A NEW PROOF OF A RESULT OF P. SZÜSZ IN THE METRICAL THEORY OF CONTINUED FRACTIONS

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We give a new proof of a result of Szüsz [6] that generalizes the classical theorem of Gauss-Kuzmin-Lévy (see [3, Chapter 2]). The proof makes use of an operator occurring in the theory of dependence with complete connections (see [4]).

AMS 2000 Subject Classification: 11K55, 60K99.

Key words: continued fraction expansion, dependence with complete connections.

1. INTRODUCTION

Any irrational number $t \in I = [0,1]$ has a unique infinite continued fraction expansion of the form

$$t = \cfrac{1}{a_1(t) + \cfrac{1}{a_2(t) + \ddots}}$$

that we shall denote $t = [a_1(t), a_2(t), \ldots]$.

Endowing $I = [0,1]$ with the σ-algebra $\mathcal{B}_I$ of its Borel subset, it is clear that the a_n, $n \geq 1$, can be viewed as random variables defined almost surely with respect to Lebesgue measure λ. Basically, the metric theory of continued fractions is the study of the sequence of random variables $(a_n)_{n\geq 1}$ and related sequences.

Let us define

$$r_n(t) = a_n(t) + [a_{n+1}(t), a_{n+2}(t), \ldots], \quad n \geq 1.$$

The first result in the metrical theory of continued fractions is a conjecture from 1812 of C.F. Gauss according to which

$$(1)\quad \lim_{n\to\infty} \lambda\left(t : r_n(t) \geq \frac{1}{x}\right) = \lim_{n\to\infty} \lambda(t : [a_n(t), a_{n+1}(t), \ldots] \leq x) = \frac{\log(1+x)}{\log 2}$$

for any $0 < x \leq 1$. The reader is referred to [3] for an authoritative recent account of this theory. All its basic results can be obtained by using the ergodic theory of random systems with complete connections (see [4]).

MATH. REPORTS **12(62)**, *4* (2010), 339–350

The nth convergent of the continued fraction $[a_1, a_2, \ldots]$ is defined as

$$\frac{p_n}{q_n} = [a_1, \ldots, a_n]$$

with $(p_n, q_n) = 1$, $n \geq 1$, and we have

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}, \quad n \geq 1, \tag{2}$$

with $p_{-1}(t) = 1$, $p_0(t) = 0$, $q_{-1}(t) = 0$ and $q_0(t) = 1$.

Clearly,

$$r_n(t) = \frac{1}{T^{n-1}(t)} \; n \geq 1, \quad t \in I,$$

where the transformation T of I is defined as

$$T(t) = \begin{cases} \left\{ \dfrac{1}{t} \right\} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Here, $\{\cdot\}$ stands for fractionary part. The transformation T does not preserve the Lebesgue measure. Instead, it preserves Gauss measure γ defined as

$$\gamma(A) = \frac{1}{\log 2} \int_A \frac{\mathrm{d}x}{1+x}, \quad A \in \mathcal{B}_I.$$

Also, T is ergodic while equation (1) implies that it is mixing. See [1].

2. A PROBLEM OF P. SZÜSZ

This problem to be stated below points to a singular random system with complete connections to which the general theory does not apply.

It follows immediately from (2) that

$$q_n p_{n-1} - p_n q_{n-1} = (-1)^n, \quad n \geq 1.$$

Hence $(q_{n-1}, q_n) = 1$, $n \geq 1$.

For $n \geq 1$ consider the set $E_n(k_1, k_2, x)$ of irrational numbers $t \in I$ such that $q_{n-1}(t) \equiv k_1$, $q_{n-2}(t) \equiv k_2 (\text{mod } r)$, $0 \leq k_1, k_2 < r$, and $r_n(t) > \frac{1}{x}$, $0 < x \leq 1$, where r is a given positive integer, and set

$$m_n(k_1, k_2, x) = \lambda\left[E_n(k_1, k_2, x)\right].$$

If $(k_1, k_2, r) \neq 1$ then $E_n(k_1, k_2, x) = \emptyset$. Indeed, if $(k_1, k_2, r) = d > 1$ and $q_{n-1} \equiv k_1$, $q_{n-2} \equiv k_2 (\text{mod } r)$, then d is a common divisor of q_{n-1} and q_{n-2}, which contradicts the fact that $(q_{n-2}, q_{n-1}) = 1$. So, we shall only consider the case $(k_1, k_2, r) = 1$.

The problem solved by Szüsz [6], using a different method, is the existence of

$$\lim_{n \to \infty} m_n(k_1, k_2, x),$$

its determination as well as the convergence rate.

We shall start by considering the function $m_1(k_1, k_2, x)$. Since $q_{-1}(t) = 0$ and $q_0(t) = 1$ for any $t \in I$, if $k_1 \neq 1$ and $k_2 \neq 0$ then $m_1(k_1, k_2, x) = 0$. Clearly,

$$m_1(1, 0, x) = \lambda\left(r_1 > \frac{1}{x}\right) = x, \quad 0 < x \leq 1.$$

We note that for any $n \geq 2$ the relations

$$q_{n-1} \equiv k_1,\ q_{n-2} \equiv k_2 (\mathrm{mod}\ r),\ 0 \leq k_1, k_2 < r \text{ and } r_n \geq \frac{1}{x},\ 0 < x \leq 1$$

are equivalent to $l \leq r_{n-1} < l + x$ and $q_{n-2} \equiv k_2$, $q_{n-3} \equiv S(l) (\mathrm{mod}\ r)$, with $S(l)$ such that $0 \leq S(l) < r$, $k_1 - lk_2 \equiv S(l) (\mathrm{mod}\ r)$, $l = 1, 2, 3, \ldots$. Indeed,

$$r_{n-1} = a_{n-1} + r_n^{-1} \quad \text{and} \quad q_{n-1} = a_{n-1} q_{n-2} + q_{n-3},$$

while a_{n-1} can take any value $l = 1, 2, 3, \ldots$.

This equivalence, that amounts to decomposing the event $E_n(k_1, k_2, x)$ into pairwise disjoint events, allows us to write the equation

$$m_n(k_1, k_2, x) = \sum_{l \geq 1} \left\{ m_{n-1}\left(k_2, S(l), \frac{1}{l}\right) - m_{n-1}\left(k_2, S(l), \frac{1}{l+x}\right) \right\}$$

for any $n \geq 2$. Hence, by differentiation with respect to x, we get

$$m'_n(k_1, k_2, x) = \sum_{l \geq 1} m'_{n-1}\left(k_2, S(l), \frac{1}{l+x}\right) \frac{1}{(l+x)^2}, \quad n \geq 2.$$

The term by term differentiation of the series is clearly allowed since

$$\sum_{l \geq 1} \frac{1}{(l+x)^2} \leq \sum_{l \geq 1} \frac{1}{l^2} < \infty$$

while m'_1 does exist, as we have seen. Putting

$$g_n(k_1, k_2, x) = (1 + x)\, m'_n(k_1, k_2, x),$$

the recurrence relation just obtained yields

$$(3) \qquad g_n(k_1, k_2, x) = \sum_{l \geq 1} \frac{1 + x}{(l + x)(l + 1 + x)}\, g_{n-1}\left(k_2, S(l), \frac{1}{1+x}\right)$$

for any $n \geq 2$. Equation (3) is fundamental in what follows.

3. A RANDOM SYSTEM WITH COMPLETE CONNECTION

The transition from g_{n-1} to g_n in (3) is made by the operator associated with a random system with complete connections whose components are

$$W = \{(k_1, k_2, x) : 0 \le k_1, k_2 < r,\ (k_1, k_2, r) = 1,\ 0 \le x \le 1\},$$
$$X = N^* = \{1, 2, \ldots\},$$
$$P(w, l) \equiv \frac{1+x}{(l+x)(l+1+x)} \quad (= p_l(x)),$$
$$u(w, l) = \left(k_2, S(l), \frac{1}{l+x}\right) \text{ for } w = (k_1, k_2, x) \in W \text{ and } l \in N^*.$$

If we metrize W by defining the metric d as

$$d(w', w'') = \delta(k_1', k_1'') + \delta(k_2', k_2'') + |x' - x''|$$

for $w' = (k_1', k_2', x')$ and $w'' = (k_1'', k_2'', x'')$ with

$$\delta(x, y) = \begin{cases} 0 & x = y, \\ 1 & x \ne y, \end{cases}$$

then

$$\sup_{w' \ne w''} \frac{d(u(w', l), u(w'', l))}{d(w', w'')} \ge 1 \tag{4}$$

for any $l \in N^*$. Indeed, if $k_2' \ne k_2''$ then

$$\sup_{w' \ne w''} \frac{d(u(w', l), u(w'', l))}{d(w', w'')} \ge \frac{d(u((r-1, k_2', x), l), u((r-1, k_2'', x), l))}{d((r-1, k_2', x), (r-1, k_2'', x))} \ge 1.$$

Inequality (4) shows that a basic hypothesis in the ergodic theory of random systems with complete connections is not satisfied. Thus, Szüsz's problem leads to a singular random system with complete connections. In what follows we shall show that an ergodic theory of this system is still possible. The result we shall prove, a generalization of the Gauss-Kuzmin-Lévy theorem, can be stated as follows

Theorem 1. *There exist positive constants C and $q < 1$ such that*

$$\left| m_n(k_1, k_2, x) - \frac{\log(1+x)}{r^2 \cdot \log 2 \cdot \prod_{p|r} \left(1 - \frac{1}{p^2}\right)} \right| \le C q^n$$

for any $n \in N^*$, $0 \le x \le 1$, $r \in N^*$, $0 \le k_1, k_2 < r$. Here p stands for prime numbers.

4. **A FEW AUXILLIARY RESULTS**

We now give a few lemmas that are essentially used in the sequel. Let us show first that

$$\sum_{l\geq 1} \left|p_l'(x)\right| \leq \frac{1}{2}, \quad 0 \leq x \leq 1. \tag{5}$$

Indeed, since $\sum_{l\geq 1} p_l'(x) = 0$, we have

$$\begin{aligned}
\sum_{l\geq 1} \left|p_l'(x)\right| &= \sum_{l\geq 1} \frac{\left|l(l-1)-(1+x)^2\right|}{(l+x)^2(l+1+x)^2} = \\
&= \frac{1}{(2+x)^2} + \frac{\left|2-(1+x)^2\right|}{(2+x)^2(3+x)^2} + \sum_{l\geq 3} p_l'(x) = \\
&= \frac{1}{(2+x)^2} + \frac{\left|2-(1+x)^2\right|}{(2+x)^2(3+x)^2} - p_1'(x) - p_2'(x) = \\
&= \frac{1}{(2+x)^2} + \frac{\left|2-(1+x)^2\right| + (1+x)^2 - 2}{(2+x)^2(3+x)^2} = \\
&= \begin{cases} \dfrac{2}{(2+x)^2} & \text{if } 0 \leq x \leq \sqrt{2}-1, \\ \dfrac{4}{(3+x)^2} & \text{if } \sqrt{2}-1 \leq x \leq 1, \end{cases}
\end{aligned}$$

and (5) follows.

For $0 \leq x \leq 1$ and $l_1, l_2, \ldots, l_k \in N^*$ let us define recursively

$$p_{l_1 l_2,\ldots,l_k}(x) = \begin{cases} p_{l_1}(x) & \text{if } k = 1, \\ p_{l_1}(x) p_{l_2,\ldots,l_k}\left(\dfrac{1}{x+l_1}\right) & \text{if } k > 1. \end{cases}$$

We have

$$\sum_{l_1,l_2\geq 1} \left|p_{l_1 l_2}'(x)\right| \leq 1, \quad 0 \leq x \leq 1. \tag{6}$$

Indeed, by (5),

$$\begin{aligned}
\sum_{l_1,l_2\geq 1} \left|p_{l_1 l_2}'(x)\right| &\leq \sum_{l_1,l_2\geq 1} \left|p_{l_1}'(x) p_{l_2}\left(\frac{1}{x+l_1}\right)\right| + \\
&+ \sum_{l_1,l_2\geq 1} \left|p_{l_1}(x) p_{l_2}'\left(\frac{1}{x+l_1}\right)\right| \frac{1}{(x+l_1)^2} \leq \frac{1}{2} + \frac{1}{2} = 1.
\end{aligned}$$

LEMMA 2. *For any* $0 \le x,\ x' \le 1$ *we have*

$$\sup_{A\subset N^*}\left|\sum_{l\in A}\big(p_l(x)-p_l(x')\big)\right| \le \frac{1}{4}\left|x-x'\right|$$

and

$$\sup_{B\subset N^*\times N^*}\left|\sum_{(l_1,l_2)\in B}\big(p_{l_1l_2}(x)-p_{l_1l_2}(x')\big)\right| \le \frac{1}{2}\left|x-x'\right|.$$

Proof. It is well known that if I is at most countable a set and $(u_i)_{i\in I}$ and $(v_i)_{i\in I}$ two probability distributions on I, then

$$\sup_{E\subset I}\left|\sum_{i\in E}(u_i-v_i)\right| = \frac{1}{2}\sum_{i\in I}\left|u_i-v_i\right|.$$

Using this equation, on account of (5), the mean value theorem yields

$$\begin{aligned}
\sup_{A\subset N^*}\left|\sum_{l\in A}\big(p_l(x)-p_l(x')\big)\right| &= \frac{1}{2}\sum_{l\in N^*}\left|p_l(x)-p_l(x')\right| = \\
&= \frac{1}{2}\sum_{l\in N^*}\big(p_l(x)-p_l(x')\big)\operatorname{sgn}\big(p_l(x)-p_l(x')\big) = \\
&= \frac{1}{2}(x-x')\sum_{l\in N^*}p_l'(\theta_{x,x'})\operatorname{sgn}\big(p_l(x)-p_l(x')\big) \le \\
&\le \frac{1}{2}\left|x-x'\right|\sum_{l\in N^*}\left|p_l'\left(\theta_{x,x'}\right)\right| \le \frac{1}{4}\left|x-x'\right|.
\end{aligned}$$

Similarly, on account of (6), we have

$$\begin{aligned}
&\sup_{B\subset N^*\times N^*}\left|\sum_{(l_1,l_2)\in B}\big(p_{l_1l_2}(x)-p_{l_1l_2}(x')\big)\right| = \\
&= \frac{1}{2}\sum_{(l_1,l_2)\in N^*\times N^*}\left|p_{l_1l_2}(x)-p_{l_1l_2}(x')\right| = \\
&= \frac{1}{2}(x-x')\sum_{(l_1,l_2)\in N^*\times N^*}p_{l_1l_2}'(\theta_{x,x'})\operatorname{sgn}\big(p_{l_1l_2}(x)-p_{l_1l_2}(x')\big) \le \\
&\le \frac{1}{2}\left|x-x'\right|\sum_{(l_1,l_2)\in N^*\times N^*}\left|p_{l_1l_2}'(\theta_{x,x'})\right| \le \frac{1}{2}\left|x-x'\right|. \qquad \square
\end{aligned}$$

LEMMA 3. *There exists a constant $a > 0$ such that*

$$\sum_{\substack{l_1,l_2,l_3,l_4\geq 1\\ l_1\equiv l_1',\, l_2\equiv l_2',\, l_3\equiv l_3',\, l_4\equiv l_4' \pmod r}} p_{l_1,l_2,l_3,l_4}(x) \geq a$$

for any $0 \leq x \leq 1$, $1 \leq l_1', l_2', l_3', l_4' < r$.

Proof. We have

$$\sum_{\substack{l\geq 1\\ l\equiv l' (\operatorname{mod} r)}} p_l(x) = \sum_{\substack{l\geq 1\\ l\equiv l' (\operatorname{mod} r)}} \frac{1+x}{(l+x)(l+1+x)} \geq$$

$$\geq \sum_{\substack{l\geq 1\\ l\equiv l' (\operatorname{mod} r)}} \frac{1}{(l+x)(l+1+x)} \geq \sum_{m\geq 0} \frac{1}{(l'+mr+1)(l'+mr+2)} \geq$$

$$\geq \sum_{m\geq 0} \frac{1}{(m+1)r\left[(m+1)r+1\right]} \geq \sum_{m\geq 0} \frac{1}{2(m+1)^2r^2} = \frac{c}{r^2}$$

with c a positive constant for any $0 \leq x \leq 1$, $1 \leq l' < r$.

The inequality from the statement follows from the definition of p_{l_1,l_2,l_3,l_4} with $a = \left(\dfrac{c}{r^2}\right)^4$. $\square$

LEMMA 4. *Consider the map $v_l : \operatorname{pr}_1 W \to \operatorname{pr}_1 W$ defined as*

$$v_l(k_1, k_2) = (k_2, S(l)).$$

Then, whatever (k_1, k_2), $(k', k'') \in \operatorname{pr}_1 W$, there exist $1 \leq l_1, l_2, l_3, l_4 < r$ such that

$$v_{l_4}\left(v_{l_3}\left(v_{l_2}\left(v_{l_1}(k_1, k_2)\right)\right)\right) = (k', k'').$$

The *proof* of this result can be found in Szüsz ([6], pp. 156–157).

5. PROOF OF THEOREM 1

The operator U associated with the random system with complete connections considered is defined by

$$(Uf)(k_1, k_2, x) = \sum_{l\geq 1} p_l(x) f\left(k_2, S(l), \frac{1}{l+x}\right).$$

We obviously have $g_n(k_1, k_2, x) = \left(U^{n-1} g_1\right)(k_1, k_2, x)$, $n \geq 2$.

The operator U takes into itself the space $L(W)$ of functions defined on W such that

$$m(f) = \sup \frac{|f(k_1, k_2, x) - f(k_1, k_2, x')|}{|x - x'|} < \infty,$$

where the upper bound is taken over all $x \neq x'$, $0 \leq x, x' \leq 1$ and k_1, k_2 such that $(k_1, k_2, r) = 1$. Indeed, we have

$$(Uf)(k_1, k_2, x) - (Uf)(k_1, k_2, x') = \sum_{l \geq 1} \left(p_l(x) - p_l(x')\right) f\left(k_2, S(l), \frac{1}{l+x}\right) +$$
$$+ \sum_{l \geq 1} p_l(x') \left[f\left(k_2, S(l), \frac{1}{l+x}\right) - f\left(k_2, S(l), \frac{1}{l+x'}\right)\right].$$

We shall now use the result below (see [4, p. 38]).

Let (Ω, K, μ) be a measure space, where μ is a finite σ-additive signed measure such that $\mu(\Omega) = 0$. If f is a bounded real-valued measurable function defined on Ω, then

$$\int_\Omega f(\omega) \mathrm{d}\mu(\omega) \leq \operatorname{osc} f \cdot \sup_{A \in K} \mu(A).$$

In conjunction with Lemma 2 this result shows that the first sum above is bounded by

$$\frac{1}{4} \left| x - x' \right| \cdot \operatorname{osc} f,$$

where

$$\operatorname{osc} f = \sup_{\substack{(k_1,k_2,r)=1 \\ 0 \leq x \leq 1}} f(k_1, k_2, x) - \inf_{\substack{(k_1,k_2,r)=1 \\ 0 \leq x \leq 1}} f(k_1, k_2, x).$$

Since

$$\sum_{l \geq 1} \frac{p_l(x)}{l^2} \leq p_l(x) + \frac{1}{4} \sum_{l \geq 2} p_l(x) = p_l(x) + \frac{1}{4}(1 - p_l(x)) \leq$$
$$\leq \frac{3}{4(2+x)} + \frac{1}{4} \leq \frac{3}{8} + \frac{1}{4} = \frac{5}{8},$$

the second sum is bounded by

$$\sum_{l \geq 1} p_l(x) m(f) \left| \frac{1}{l+x} - \frac{1}{l+x'} \right| \leq \left| x - x' \right| m(f) \sum_{l \geq 2} \frac{p_l(x)}{l^2} < \frac{5}{8} \left| x - x' \right| m(f).$$

Therefore,

$$\frac{(Uf)(k_1, k_2, x) - (Uf)(k_1, k_2, x')}{|x - x'|} \leq \frac{1}{4} \operatorname{osc} f + \frac{5}{8} m(f),$$

whence

$$m(Uf) \leq \frac{1}{4} \operatorname{osc} f + \frac{5}{8} m(f).$$

As

$$\sup_{\substack{(k_1,k_2,r)=1 \\ 0 \leq x \leq 1}} (Uf) \leq \sup f \quad \text{and} \quad \inf_{\substack{(k_1,k_2,r)=1 \\ 0 \leq x \leq 1}} (Uf) \geq \inf f,$$

we have

$$\operatorname{osc}(U^n f) \geq \operatorname{osc}\left(U^{n+1} f\right), \quad n \geq 1.$$

We shall prove that there exists a constant $U^\infty f$ such that both $m\left(U^n f - U^\infty f\right)$ and $|U^n f - U^\infty f|$ converge geometrically to zero as $n \to \infty$. The proof consists of two steps. In the first step we derive a bound of $m(U^n f)$.

Remark that

$$(U^n f)(k_1, k_2, x) = \sum_{l_1 \geq 1} p_{l_1}(x)(U^{n-1} f)\left(k_2, S(l_1), \frac{1}{l_1 + x}\right) =$$

$$= \sum_{l_1, l_2 \geq 1} p_{l_1 l_2}(x)(U^{n-2} f)\left(v_{l_2}(v_{l_1}(k_1, k_2,)), \frac{1}{l_2 + \dfrac{1}{l_1 + x}}\right),$$

so that

$$(U^n f)(k_1, k_2, x) - (U^n f)(k_1, k_2, x') =$$

$$= \sum_{l_1, l_2 \geq 1} \left(p_{l_1 l_2}(x) - p_{l_1 l_2}(x')\right) U^{n-2} f\left(v_{l_2}(v_{l_1}(k_1, k_2,)), \frac{l_1 + x}{l_1 l_2 + l_2 x + 1}\right) +$$

$$+ \sum_{l_1, l_2 \geq 1} p_{l_1 l_2}(x') \left[U^{n-2} f\left(v_{l_2}(v_{l_1}(k_1, k_2,)), \frac{l_1 + x}{l_1 l_2 + l_2 x + 1}\right) - \right.$$

$$\left. - U^{n-2} f\left(v_{l_2}(v_{l_1}(k_1, k_2,)), \frac{l_1 + x'}{l_1 l_2 + l_2 x' + 1}\right)\right].$$

The reasoning used to prove that $m(Uf) < \infty$ shows that the first sum is bounded by

$$\frac{1}{2}\left|x - x'\right| \operatorname{osc}(U^{n-2} f)$$

while the second sum is bounded by

$$\sum_{l_1, l_2 \geq 1} p_{l_1 l_2}(x') m(U^{n-2} f)\left|x - x'\right| \frac{1}{(l_1 l_2)^2} \leq \frac{1}{4} m(U^{n-2} f)\left|x - x'\right|.$$

Therefore,

$$\frac{(U^n f)(k_1, k_2, x) - (U^n f)(k_1, k_2, x')}{|x - x'|} \leq \frac{1}{2} \operatorname{osc}(U^{n-2} f) + \frac{1}{4} m(U^{n-2} f),$$

whence

$$m(U^n f) \leq \frac{1}{2} \operatorname{osc}(U^{n-2} f) + \frac{1}{4} m(U^{n-2} f).$$

Since $\operatorname{osc}(U^{n-2}f) \le \operatorname{osc}(U^{n-4}f)$, we conclude that

$$m(U^n f) \le \frac{1}{2}\operatorname{osc}(U^{n-2}f) + \frac{1}{4}\left(\frac{1}{2}\operatorname{osc}(U^{n-4}f) + \frac{1}{4}\,m(U^{n-4}f)\right) \le \tag{7}$$

$$\le \frac{1}{16}\,m(U^{n-4}f) + \frac{5}{8}\operatorname{osc}(U^{n-4}f), \quad n \ge 4.$$

The second step amounts to deriving a bound of $\operatorname{osc}(U^n f)$. Let us denote

$$v_{l_1 l_2 l_3 l_4}(k_1,k_2) = v_{l_4}(v_{l_3}(v_{l_2}(v_{l_1}(k_1,k_2)))),$$

$$x\cdot(l_1,l_2,l_3,l_4) = \cfrac{1}{l_4+\cfrac{1}{l_3+\cfrac{1}{l_2+\cfrac{1}{l_1+x}}}}\cdot$$

We can then write

$$(U^n f)(k_1,k_2,x) = \sum p_{l_1l_2l_3l_4}(x)(U^{n-4}f)\left(v_{l_1l_2l_3l_4}(k_1,k_2),\ x\cdot(l_1l_2l_3l_4)\right).$$

Let $(k',k'') \in \mathrm{pr}_1 W$, $x = x_{n-4}$ be values for which the minimum of $U^{n-4}(f)$ is reached. By Lemma 4 there exist $1 \le l_1', l_2', l_3', l_4' < r$ such that

$$v_{l_1'l_2'l_3'l_4'}(k_1,k_2) = (k',k'').$$

Clearly, we shall have $v_{l_1l_2l_3l_4}(k_1,k_2) = (k',k'')$ for any $(l_1,l_2,l_3,l_4) \in A$, where

$$A = \left\{(m_1,m_2,m_3,m_4) : m_1 \equiv l_1',\ m_2 \equiv l_2',\ m_3 \equiv l_3',\ m_4 \equiv l_4' \,(\operatorname{mod} r)\right\}.$$

Therefore,

$$(U^n f)(k_1,k_2,x) = \sum_{(l_1,l_2,l_3,l_4)\in A} p_{l_1l_2,l_3,l_4}(x)\cdot$$

$$\cdot\left[(U^{n-4}f)\left(v_{l_1l_2l_3l_4}(k_1,k_2),\ x\cdot(l_1l_2l_3l_4)\right) - (U^{n-4}f)(k',k'',x_{n-4})\right] +$$

$$+(U^{n-4}f)(k',k'',x_{n-4}) \sum_{(l_1,l_2,l_3,l_4)\in A} p_{l_1l_2l_3l_4}(x) +$$

$$+ \sum_{(l_1,l_2,l_3,l_4)\notin A} p_{l_1l_2l_3l_4}(x)\left(U^{n-4}f\right)\left(v_{l_1l_2l_3l_4}(k_1,k_2),\ x\cdot(l_1l_2l_3l_4)\right),$$

whence putting

$$A(x) = \sum_{(l_1,l_2,l_3,l_4)\in A} p_{l_1l_2,l_3,l_4}(x)$$

that by Lemma 3 exceeds $a > 0$, we have

$$(U^n f)\,(k_1,k_2,x) \le A(x)\inf(U^{n-4}f) + (1-A(x))\sup(U^{n-4}f) +$$
$$+ \sum_{(l_1,l_2,l_3,l_4)\in A} p_{l_1 l_2,l_3,l_4}(x)\,[\max(x\cdot(l_1l_2l_3l_4),\ 1-x\cdot(l_1l_2l_3l_4))]\,m(U^{n-4}f).$$

Remark that

$$\sum_{(l_1,l_2,l_3,l_4)\in A} p_{l_1 l_2,l_3,l_4}(x)\,[\max(x\cdot(l_1l_2l_3l_4),\ 1-x\cdot(l_1l_2l_3l_4))] \le$$
$$\le b\cdot \sum_{(l_1,l_2,l_3,l_4)\in A} p_{l_1 l_2,l_3,l_4}(x),$$

where b is a positive constant strictly less than 1. Therefore,

$$(U^n f)(k_1,k_2,x) \le A(x)\inf(U^{n-4}f) + (1-A(x))\sup(U^{n-4}f) + bA(x)m(U^{n-4}f),$$

whence

$$(U^n f)(k_1,k_2,x) - \inf(U^{n-4}f) \le$$
$$\le (1-A(x))\sup(U^{n-4}f) - \inf(U^{n-4}f) + bA(x)m(U^{n-4}f) \le$$
$$\le a_n \mathrm{osc}(U^{n-4}f) + b_n m(U^{n-4}f)$$

with $a \le a_n \le 1-a$, $0 \le b_n \le b$, $a_n + b_n \le 1 - a(1-b) < 1$.

Since $\inf(Uf) \ge \inf f$, whence $\inf(U^n f) \ge \inf(U^{n-4}f)$, we finally have

$$\mathrm{osc}\,(U^n f) \le a_n \mathrm{osc}\left(U^{n-4}f\right) + b_n m(U^{n-4}f), \quad n \ge 4. \tag{8}$$

Equations (7) and (8) imply that $\mathrm{osc}(U^n f) + m(U^n f) \le O(q^n)$, where

$$q \le \max\left(\frac{11}{16}, 1 - a(1-b)\right) < 1.$$

As the sequences $(\sup(U^n f))_{n\ge1}$ and $(\inf(U^n f))_{n\ge1}$ are monotonic, the existence of the constant $U^\infty f$ is immediate.

Coming back to Szüsz's problem, the result obtained shows that

$$\lim_{n\to\infty}(1+x)m_n'(k_1,k_2,x) = c_0$$

does exist and does not depend on k_1 and k_2. Therefore,

$$\left|(1+x)m_n'(k_1,k_2,x) - c_0\right| \le cq^n,$$

whence

$$|m_n(k_1,k_2,x) - c_0\log(1+x)| \le Cq^n,$$

with suitable positive constants c and C.

Since

$$\sum_{\substack{(k_1,k_2,r)=1\\ 0\le k_1,k_2<r}} m_n(k_1,k_2,x) = \lambda\left(t : r_n(t) \ge \frac{1}{x}\right) \to \frac{\log(1+x)}{\log 2}$$

as $n \to \infty$, we have

$$c_0 = \frac{1}{\log 2} \Bigg/ \sum_{\substack{(k_1,k_2,r)=1\\ 0\le k_1,k_2<r}} 1 = \frac{1}{r^2 \log 2} \prod_{\substack{p|r\\ p\,\text{prime}}} \left(1 - \frac{1}{p^2}\right)^{-1}.$$

The proof is now complete. $\square$

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Received 14 October 2009

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