PROPERTIES OF INFINITE DIVISIBILITY

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We study triplets of natural numbers whose sum divides the sum of their powers for infinitely many exponents.

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We start with

PROPOSITION 1. *There exist infinitely many* $x, y, z \in \mathbb{N}^*$ *with* $(x, y, z) = 1$ *such that*

$$x + y + z \mid x^{2^n} + y^{2^n} + z^{2^n}$$

for every $n \in \mathbb{N}$.

Proof. In the identity

$$a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2)$$

we take $a^2 = x^k$, $ab = y^k$, $b^2 = z^k$, with k a positive integer. We obtain

$$x^k + y^k + z^k \mid x^{2k} + y^{2k} + z^{2k}.$$

Letting k take the values $1, 2, 4, \ldots, 2^{n-1}$, we obtain

$$x + y + z \mid x^{2^n} + y^{2^n} + z^{2^n}.$$

If we take $(a, b) = 1$, we also obtain $(x, y, z) = 1$. $\square$

PROPOSITION 2. *There exist infinitely many* $x, y, z \in \mathbb{N}^*$ *with* $(x, y, z) = 1$ *such that*

$$x + y + z \mid x^{2n+1} + y^{2n+1} + z^{2n+1}$$

for every $n \in \mathbb{N}$.

This follows from the following two auxiliary propositions.

A. *Let* $x, y, z \in \mathbb{N}^*$. *If* $x + y + z \mid xyz$ *then*

$$x + y + z \mid x^{2n+1} + y^{2n+1} + z^{2n+1}$$

for every $n \in \mathbb{N}$.

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Proof. It follows by induction from the identity

$$x^{2n+1}+y^{2n+1}+z^{2n+1}=(x+y+z)(x^{2n}+y^{2n}+z^{2n})-$$
$$-(xy+yz+zx)(x^{2n-1}+y^{2n-1}+z^{2n-1})+xyz(x^{2n-2}+y^{2n-2}+z^{2n-2})$$

and the assumption $x+y+z \mid xyz$. □

B. *There are infinitely many* $x,y,z\in\mathbb{N}^*$ *with* $(x,y,z)=1$ *such that*

$$x+y+z \mid xyz.$$

Proof. Denote $x+y+z=s$. From $s \mid xyz$ we deduce $s \mid xy(x+y)$. Take $s=xy(x+y)$ and then $x=a, y=b$ with a,b positive integers with $(a,b)=1$. It follows that $z=ab(a+b)-a-b$. Thus we have infinitely many triplets

$$(x,y,z)=(a,b,a^2b+ab^2-a-b). \quad \square$$

Proposition 2 follows immediately from A and B.

This property holds for infinitely many exponents consisting of the set of odd numbers, or the set of powers of 2. Does it hold for other infinitely many sets of even numbers? For example,

$$\{2q;\ q\in\mathbb{N}^*\},\ \{4q;\ q\in\mathbb{N}^*\},\ \{8q;\ q\in\mathbb{N}^*\},\ldots,\{2^kq;\ q\in\mathbb{N}^*\}.$$

The negative answer is given by the following

Proposition 3. *Let* $x,y,z\in\mathbb{N}^*$ *with* $(x,y,z)=1$. *If*

$$x+y+z \mid x^{2^kq}+y^{2^kq}+z^{2^kq} \tag{1}$$

with $k\geq 1$ *fixed and* $q=1,2,3,4,\ldots$, *then* $x+y+z$ *has at most a finite number of values.*

Proof. Denote $x+y+z=s$ and let p be a prime factor of s, $p\mid s$. In the identity

$$a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)$$

we set $a=x^{2^k}$, $b=y^{2^k}$, $c=z^{2^k}$. Using (1) for $q=2$ and $q=1$, it follows

$$2\big(x^{2^k}y^{2^k}+y^{2^k}z^{2^k}+z^{2^k}x^{2^k}\big)\equiv 0 \bmod s. \tag{2}$$

Suppose there exists a prime $p\geq 3$ with $p\mid s$. If $p\mid x$ then (2) implies $p\mid yz$, and $x+y+z=s$ and $p\mid s$ implies $p\mid y$ and $p\mid z$. Contradiction, since $(x,y,z)=1$. Hence $p\nmid x$, and similarly $p\nmid y$,$p\nmid z$. Observe that if $p\geq 3$ divides s, then $p\nmid xyz$.

In the identity

$$a^3+b^3+c^3=(a+b+c)(a^2+b^2+c^2-ab-bc-ac)+3abc$$

we set $a=x^{2^k}$, $b=y^{2^k}$, $c=z^{2^k}$. Using (1) for $q=3$ and $q=1$, we obtain

$$s \mid 3x^{2^k}y^{2^k}z^{2^k}.$$

From here and the previous observation that if $p \geq 3$ and $p \mid s$, then $p \nmid xyz$, we deduce that in the canonical decomposition of s there are no prime factors > 3, and $3^2 \nmid s$.

Suppose now that $2 \mid s$. It follows from $x + y + z = s$ and $(x, y, z) = 1$ that two of x, y, z are odd and one even and then the number in parantheses in (2) is odd. Hence $2^2 \nmid s$.

Therefore, $s \mid 6$, hence $x + y + z$ has a finite number of values. And the triplets which verify (1) are $(x, y, z) = (1, 1, 1), (1, 1, 4), (1, 4, 1), (4, 1, 1)$. □

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