THE $L(2,1)$-LABELING ON TOTAL GRAPHS OF COMPLETE BIPARTITE GRAPHS

GABRIELA MIHAI

An $L(2,1)$-labeling of a connected graph G is defined as a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(u) - f(v)| \geq 2$ if $d_G(u,v) = 1$ and $|f(u) - f(v)| \geq 1$ if $d_G(u,v) = 2$, where $d_G(u,v)$ denotes the distance between vertices u and v in G. The $L(2,1)$-labeling number of G, denoted by $\lambda(G)$, is the smallest number k such that G has an $L(2,1)$-labeling with $\max\{f(v) : v \in V(G)\} = k$. In this paper, we consider the total graphs of the complete bipartite graphs and provide exact value for their λ-numbers.

AMS 2000 Subject Classification: 05C78, 05C12.

Key words: complete bipartite graph, total graph, graph labeling, $L(2,1)$-labeling.

1. INTRODUCTION

Motivated by the frequency assignment problem, Yeh [6] and Griggs and Yeh [3] proposed the notion of $L(2,1)$-labeling of a simple graph. An $L(2,1)$-labeling of a graph is a coloring of the vertices with nonnegative integers such that the labels on adjacent vertices differ by at least 2 and the labels on vertices at distance two differ by at least 1. This concept generalizes the notion of vertex coloring, because vertex coloring is the same as $L(1,0)$-labeling.

The $L(2,1)$-labeling number of G, denoted by $\lambda(G)$, is the smallest number k, such that G has a $L(2,1)$-labeling with no label greater than k.

Griggs and Yeh [3] showed that every graph with maximum degree Δ has an $L(2,1)$-labeling for which the value λ is at most $\Delta^2 + 2\Delta$. Chang and Kuo [1] provided a better upper bound $\Delta^2 + \Delta$.

Griggs and Yeh [3] conjectured that the best bound is Δ^2 for any graph G with maximum degree $\Delta \geq 2$; this bound is valid for graphs having diameter 2.

There are many articles that are studying the problem of $L(2,1)$ – labelings ([1–3; 5–7]). Most of these papers consider the values of λ on particular classes of graphs. For example, Shao, Yeh and Zhang [7] determined the λ-numbers for the total graphs of complete graphs.

Determining the value of λ was proved to be *NP*-complete [3].

The goal of this paper is to determine the exact value of λ for total graphs of the complete bipartite graphs. Griggs and Yeh's conjecture is true for all cases of these particular graphs. It also provides a better upper bound for λ-numbers corresponding to this class of graphs.

For basic terminology and notation in graph theory we refer to [4].

2. TOTAL GRAPHS OF COMPLETE BIPARTITE GRAPHS

The total graph $T(G)$ of a graph G is the graph whose vertices correspond to the vertices and edges of G, and whose two vertices are joint if and only if the corresponding vertices are adjacent, edges are adjacent or vertices and edges are incident in G.

In this paper we consider the complete bipartite graphs $K_{n,m}$ with $n \leq m$. Next, we will use the following notation. If vertices x and y are adjacent in $K_{n,m}$, then the edge $[x, y]$ will be a vertex in the total graph $T(K_{n,m})$, denoted by xy.

LEMMA 1. *If G is the total graph $T(K_{n,m})$ then,*

$$|V(G)| = n + m + nm,$$

$$|E(G)| = 3nm + n\binom{m}{2} + m\binom{n}{2}.$$

Proof. We know that the total graph $T(K_{n,m})$ has the vertices corresponding to the vertices and edges of complete bipartite graph $K_{n,m}$. This implies that $|V(G)| = |V(K_{n,m})| + |E(K_{n,m})| = n + m + nm$. Next, we will find the number of edges of the total graph $T(K_{n,m})$.

As shown in Figure 1, $|E(G)| = 3$ for $n = m = 1$.

Consider the bipartition $V(K_{n,m}) = V_1 \cup V_2$, where partite sets V_1 and V_2 are disjoint.

(1) $n = 1$ and $m \geq 2$. In this case V_1 contains one vertex denoted by x, and V_2 contains m vertices denoted by $y_1, y_2, \ldots, y_m$.

Vertices x and y_j are adjacent in $K_{1,m}$ for all $1 \leq j \leq m$. This implies that in the total graph $T(K_{1,m})$ the vertex x is adjacent to the vertices y_j and xy_j, vertex y_j is adjacent to vertex xy_j and the vertices xy_{j_1} and xy_{j_2} are adjacent if and only if $j_1 \neq j_2$, for all $1 \leq j, j_1, j_2 \leq m$. Therefore, $|E(G)| = 3m + \binom{m}{2}$.

(2) $m \geq n \geq 2$. In this case V_1 contains n vertices denoted by $x_1, x_2, \ldots,$ x_n, and V_2 contains m vertices denoted by $y_1, y_2, \ldots, y_m$. Vertices x_i and y_j are adjacent in $K_{n,m}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. This implies that in the total graph $T(K_{n,m})$ vertex x_i is adjacent to vertices y_j and x_iy_j, vertex y_j is adjacent to vertices x_iy_j, and vertices $x_{i_1}y_{j_1}$ and $x_{i_2}y_{j_2}$ are adjacent if

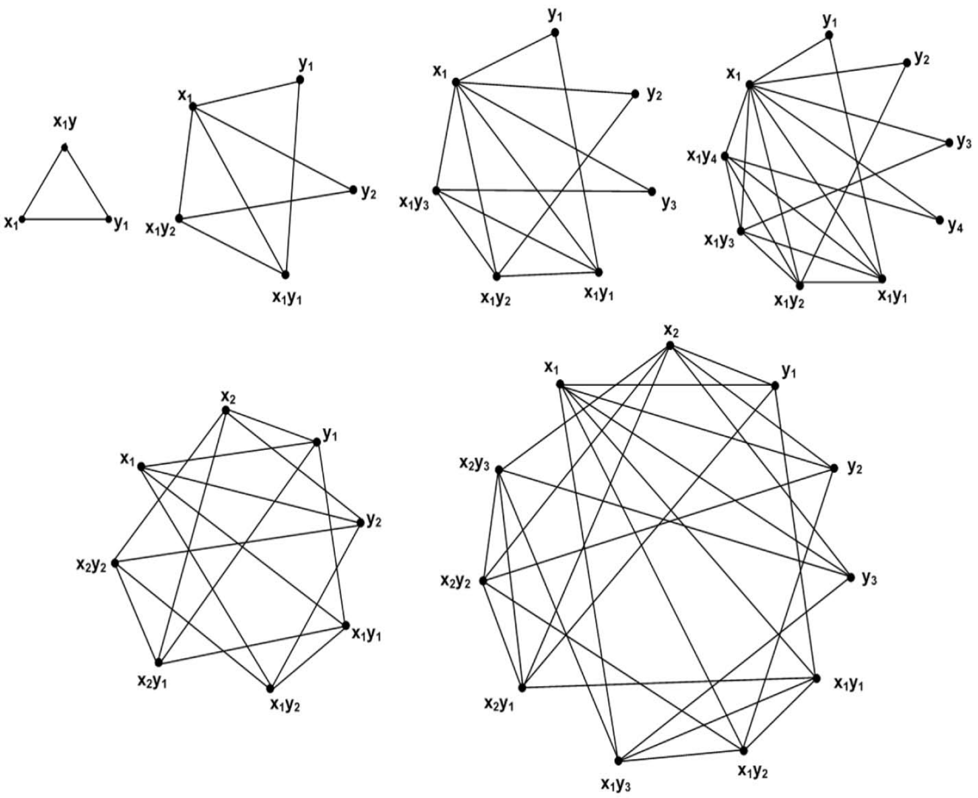

Fig. 1. Total graphs of $K_{1,1}$, $K_{1,2}$, $K_{1,3}$, $K_{1,4}$, $K_{2,2}$ and $K_{2,3}$.

and only if they have one common extremity, where $1 \le i, i_1, i_2 \le n$ and $1 \le j, j_1, j_2 \le m$. Therefore, $|E(G)| = 3nm + n\binom{m}{2} + m\binom{n}{2}$. □

Let u be a vertex of the total graph $T(G)$. If u corresponds to a vertex in graph G, then it is called a v-vertex. Otherwise, if u corresponds to an edge in G, then it is called an e-vertex [7].

Lemma 2. *The total graph $T(K_{n,m})$ has the diameter*

$$\operatorname{diam}(T(K_{n,m})) = \begin{cases} 1 & \text{if } n = m = 1, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. As shown in Figure 1 the total graph $T(K_{1,1})$ is K_3; therefore, in this case $\operatorname{diam}(T(K_{1,1})) = 1$. Otherwise, from the definition of the total graph $T(K_{n,m})$, $d_{T(K_{n,m})}(u_1, u_2) = 1$ if and only if u_1 and u_2 are v-vertices in different partite sets, or u_1 is a v-vertex and u_2 is an e-vertex that has one extremity equal to u_1 in $K_{n,m}$, or u_1 and u_2 are e-vertices that have one common extremity in $K_{n,m}$. Otherwise, $d_{T(K_{n,m})}(u_1, u_2) = 2$ because in all cases there is a vertex that is adjacent to both vertices u_1 and u_2 or an edge adjacent to vertices u_1 and u_2.

Therefore, in this case $\operatorname{diam}(T(K_{n,m})) = 2$. □

3. λ-NUMBERS FOR TOTAL GRAPHS $T(K_{n,m})$

Before proving Theorem 7, we need the following results.

THEOREM 3 (Dirac). *Let G be a graph with minimum degree δ. If $\delta \geq |V(G)|/2$ then there is a Hamiltonian cycle in G.*

THEOREM 4 ([2]). *Let G be a graph of order n and $\overline{G}$ its complement. Let $c(\overline{G})$ be the smallest number of vertex-disjoint paths in $\overline{G}$ needed to cover the vertex set. Then*

(i) $\lambda(G) \leq n-1$ *if and only if* $c(\overline{G}) = 1$ *(i.e., $\overline{G}$ has a Hamiltonian path);*

(ii) $\lambda(G) = n + r - 2$ *if and only if* $c(\overline{G}) = r$ *and* $r \geq 2$.

THEOREM 5 ([3]). *Let G be a graph that has diameter 2. If the complement of the graph G has a Hamiltonian path, then $\lambda(G) = |V(G)| - 1$.*

LEMMA 6. *If G is the total graph $T(K_{n,m})$, then the minimum degree of its complement is $\delta(\overline{G}) = nm - 1 + n - m$.*

Proof. Let the bipartition $V(K_{1,m}) = V_1 \cup V_2$, where partite sets V_1 and V_2 are disjoint.

(1) $n = m = 1$. As shown in Figure 1, $|V(T(K_{1,1}))| = 3$ and $d_{T(K_{1,1})}(u) = 2$ for all vertices u. This implies that $d_{\overline{T(K_{1,1})}}(u) = |V(T(K_{1,1}))|-1-d_{T(K_{1,1})}(u) = 0$ for all vertices u of $\overline{T(K_{1,1})}$. Thus, $\delta(\overline{T(K_{1,1})}) = 0$.

(2) $n = 1$ and $m \geq 2$. In this case, V_1 contains one vertex denoted by x, and V_2 contains m vertices denoted by $y_1, y_2, \ldots, y_m$. Vertices x and y_j are adjacent in $K_{1,m}$ for all $1 \leq j \leq m$. This implies that in the total graph $T(K_{1,m})$ vertex x is adjacent to vertices y_j and xy_j for all $1 \leq j \leq m$. This implies that $d_{T(K_{1,m})}(x) = 2m$. Then, $d_{\overline{T(K_{1,m})}}(x) = |V(T(K_{1,m}))| - 1 - d_{T(K_{1,m})}(x) = 0$, thus implying $\delta(\overline{T(K_{1,m})}) = 0$.

(3) $m \geq n \geq 2$. In this case, V_1 contains n vertices denoted by $x_1, x_2, \ldots, x_n$, and V_2 contains m vertices denoted by $y_1, y_2, \ldots, y_m$. Vertices x_i and y_j are adjacent in $K_{n,m}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Thus, in the total graph $T(K_{n,m})$ vertex x_i is adjacent to vertices y_j and x_iy_j for all $1 \leq j \leq m$. This implies that $d_{T(K_{n,m})}(x_i) = 2m$ and $d_{\overline{T(K_{n,m})}}(x_i) = |V(T(K_{n,m}))| - 1 - d_{T(K_{n,m})}(x_i) = nm - 1 + n - m$. Similary, we obtain $d_{T(K_{n,m})}(y_j) = 2n$ and $d_{\overline{T(K_{n,m})}}(y_j) = |V(T(K_{n,m}))| - 1 - d_{T(K_{n,m})}(y_j) = nm - 1 + m - n$. Also,

in $T(K_{n,m})$ vertex x_iy_j is adjacent to vertices x_i, y_j, $x_iy_{j_1}$ and $x_{i_1}y_j$, where $1 \le i_1 \le n$, $i_1 \ne i$ and $1 \le j_1 \le m$, $j_1 \ne j$. This implies that $d_{T(K_{n,m})}(x_iy_j)$ $= n+m$ and $d_{\overline{T(K_{n,m})}}(x_iy_j) = |V(T(K_{n,m}))| - 1 - d_{T(K_{n,m})}(x_iy_j) = nm - 1$. Since $m \ge n$ one deduces $\delta(\overline{T(K_{n,m})}) = nm - 1 + n - m$. □

THEOREM 7. *We have*

$$\lambda(T(K_{n,m})) = \begin{cases} 4 & \text{if } n = m = 1, \\ 2m+1 & \text{if } n = 1 \text{ and } m \ge 2, \\ nm+n+m-1 & \text{if } m \ge n \ge 2. \end{cases}$$

Proof. We distinguish several cases.

(1) $n = m = 1$. It is easy to see that the function $f : V(T(K_{1,1})) \to Z^*$, with $f(x_1) = 0$, $f(y_1) = 2$ and $f(x_1y_1) = 4$ is an $L(2,1)$-labeling of the total graph $T(K_{1,1})$ and the label 4 cannot be decreased. This implies that $\lambda(T(K_{1,1})) = 4$.

(2) $n = 1$ and $m \ge 2$. In this case, V_1 contains one vertex denoted by x_1, and V_2 contains m vertices denoted by $y_1, y_2, \ldots, y_m$. As we have seen before, $d_{\overline{T(K_{1,m})}}(x_1) = 0$. It follows that by Theorem 5, $\lambda(\overline{T(K_{1,m})}) \ge |V(T(K_{1,m}))| = 2m+1$ since $c(\overline{T(K_{1,m})}) \ge 2$.

For $m = 2$ and $m = 3$ the theorem will be verified directly by proving the opposite inequality. If $m = 2$ then let x_1, y_1, y_2, x_1y_1 and x_1y_2 be vertices of the total graph $T(K_{1,2})$ as shown in Figure 1. The function $f : V(T(K_{1,2})) \to Z^*$ represented by the table

u	x_1	y_1	y_2	x_1y_1	x_1y_2
$f(u)$	0	3	4	5	2

is an $L(2,1)$-labeling of the total graph $T(K_{1,2})$. This implies that

$$\lambda(T(K_{1,2})) = 5.$$

If $m = 3$ then let $x_1, y_1, y_2, y_3, x_1y_1, x_1y_2$ and x_1y_3 be the vertices of the total graph $T(K_{1,3})$ as shown in Figure 1. The function $f : V(T(K_{1,3})) \to Z^*$ represented by the table

u	x_1	y_1	y_2	y_3	x_1y_1	x_1y_2	x_1y_3
$f(u)$	0	4	6	2	7	3	5

is an $L(2,1)$-labeling of the total graph $T(K_{1,3})$. This implies that

$$\lambda(T(K_{1,3})) = 7.$$

For $m \ge 4$, we will prove that in $\overline{T(K_{1,m})}$ the set of vertices $V(\overline{T(K_{1,m})}) \setminus \{x_1\}$ includes a Hamiltonian path of length $2m - 1$ denoted by L_{2m}, with extremities x_1y_m and y_m. The proof is by induction on m.

For $m = 4$ this property is true because in the graph $\overline{T(K_{1,4})}$ the set of vertices $V(\overline{T(K_{1,4})}) \setminus \{x_1\}$ includes the Hamiltonian path $L_8 = x_1y_4$, y_3, x_1y_2, y_1, x_1y_3, y_2, x_1y_1, y_4 that has length 7 and extremities x_1y_4 and y_4 (see Figure 1, $T(K_{1,4})$).

Let $m \geq 4$ and assume that $\overline{T(K_{1,m})} \setminus \{x_1\}$ has a Hamiltonian path of length $2m-1$ denoted by $L_{2m} = x_1y_m, \dots, y_m$ with extremities x_1y_m and y_m.

Since $V(\overline{T(K_{1,m+1})}) = V(\overline{T(K_{1,m})}) \cup \{x_1y_{m+1}, y_{m+1}\}$ and y_{m+1} is adjacent to vertex x_1y_m and x_1y_{m+1} is adjacent to vertex y_m, we can define the path $L_{2(m+1)} = y_{m+1}, L_{2m}, x_1y_{m+1}$ that verify the induction hypothesis. Since in $\overline{T(K_{1,m})}$ the vertex x_1 is isolated and the set $V(\overline{T(K_{1,m})}) \setminus \{x_1\}$ includes a Hamiltonian path, the smallest number of vertex-disjoint paths in $\overline{T(K_{1,m})}$ needed to cover the vertex set $V(\overline{T(K_{1,m})})$ is 2. By Theorem 4 we obtain $\lambda(T(K_{1,m})) = 2m + 1$.

(3) $n = 2$ and $m \geq 2$. In this case, V_1 contains two vertices, x_1 and x_2, and V_2 contains m vertices, $y_1, y_2, \dots, y_m$. We will prove that there is a Hamiltonian path in $\overline{T(K_{2,m})}$ and then we can apply Theorem 5.

For $m = 2$ the result will be verified directly: the path $L_7 = x_1y_2$, y_1, x_2y_2, x_1, x_2y_1, y_2, x_1y_1, x_2 is a Hamiltonian path in $\overline{T(K_{2,2})}$ (see Figure 1, $T(K_{2,2})$).

For $m \geq 3$ we will prove that $\overline{T(K_{2,m})}$ has a Hamiltonian path of length $3m+1$ denoted by L_{3m+2}, with extremities x_iy_m and y_m, and containg an edge $[x_jy_m, y_p]$, where $i, j \in \{1, 2\}$, $i \neq j$ and $1 \leq p \leq m-1$. The proof is by induction on m.

For $m = 3$ the graph $\overline{T(K_{2,3})}$ has a Hamiltonian path $L_{11} = x_1y_3$, y_1, x_2y_3, y_2, x_1y_1, x_2, x_1y_2, x_2y_1, x_1, x_2y_2, y_3 with extremities x_1y_3 and y_3, and containg the edge $[x_2y_3,\ y_2]$ (see Figure 1, $T(K_{2,3})$).

Let $m \geq 3$ and assume that $\overline{T(K_{2,m})}$ has a Hamiltonian path of length $3m+1$ denoted by L_{3m+2}, with extremities x_iy_m and y_m and containg an edge $[x_jy_m, y_p]$, where $i, j \in \{1, 2\}$, $i \neq j$ and $1 \leq p \leq m-1$. We have $V(\overline{T(K_{2,m+1})}) = V(\overline{T(K_{2,m})}) \cup \{y_{m+1}, x_1y_{m+1}, x_2y_{m+1}\}$.

If $i = 1$ then $j = 2$ and this implies that $L_{3m+2} = x_1y_m, \dots, x_2y_m, y_p, \dots, y_m$. We denote $L^1_{3m+2} = x_1y_m, \dots, x_2y_m$ and $L^2_{3m+2} = y_p, \dots, y_m$. In $\overline{T(K_{2,m+1})}$ the vertex y_{m+1} is adjacent to vertex x_1y_m, the vertex x_1y_{m+1} is adjacent to vertices x_2y_m and y_p, and the vertex x_2y_{m+1} is adjacent to vertex y_m. So, we can define the path $L_{3(m+1)+2} = y_{m+1}$, L^1_{3m+2}, x_1y_{m+1}, L^2_{3m+2}, x_2y_{m+1} that verifies the induction hypothesis.

If $i = 2$ then $j = 1$ and this implies that $L_{3m+2} = x_2y_m, \dots, x_1y_m, y_p, \dots, y_m$. By denoting $L^1_{3m+2} = x_2y_m, \dots, x_1y_m$ and $L^2_{3m+2} = y_p, \dots, y_m$, since in $\overline{T(K_{2,m+1})}$ the vertex y_{m+1} is adjacent to vertex x_2y_m, the vertex x_2y_{m+1} is adjacent to vertices x_1y_m and y_p, and the vertex x_1y_{m+1} is adjacent to vertex y_m, the path $L_{3(m+1)+2} = y_{m+1}$, L^1_{3m+2}, x_2y_{m+1}, L^2_{3m+2}, x_1y_{m+1} verifies the induction hypothesis. It follows that $\overline{T(K_{2,m})}$, for all $m \geq 3$, has a Hamiltonian path.

Consequently, for all $m \geq 2$ the graph $\overline{T(K_{2,m})}$ has a Hamiltonian path. Since the total graph $T(K_{2,m})$ is a diameter 2 graph, by Theorem 5 we obtain $\lambda(T(K_{2,m})) = 3m + 1$.

(4) $m \geq n \geq 3$. By Lemma 6, $G = \overline{T(K_{n,m})}$ has minimum degree $\delta = nm - 1 + n - m$ and $|V(G)| = nm + n + m$. This implies that $\delta \geq |V(G)|/2$. By Theorem 3, there is a Hamiltonian cycle in $\overline{T(K_{n,m})}$. By Theorem 5, we have $\lambda(T(K_{n,m})) = n + m + nm - 1$. □

Since the total graph $T(K_{n,m})$ has maximum degree $\Delta = 2m$ we get:

Corollary 8. $\lambda(T(K_{n,m})) \leq \frac{1}{4}\Delta^2 + \Delta - 1$ *for all* $m \geq n \geq 2$.

By Theorem 7 we also obtain $\lambda(T(K_{1,1})) = \Delta^2 = 4$ and $\lambda(T(K_{1,m})) = \Delta + 1$ for all $m \geq 2$, values that agree with Griggs and Yeh's conjecture.

REFERENCES

[1] G.J. Chang and D. Kuo, *The $L(2,1)$-labeling on graphs.* SIAM J. Discrete Math. **9** (1996), 309–316.

[2] J. Georges, D.W. Mauro and M. Whittlesey, *Relating path covering to vertex labelings with a condition at distance two.* Discrete Math. **135** (1994), 103–111.

[3] J.R. Griggs and R.K. Yeh, *Labeling graphs with a condition at distance two.* SIAM J. Discrete Math. **5** (1992), 586–595.

[4] F. Harary, *Graph Theory.* Addison-Wesley, Reading, MA, 1969.

[5] D.D.-F. Liu and R.K. Yeh, *On distance-two labelings of graphs.* Ars Combin. **47** (1997), 13–22.

[6] R.K. Yeh, *Labeling Graphs with a Condition at Distance Two.* Ph.D. Thesis, Dept. of Math., Univ. of South Carolina, Columbia, SC, USA, 1990.

[7] Z. Shao, R.K. Yeh and D. Zhang, *The $L(2,1)$-labeling on graphs and the frequency assignment problem.* Appl. Math. Lett. **21** (2008), 37–41.

Received 9 July 2009

University of Bucharest
Faculty of Mathematics and Computer Science
Str. Academiei 14
010014 Bucharest, Romania
criogab@fmi.unibuc.ro