization.

Corollary 2.4. *Let X be a Banach space. Then for each operator $T : c_0 \to X$ (resp. $T : c \to X$) the following assertions are equivalent:*

(1) *T is b-weakly compact.*
(2) *T is Dunford-Pettis.*
(3) *T is weakly compact.*
(4) *T is M-weakly compact.*
(5) *T is compact.*

Recall from [1] that a Banach lattice E is said to be lattice embeddable into another Banach lattice F whenever there exists a lattice homomorphism $T : E \to F$ and there exist two positive constants K and M satisfying

$$K\|x\| \leq \|T(x)\| \leq M\|x\| \text{ for all } x \in E.$$

In this case T is called a lattice embedding from E into F, and $T(E)$ is a closed sublattice of F which can be identified with E. A Banach lattice E is said to be a KB-space whenever every increasing norm bounded sequence of E^+ is norm convergent. As an example, each reflexive Banach lattice (resp. AL-space) is a KB-space. On the other hand, each KB-space has an order continuous norm, but a Banach lattice with an order continuous norm is not necessary a KB-space. In fact, the Banach lattice c_0 has an order continuous norm but it is not a KB-space. However, if E is a Banach lattice, the topological dual E' is a KB-space if and only if its norm is order continuous. In [6, Proposition 2.1], it was proved that a Banach lattice E is a KB-space iff every operator from E into X is b-weakly compact, for each Banach space X. In [8, Proposition 2.3], we showed the following characterization.

Proposition 2.5 ([8]). *Let X be a Banach space. Then the following assertions are equivalent:*

(1) *For each Banach lattice E, every operator from E into X is b-weakly compact.*

(2) *Each operator from c_0 into X is b-weakly compact (resp. compact).*

(3) *c_0 is not embeddable in X.*

Proof. (1) ⇒ (2) follows immediately from Corollary 2.4.

(2) ⇒ (3). The proof is the same as Proposition 2.3 of [8]. Assume by way of contradiction that c_0 is embeddable in X and let $T : c_0 \to X$ be an embedding. Then there exist two positive constants K and M satisfying

$$K \left\|(\lambda_n)\right\|_\infty \leq \left\|T\left((\lambda_n)\right)\right\| \leq M \left\|(\lambda_n)\right\|_\infty \text{ for all } (\lambda_n) \in c_0.$$

By Corollary 2.4, it suffices to show that T is not M-weakly compact. To this end, let (e_n) be the sequence of real numbers with all terms are zero except for the nth which is 1. Since (e_n) is a b-order bounded disjoint sequence of c_0 and

$$\|T(e_n)\| \geq K \left\|(e_n)\right\|_\infty = K > 0 \text{ for each } n,$$

we conclude that T is not M-weakly compact.

(3) ⇒ (1). Let E be a Banach lattice and let $T : E \to X$ be an operator. Since c_0 is not embeddable in X, by a theorem of Ghoussoub-Johnson [1, Theorem 4.63] the existence of a KB-space F, a lattice homomorphism $Q : E \to F$ and an operator $S : F \to X$ such that $T = S \circ Q$ are guaranted. To finish the proof, we have to show that T is b-weakly compact. Let A be a b-order bounded subset of E. Since Q is a lattice homomorphism, Q is a positive operator and therefore it is b-order bounded. Then $Q(A)$ is a b-order bounded subset of F. On the other hand, since F is a KB-space, F has property (b). So, $Q(A)$ is an order bounded subset of F and therefore $Q(A)$ is relatively compact for the topology $\sigma(F, F')$ in F. So, $S(Q(A))$ is also relatively compact for $\sigma(X, X')$ in X. Hence, $T = S \circ Q$ is b-weakly compact. This completes the proof. □

Let us recall from [11] that an operator T from a Banach space E into a Banach lattice F is said to be semi-compact if for each $\varepsilon > 0$ there exists some $u \in F^+$ such that $T(B_E) \subset [-u, u] + \varepsilon B_F$, where B_H is the closed unit ball of $H = E, F$ and $F^+ = \{y \in F : 0 \leq y\}$. Note that the class of semi-compact operators does not satisfy the duality problem, i.e., there exist semi-compact operators whose dual operators are not semi-compact, and conversely, there exist operators which are not semi-compact but their dual operators are semi-compact. However, it was established that if $T : E \to F$ is an operator between two Banach lattices such that its adjoint $T' : F' \to E'$ is semi-compact, then T is order weakly compact, and conversely, if the operator $T : E \to F$ is semi-compact, then its adjoint $T' : F' \to E'$ is order weakly compact [11, Proposition 3.6.18].

The following is a generalization of Proposition 3.6.18 of [11]. Also, it is a generalization of a Corollary of [7, p. 580].

PROPOSITION 2.6. *Let E be a Banach lattice and X a Banach space.*

(1) *If T is a semi-compact operator from X into E, then its adjoint T' is b-weakly compact.*

(2) *If T is an operator from E into X such that its adjoint T' is semi-compact, then T is b-weakly compact.*

Proof. (1) If T is a semi-compact operator from X into E, then by Proposition 3.6.18 (ii) of [11] its adjoint $T' : E' \to X'$ is order weakly compact. On the other hand, since E' has property (b), by Proposition 2.1 each order weakly compact operator from E' into X' is b-weakly compact. This proves the result.

(2) Let T be an operator from E into X such that T' is semi-compact. Then Proposition 3.6.18 (ii) of [11] implies that its second adjoint T'' from E'' into X'' is order weakly compact. Now, from Theorem 3.5.8 of [11] we have the existence of a KB-space F and two operators $Q : E \to F$ and $S : F \to X$ such that $T = S \circ Q$. Next, since F is a KB-space, c_0 is not embeddable in F [1, Theorem 4.60]. Finally, Proposition 2.5 implies that Q is b-weakly compact, and hence $T = S \circ Q$ is b-weakly compact. □

Another characterization is giving by the following result:

PROPOSITION 2.7. *Let E be a Banach lattice with an order continuous norm, then the following statements are equivalent:*

(1) *E is a KB-space.*

(2) *Each operator from E into c_0 is b-weakly compact.*

Proof. (1) $\Rightarrow$ (2) follows from Proposition 2.1 of [6].

(2) $\Rightarrow$ (1). Suppose that E is not a KB-space. Since the norm of E is order continuous, Theorem 2.4.12 and Corollary 2.4.3 of [11] imply that E contains a complemented copy of c_o and there exists a positive projection $P : E \to c_0$. By assumption, $P : E \to c_0$ is b-weakly compact, and hence the composed operator $P \circ i : c_0 \to E \to c_0$, which is just the identity operator of c_0, is b-weakly compact. And then it follows from Corollary 2.9 of [2] that c_0 is a KB-space. But this is false. □

To give other characterizations of b-weakly compact operators, we will need the following Lemmas:

LEMMA 2.8. *Let E be a Banach lattice. Then every positive norm bounded increasing net (x_α) of E is b-order bounded, i.e., (x_α) is order bounded in the topological bidual E''.*

Proof. Pick a net $(x_\alpha) \subset E$ satisfying $0 \leq x_\alpha \uparrow$ and $\sup\{\|x_\alpha\|\} < \infty$. Since $\phi(f) = \sup\{f(x_\alpha)\} < \infty$ for each $0 \leq f \in E'$, the mapping $\phi : (E')^+ \to \mathbf{R}^+$ is additive, and hence it defines a positive linear functional on E'. Since $0 \leq x_\alpha \uparrow \phi$ in E'', (x_α) is b-order bounded. □

Recall from [2] the following notation: if E is a Banach lattice and if $0 \leq x'' \in E''$, then the principal ideal $I_{x''}$ generated by $x'' \in E''$ under the norm $\|\cdot\|_\infty$ defined by

$$\|y''\|_\infty = \inf\{\lambda > 0 : |y''| \leq \lambda x''\}, \quad y'' \in I_{x''},$$

is an AM-space with unit x'', whose closed unit ball is the order interval $[-x'', x''] \subset \|x''\| \cdot B_{E''}$ [1, Theorem 4.21].

Proposition 2.9. *Let T be an operator from a Banach lattice E into a Banach space X. Then the following assertions are equivalent:*

(1) *T is b-weakly compact.*

(2) *For each $0 \leq x'' \in E''$, the restriction of T to the vector lattice $Y = I_{x''} \cap E$ is weakly compact, where Y is equipped by the norm induced on Y by $I_{x''}$.*

Proof. (1) ⇒ (2). Let $0 \leq x'' \in E''$ and let S be the restriction of the operator T to the vector lattice $Y = I_{x''} \cap E$. Note that Y is an AM-space for the norm $\|\cdot\|_\infty$ induced on Y by $I_{x''}$. As $B_Y = Y \cap [-x'', x''] = E \cap [-x'', x'']$, B_Y is b-order bounded. Since T is b-weakly compact, $S(B_Y)$ is relatively weakly compact. And hence $S : Y \to X$ is weakly compact.

(2) ⇒ (1). Let A be a b-order bounded subset of E. We have to show that $T(A)$ is relatively weakly compact. Choose $0 \leq x'' \in E''$ such that $A \subset [-x'', x'']$ and let $I_{x''}$ and Y be defined in the above. Then $S : Y \to X$ is weakly compact, where S is the restriction of the operator T to Y. Since $A \subset B_Y$, $S(A)$ is relatively weakly compact, i.e., $T(A)$ is relatively weakly compact. This completes the proof. □

Proposition 2.10. *Let T be an operator from a Banach lattice E into a Banach space X. Then the following assertions are equivalent:*

(1) *T is b-weakly compact.*

(2) *$(T(x_n))$ is norm convergent for every positive increasing sequence (x_n) in the closed unit ball B_E of E.*

(3) *T preserves no subspace isomorphic to c_0.*

(4) *T preserves no sublattice isomorphic to c_0.*

Proof. (2) ⇔ (3) ⇔ (4) It is just Theorem 3.4.11 of [11].

(1) ⇒ (2). Let (x_n) be a positive increasing sequence of B_Y. It follows from Lemma 2.8 that (x_n) is b-order bounded. Hence $(x_n) \subset [-x'', x'']$ for some $0 \leq x'' \in E''$. Let Y be the vector lattice $I_{x''} \cap E$. Then Y is an AM-space for the norm $\|\cdot\|_\infty$ and B_Y is b-order bounded. Let S be the restriction of the

operator T to Y. Since T is b-weakly compact, $S(B_Y)$ is relatively weakly compact. Hence $S : Y \to X$ is weakly compact, and then Proposition 2.3 (5) implies that $(S(x_n))$ is norm convergent, i.e., $(T(x_n))$ is norm convergent.

(2) $\Rightarrow$ (1). By Proposition 2.9 it suffices to show that for each $0 \le x'' \in E''$, the operator $S : Y \to X$ is weakly compact where S is the restriction of the operator T to the vector lattice $Y = I_{x''} \cap E$. Since Y is an AM-space, it suffices to prove that the condition (5) of Proposition 2.3 holds. To this end, let (x_n) be a positive increasing sequence of B_Y. Note that

$$B_Y = E \cap \left[-x'', x''\right] \subset \|x''\| \cdot B_{E''}.$$

Then $\left(\frac{x_n}{\|x''\|+1}\right)$ is a positive increasing sequence of B_Y. By (2), the sequence $\left(T\left(\frac{x_n}{\|x''\|+1}\right)\right)$ is norm convergent. So, $(S(x_n))$ is norm convergent. This completes the proof. □

Proposition 2.11. *Let E and F be Banach lattices, X a Banach space, and let $E \xrightarrow{T} F \xrightarrow{S} X$ be operators. If E' and F have order continuous norms and S is b-weakly compact, then $S \circ T$ is weakly compact.*

Proof. By Proposition 2 of [7], there exist a KB-space G, an interval preserving lattice homomorphism $Q : F \to G$ and an operator $R : G \to X$ such that $S = R \circ Q$. Since E' has an order continuous norm and G is a KB-space, by Theorem 5.27 of [1], $Q \circ T : E \to G$ is weakly compact, and so $S \circ T = R \circ (Q \circ T)$ is weakly compact. □

As a consequence, we obtain

Corollary 2.12. *Let E be a Banach lattice, X a Banach space, and let $T : E \to X$ be an operator. If E and E' have order continuous norms and T is b-weakly compact, then T is weakly compact.*

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