ON IDEAL CONVERGENCE OF DOUBLE SEQUENCES IN PROBABILISTIC NORMED SPACES

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One of the generalizations of statistical convergence is $I$-convergence which was introduced by Kostyrko et al. [12]. In this paper, we define and study the concept of $I$-convergence, $I^*$-convergence, $I$-limit points and $I$-cluster points of double sequences in probabilistic normed space. We discuss the relationship between $I_2$-convergence and $I^*_2$-convergence, i.e., we show that $I^*_2$-convergence implies $I_2$-convergence in probabilistic normed space. Furthermore, we have also demonstrated through an example that, in general, $I_2$-convergence does not imply $I^*_2$-convergence in probabilistic normed space.

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1. INTRODUCTION AND PRELIMINARIES

The concept of statistical convergence for sequences of real numbers was introduced by Fast [4] and Steinhaus [22] independently in the same year 1951 and since then several generalizations and applications of this notion have been investigated by various authors, namely [3], [7], [15], [16], [17], [18], [19]. One of its interesting generalization is $I$-convergence which was given by Kostyrko et al. [12]. Recently $I$-convergence for sequences of functions has been studied by Balcerzak et al. [2] and by Komisarski [13].

The theory of probabilistic normed spaces [5] originated from the concept of statistical metric spaces which was introduced by Menger [14] and further studied by Schweizer and Sklar [20, 21]. It provides an important method of generalizing the deterministic results of normed linear spaces. It has also very useful applications in various fields, e.g., continuity properties [1], topological spaces [5], linear operators [8], study of boundedness [9], convergence of random variables [10] etc.

In this paper we study the concept of $I$-convergence and $I^*$-convergence in a more general setting, i.e., in probabilistic normed spaces. We also define
I-limit points and I-cluster points in probabilistic normed space and prove some interesting results.

We recall some notations and basic definitions used in this paper.

**Definition 1.1 ([21]).** A triangular norm ($t$-norm) is a continuous mapping $*: [0, 1] \times [0, 1] \to [0, 1]$ such that $([0, 1], *)$ is an abelian monoid with unit one and $c * d \geq a * b$ if $c \geq a$ and $d \geq b$ for all $a, b, c, d \in [0, 1]$.

**Definition 1.2 ([5]).** A function $f: \mathbb{R} \to \mathbb{R}_0^+$ is called a distribution function if it is non-decreasing and left-continuous with $\inf_{t \in \mathbb{R}} f(t) = 0$ and $\sup_{t \in \mathbb{R}} f(t) = 1$.

By $D$, we denote the set of all distribution functions.

**Definition 1.3 ([5]).** Let $X$ be a real linear space and $\nu: X \to D$. A probabilistic norm or $\nu$-norm is a $t$-norm satisfying the following conditions:

(i) $\nu_x(0) = 0$,
(ii) $\nu_x(t) = 1$ for all $t > 0$ iff $x = 0$,
(iii) $\nu_{\alpha x}(t) = \nu_x(\frac{t}{|\alpha|})$ for all $\alpha \in \mathbb{R} \setminus \{0\}$ and for all $t > 0$,
(iv) $\nu_{x+y}(s+t) \geq \nu_x(s) * \nu_y(t)$ for all $x, y \in X$ and $s, t \in \mathbb{R}_0^+$,

where $\nu_x$ means $\nu(x)$ and $\nu_x(t)$ is the value of $\nu_x$ at $t \in \mathbb{R}$.

The $(X, \nu, *)$ is called a probabilistic normed space (for short, PNS).

**Definition 1.4 ([11]).** Let $(X, \nu, *)$ be an PNS. A sequence $x = (x_k)$ is said to be convergent to $\xi \in X$ with respect to the probabilistic norm $\nu$, that is, $x_k \xrightarrow{\nu} \xi$ if for every $t > 0$ and $\epsilon \in (0, 1)$, there is a positive integer $k_0$ such that $\nu_{x_k - \xi}(t) > 1 - \epsilon$ whenever $k \geq k_0$. In this case we write $\nu$-lim $x = \xi$.

**Remark 1.1.** Let $(X, \| \cdot \|)$ be a real normed linear space, and

$$\nu(x, t) := \frac{t}{t + \|x\|}$$

for all $x \in X$ and $t > 0$. Then $x_n \xrightarrow{\| \cdot \|} x$ if and only if $x_n \xrightarrow{\nu} x$.

**Definition 1.5 ([6]).** Let $K$ be a subset of $\mathbb{N}$, the set of natural numbers. The asymptotic density of $K$ denoted by $\delta(K)$, is defined as

$$\delta(K) = \lim_{n \to \infty} \frac{1}{n} |\{k \leq n : k \in K\}|,$$

where the vertical bars denote the cardinality of the enclosed set.

**Definition 1.6 ([4, 22]).** A number sequence $x = (x_k)$ is said to be statistically convergent to the number $\ell$ if for each $\epsilon > 0$, the set $K(\epsilon) = \{k \leq n :
\[|x_k - \ell| > \epsilon\} \text{ has asymptotic density zero, i.e.,} \]
\[
\lim_{n \to \infty} \frac{1}{n} \{|k \leq n : |x_k - \ell| > \epsilon\} = 0.
\]
In this case we write \(st\)-\(\lim x = \ell\).

**Definition 1.7** ([11]). Let \((X, \nu, \ast)\) be a PNS. A sequence \(x = (x_k)\) is said to be statistically convergent to \(\xi \in X\) with respect to the probabilistic norm \(\nu\) provided that, for every \(t > 0\) and \(\epsilon > 0\),
\[
\delta(\{k \leq n : \nu_{x_k - \xi}(t) \leq 1 - \epsilon\}) = 0
\]
or, equivalently,
\[
\lim_{n \to \infty} \frac{1}{n} \{|k \leq n : \nu_{x_k - \xi}(t) \leq 1 - \epsilon\} = 0.
\]
In this case we write \(st_{\nu\ast}\)-\(\lim x = \xi\).

**Definition 1.8** ([12]). If \(X\) is a non-empty set, a family \(I \subseteq 2^X\) of subsets of \(X\) is called an ideal in \(X\) if
1. \(\emptyset \in I\),
2. \(A, B \in I\) implies \(A \cup B \in I\),
3. for each \(A \in I\) and \(B \subseteq A\) we have \(B \in I\).

An ideal \(I\) is called nontrivial if \(X \notin I\).

**Definition 1.9** ([12]). Let \(X\) be a non-empty set. A non-empty family of sets \(F \subseteq P(X)\), the power set of \(X\), is called a filter on \(X\) if and only if
1. \(\emptyset \notin F\),
2. \(A, B \in F\) implies \(A \cap B \in F\),
3. for each \(A \in F\) and \(B \supset A\) we have \(B \in F\).

**Definition 1.10** ([12]). A non-trivial ideal \(I\) in \(X\) is called an admissible ideal if it is different from \(P(\mathbb{N})\) and it contains all singletons, i.e., \(\{x\} \in I\) for each \(x \in X\).

Let \(I \subseteq P(X)\) be a non-trivial ideal. A class \(F(I) = \{M \subseteq X : M = X \setminus A\text{, for some } A \in I\}\) is a filter on \(X\), called the filter associated with the ideal \(I\).

**Definition 1.11** ([12]). An admissible ideal \(I \subseteq P(\mathbb{N})\) is said to satisfy the condition \((AP)\) if for every sequence \((A_n)_{n \in \mathbb{N}}\) of pairwise disjoint sets from \(I\) there are sets \(B_n \subseteq \mathbb{N}, n \in \mathbb{N}\), such that the symmetric difference \(A_n \Delta B_n\) is a finite set for every \(n\) and \(\bigcup_{n \in \mathbb{N}} B_n \in I\).

**Definition 1.12** ([12]). Let \(I \subseteq 2^\mathbb{N}\) be a non-trivial ideal in \(\mathbb{N}\). A sequence \(x = (x_k)\) is said to be \(I\)-convergent to \(L\) if for every \(\epsilon > 0\), the set \(\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I\). In this case we write \(I\)-\(\lim x = L\).
2. I₂-CONVERGENCE IN PNS

In this section, we study the concept of ideal convergence of double sequences in probabilistic normed space. Throughout the paper we take $I_2$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

We define

**Definition 2.1.** Let $I$ be a non trivial ideal of $\mathbb{N} \times \mathbb{N}$ and $(X, \nu, \ast)$ be a probabilistic normed space. A double sequence $x = (x_{jk})$ of elements of $X$ is said to be $I_2$-convergent to $\xi \in X$ with respect to the probabilistic norm $\nu$ (or $I_2 \nu$-convergent to $\xi$) if for each $\epsilon > 0$ and $t > 0$,

$$\nu_{x_{jk} - \xi}(t) \leq 1 - \epsilon \quad \forall (j, k) \in \mathbb{N} \times \mathbb{N}$$

We get

$$K_{x_{jk} - \xi}(t) = \left\{ (j, k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk} - \xi}(t) \leq 1 - \epsilon \right\} \in I_2.$$

In this case we write $I_2 \nu$-lim $x = \xi$.

**Theorem 2.1.** Let $(X, \nu, \ast)$ be a PNS. Then, the following statements are equivalent:

(i) $I_2 \nu$-lim $x = \xi$;
(ii) $\nu_{x_{jk} - \xi}(t) \leq 1 - \epsilon \quad \forall (j, k) \in \mathbb{N} \times \mathbb{N}$;
(iii) $\nu_{x_{jk} - \xi}(t) > 1 - \epsilon \quad \forall (j, k) \in \mathbb{N} \times \mathbb{N}$;
(iv) $I_2 \nu$-lim $x = \xi$.

The proof is standard. □

**Theorem 2.2.** Let $(X, \nu, \ast)$ be a PNS. If a double sequence $x = (x_{jk})$ is $I_2 \nu$-convergent then $I_2 \nu$-limit is unique.

Proof. Suppose that $I_2 \nu$-lim $x = \xi_1$ and $I_2 \nu$-lim $x = \xi_2$. Given $\epsilon > 0$ and $t > 0$, choose $r > 0$ such that $(1 - r) * (1 - r) \geq 1 - \epsilon$. Then, we define the following sets as

$$K_{\nu,1}(r, t) = \left\{ (j, k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk} - \xi_1}(t) \leq 1 - r \right\}$$

$$K_{\nu,2}(r, t) = \left\{ (j, k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk} - \xi_2}(t) \leq 1 - r \right\}.$$

Since $I_2 \nu$-lim $x = \xi_1$, we have $K_{\nu,1}(r, t) \in I_2$. Furthermore, using $I_2 \nu$-lim $x = \xi_2$, we get $K_{\nu,2}(r, t) \in I_2$. Now, let $K_\nu(r, t) = K_{\nu,1}(r, t) \cup K_{\nu,2}(r, t) \in I_2$. Then we see that $K_\nu(r, t) \in I_2$. This implies that its complement $K_\nu^C(r, t)$ is non empty in $F(I_2)$. If $(j, k) \in K_\nu^C(r, t)$, then we have $F(j, k) \in K_{\nu,1}(r, t) \cap K_{\nu,2}(r, t)$, and so

$$\nu_{\xi_1 - \xi_2}(t) \geq \nu_{x_{jk} - \xi_1}(t/2) \ast \nu_{x_{jk} - \xi_2}(t/2) > (1 - r) \ast (1 - r).$$

Since $(1 - r) \ast (1 - r) \geq 1 - \epsilon$, we have $\nu_{\xi_1 - \xi_2}(t) > 1 - \epsilon$. Since $\epsilon > 0$ was arbitrary, we get $\nu(\xi_1 - \xi_2, t) = 1$ for all $t > 0$, which yields $\xi_1 = \xi_2$.

This completes the proof of the theorem. □

**Theorem 2.3.** Let $(X, \nu, \ast)$ be a PNS.
(i) If \( \nu \)-\( \lim \) \( x_{jk} = \xi \) then \( I_2^- \)-\( \lim \) \( x_{jk} = \xi \).

(ii) If \( I_2^- \)-\( \lim \) \( x_{jk} = \xi_1 \) and \( I_2^- \)-\( \lim \) \( y_{jk} = \xi_2 \) then \( I_2^- \)-\( \lim \) \((x_{jk} + y_{jk}) = (\xi_1 + \xi_2) \).

(iii) If \( I_2^- \)-\( \lim \) \( x_{jk} = \xi \) then \( I_2^- \)-\( \lim \) \( ax_{jk} = a\xi \).

**Proof.** (i) Suppose that \( \nu \)-\( \lim \) \( x_{jk} = \xi \). Then for each \( \epsilon > 0 \) and \( t > 0 \) there exists a positive integer \( N \) such that

\[
\nu_{x_{jk}-\xi}(t) > 1 - \epsilon
\]

for each \( j, k > N \). Since the set

\[
A(t) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk}-\xi}(t) \leq 1 - \epsilon\}
\]

is contained in \( \{1, 2, 3, \ldots, N - 1\} \) and the ideal \( I_2 \) is admissible we have \( A(t) \in I_2 \). Hence \( I_2^- \)-\( \lim x_{jk} = \xi \).

(ii) Let \( I_2^- \)-\( \lim x_{jk} = \xi_1 \) and \( I_2^- \)-\( \lim y_{jk} = \xi_2 \). For given \( \epsilon > 0 \) and \( t > 0 \), choose \( r > 0 \) such that \((1 - r) \ast (1 - r) > 1 - \epsilon \). Define the sets

\[
K_{\nu,1}(r, t) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk}-\xi_1}(t) \leq 1 - r\},
\]

\[
K_{\nu,2}(r, t) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \nu_{y_{jk}-\xi_2}(t) \leq 1 - r\}.
\]

Since \( I_2^- \)-\( \lim x_{jk} = \xi_1 \), we have

\[
K_{\nu,1}(r, t) \in I_2.
\]

Furthermore, using \( I_2^- \)-\( \lim x = \xi_2 \), we get

\[
K_{\nu,2}(r, t) \in I_2.
\]

Now, let \( K_{\nu}(r, t) = K_{\nu,1}(r, t) \cup K_{\nu,2}(r, t) \). Then \( K_{\nu}(r, t) \in I_2 \) which implies that \( K_{\nu}^+(r, t) \) is non empty in \( F(I_2) \). Now, we have to show that \( K_{\nu}^+(r, t) \subset \{(j, k) \in \mathbb{N} \times \mathbb{N} : \nu(x_{jk} + y_{jk} - (\xi_1 + \xi_2))(t) > 1 - \epsilon \} \). If \( (j, k) \in K_{\nu}^+(r, t) \), then we have \( \nu_{x_{jk}-\xi_1}(t) \geq \nu_{x_{jk}} - \xi_1 + \nu_{y_{jk}-\xi_2}(t) > 1 - r \) and \( \nu_{y_{jk}-\xi_2}(t) > 1 - r \). Therefore,

\[
\nu(x_{jk} + y_{jk} - (\xi_1 + \xi_2))(t) \geq \nu_{x_{jk}} - \xi_1 \left(\frac{t}{2}\right) \ast \nu_{y_{jk}} - \xi_2 \left(\frac{t}{2}\right) > (1 - r) \ast (1 - r) > 1 - \epsilon.
\]

This shows that

\[
K_{\nu}^+(r, t) \subset \{(j, k) \in \mathbb{N} \times \mathbb{N} : \nu(x_{jk} + y_{jk} - (\xi_1 + \xi_2))(t) > 1 - \epsilon \}.
\]

Since \( K_{\nu}^+(r, t) \in F(I_2) \), we have \( I_2^- \)-\( \lim x_{jk} + y_{jk} = (\xi_1 + \xi_2) \).

(iii) It is trivial for \( \alpha = 0 \). Now let \( \alpha \neq 0 \). Then for given \( \epsilon > 0 \) and \( t > 0 \),

\[
B(t) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk}-\xi}(t) > 1 - \epsilon\} \in F(I_2).
\]

It is sufficient to prove that for each \( \epsilon > 0 \) and \( t > 0 \),

\[
B(t) \subset \{(j, k) \in \mathbb{N} \times \mathbb{N} : \nu_{ax_{jk}-a\xi}(t) > 1 - \epsilon\}.
\]
Let \((j, k) \in B(t)\). Then we have \(\nu_{x_{jk} - \xi}(t) > 1 - \epsilon\). Now,

\[
\nu_{\alpha x_{jk} - \alpha \xi}(t) = \nu_{x_{jk} - \xi}\left(\frac{t}{|\alpha|}\right) \geq \nu_{x_{jk} - \xi}(t) \cdot \nu_1\left(\frac{t}{|\alpha|} - 1\right) = \nu_{x_{jk} - \xi}(t) \cdot 1 = \nu_{x_{jk} - \xi}(t) > 1 - \epsilon.
\]

Hence

\[
B(t) \subset \{(j, k) \in \mathbb{N} : \nu_{\alpha x_{jk} - \alpha \xi}(t) > 1 - \epsilon\}
\]

and from (3), we conclude that \(I_2^*\)-lim \(\alpha x_{jk} = \alpha \xi\).

This completes the proof of the theorem. □

3. \(I_2\)-CONVERGENCE IN PNS

In this section, we introduce the concept of \(I_2^*\)-convergence of double sequences in probabilistic normed space and show that \(I_2^*\)-convergence implies \(I_2\)-convergence but not conversely.

**Definition 3.1.** Let \((X, \nu, \ast)\) be a probabilistic normed space. We say that a sequence \(x = (x_{jk})\) of elements in \(X\) is \(I_2^*\)-convergent to \(\xi \in X\) with respect to the probabilistic norm \(\nu\) if there exists a subset \(K = \{(j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots \}\) of \(\mathbb{N} \times \mathbb{N}\) such that \(K \in F(I_2)\) (i.e., \(\mathbb{N} \times \mathbb{N}\backslash K \in I_2\)) and \(\nu\)-\(m\)-lim \(x_{j_m k_m} = \xi\).

In this case we write \(I_2^*\)-lim \(x = \xi\) and \(\xi\) is called the \(I_2^*\)-limit of the double sequence \(x = (x_{jk})\).

**Theorem 3.1.** Let \((X, \nu, \ast)\) be a PNS and \(I_2\) be an admissible ideal. If \(I_2^*\)-lim \(x = \xi\) then \(I_2\)-lim \(x = \xi\).

**Proof.** Suppose that \(I_2^*\)-lim \(x = \xi\). Then \(K = \{(j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots \} \in F(I_2)\) (i.e., \(\mathbb{N} \times \mathbb{N}\backslash K = H\)(say) \(\in I_2\)) such that \(\nu\)-\(m\)-lim \(x_{j_m k_m} = \xi\). But then for any \(\epsilon > 0\) and \(t > 0\) there exists a positive integer \(N\) such that \(\nu_{x_{j_m k_m} - \xi}(t) > 1 - \epsilon\) for all \(m > N\). Since \(\{(j_m, k_m) \in K : \nu_{x_{j_m k_m} - \xi}(t) \leq 1 - \epsilon\}\) is contained in \(\{j_1 < j_2 < \cdots < j_{N-1} ; k_1 < k_2 < \cdots < k_{N-1}\}\) and the ideal \(I_2\) is admissible, we have

\[
\{(j_m, k_m) \in K : \nu_{x_{j_m k_m} - \xi}(t) \leq 1 - \epsilon\} \in I_2.
\]

Hence

\[
\{(j, k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk} - \xi}(t) \leq 1 - \epsilon\} \subseteq H \cup \{j_1 < j_2 < \cdots < j_{N-1} ; k_1 < k_2 < \cdots < k_{N-1}\} \in I_2
\]

for all \(\epsilon > 0\) and \(t > 0\). Therefore, we conclude that \(I_2\)-lim \(x = \xi\).
Remark 3.1. The following example shows that the converse of Theorem 3.1 need not be true.

Example 3.1. Let \((\mathbb{R}, | \cdot |)\) denote the space of all real numbers with the usual norm, and let \(a \ast b = ab\) for all \(a, b \in [0, 1]\). For all \(x \in \mathbb{R}\) and every \(t > 0\), consider

\[\nu_x(t) := \frac{t}{t + |x|}.\]

Then \((\mathbb{R}, \nu, \ast)\) is a PNS.

Let \(N \times N = \bigcup_{i,j} \triangle_{ij}\) be a decomposition of \(N \times N\) such that for any \((m, n) \in N \times N\) each \(\triangle_{ij}\) contains infinitely many \((i, j)\)'s where \(i \geq m, j \geq n\) and \(\triangle_{ij} \cap \triangle_{mn} = \emptyset\) for \((i, j) \neq (m, n)\). Let \(I_2\) be the class of all subsets of \(N \times N\) which intersect at most a finite number of \(\triangle_{ij}\)'s. Then \(I_2\) is an admissible ideal. Now, we define a double sequence \(x_{mn} = \frac{1}{ij}\) if \((m, n) \in \triangle_{ij}\). Then

\[\nu_{x_{mn}}(t) = \frac{t}{t + |x_{mn}|} \to 1\]

as \(m, n \to \infty\). Hence \(I_2^\ast\)-lim \(x_{mn} = 0\).

Now, suppose that \(I_2^\ast\)-lim \(x_{mn} = 0\). Then there exists a subset \(K = \{ (m_j, n_j) : m_1 < m_2 < \cdots ; n_1 < n_2 < \cdots \}\) of \(N \times N\) such that \(K \in F(I_2)\) and \(\nu\)-lim \(x_{m_jn_j} = 0\). Since \(K \in F(I_2)\), there is a set \(H \in I_2\) such that \(K = N \times N \setminus H\). Now, from the definition of \(I_2\), there exist, say \(p, q \in N\), such that

\[H \subset \left( \bigcup_{m=1}^{p} \left( \bigcup_{n=1}^{\infty} \triangle_{mn} \right) \right) \cup \left( \bigcup_{n=1}^{q} \left( \bigcup_{m=1}^{\infty} \triangle_{mn} \right) \right),\]

But then \(\triangle_{p+1,q+1} \subset K\), and therefore

\[x_{m_jn_j} = \frac{1}{(p+1)(q+1)} > 0\]

for infinitely many \((m_j, n_j)\)'s from \(K\) which contradicts \(\nu\)-lim \(x_{m_jn_j} = 0\). Therefore, the assumption \(I_2^\ast\)-lim \(x_{mn} = 0\) leads to the contradiction.

Hence the converse of the theorem need not be true.

Remark 3.2. From the above result we have seen that \(I_2^\ast\)-convergence implies \(I_2\)-convergence but not conversely. Now the question arises under what condition the converse may hold. The following theorem shows that the converse holds if the ideal \(I_2\) satisfies condition (AP).

Definition 3.2. An admissible ideal \(I_2 \subset P(N \times N)\) is said to satisfy the condition (AP) if for every sequence \((A_n)_{n \in N}\) of pairwise disjoint sets from \(I_2\)
there are sets $B_n \subset \mathbb{N}$, $n \in \mathbb{N}$, such that the symmetric difference $A_n \triangle B_n$ is a finite set for every $n$ and $\bigcup_{n \in \mathbb{N}} B_n \in I_2$.

**Theorem 3.2.** Let $(X, \nu, \ast)$ be a PNS and the ideal $I_2$ satisfy the condition (AP). If $x = (x_{jk})$ is a double sequence in $X$ such that $I^\nu_2 \cdot \lim x = \xi$, then $I^\nu_2 \cdot \lim x = \xi$.

**Proof.** Suppose $I_2$ satisfies condition (AP) and $I^\nu_2 \cdot \lim x = \xi$. Then for each $\epsilon > 0$ and $t > 0$,

$$\{ (j, k) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{jk}-\xi}(t) \leq 1 - \epsilon \} \in I_2.$$  

We define the set $A_p$ for $p \in \mathbb{N}$ and $t > 0$ as

$$A_p = \left\{ (j, k) \in \mathbb{N} \times \mathbb{N} : 1 - \frac{1}{p} \leq \nu_{x_{jk}-\xi}(t) < 1 - \frac{1}{p+1} \right\}.$$  

Obviously, $\{A_1, A_2, \ldots\}$ is countable and belongs to $I_2$, and $A_i \cap A_j = \emptyset$ for $i \neq j$. By condition (AP), there is a countable family of sets $\{B_1, B_2, \ldots\} \in I_2$ such that the symmetric difference $A_i \triangle B_i$ is a finite set for each $i \in \mathbb{N}$ and $B = \bigcup_{i=1}^{\infty} B_i \in I_2$. From the definition of associate filter $F(I_2)$ there is a set $K \in F(I_2)$ such that $K = \mathbb{N} \times \mathbb{N} \setminus B$. To prove the theorem it is sufficient to show that the subsequence $(x_{jk})_{(j,k) \in K}$ is convergent to $\xi$ with respect to the probabilistic norm $\nu$. Let $\eta > 0$ and $t > 0$. Choose $q \in \mathbb{N}$ such that $\frac{1}{q} < \eta$. Then

$$\{ (j, k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk}-\xi}(t) \leq 1 - \eta \} \subset \bigcup_{i=1}^{q+1} A_i.$$  

Since $A_i \triangle B_i$, $i = 1, 2, \ldots, q+1$ are finite, there exists $(j_0, k_0) \in \mathbb{N} \times \mathbb{N}$ such that

$$(4) \quad \left( \bigcup_{i=1}^{q+1} B_i \right) \cap \{(j, k) : j \geq j_0 \text{ and } k \geq k_0 \} = \left( \bigcup_{i=1}^{q+1} A_i \right) \cap \{(j, k) : j \geq j_0 \text{ and } k \geq k_0 \}.$$  

If $j \geq j_0, k \geq k_0$ and $(j, k) \in K$ then $(j, k) \notin \bigcup_{i=1}^{q+1} B_i$. Therefore by (4), we have $(j, k) \notin \bigcup_{i=1}^{q+1} A_i$. Hence for every $j \geq j_0, k \geq k_0$ and $(j, k) \in K$ we have

$$\nu_{x_{jk}-\xi}(t) > 1 - \eta.$$  

Since $\eta > 0$ was arbitrary, we have $I^\nu_2 \cdot \lim x = \xi$. This completes the proof of the theorem. □
Theorem 3.3. Let \((X, \nu, \ast)\) be a PNS. Then the following conditions are equivalent:

(i) \(I_2^{\ast}\nu\)-\(\lim x = \xi\).

(ii) There exist two sequences \(y = (y_{jk})\) and \(z = (z_{jk})\) in \(X\) such that \(x = y + z\), \(\nu\)-\(\lim y = \xi\) and the set \(\{(j, k) : z_{jk} \neq \theta\} \in I_2\), where \(\theta\) denotes the zero element of \(X\).

Proof. Suppose that the condition (i) holds. Then there exists a subset \(K = \{(j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots \}\) of \(\mathbb{N} \times \mathbb{N}\) such that

\[
K \in F(I_2) \quad \text{and} \quad \nu\lim_m x_{j_mk_m} = \xi.
\]

We define the sequences \(y = (y_{jk})\) and \(z = (z_{jk})\) as

\[
y_{jk} = \begin{cases} 
x_{jk} & \text{if } (j, k) \in K, \\
\xi & \text{if } (j, k) \in K^C,
\end{cases}
\]

and \(z_{jk} = x_{jk} - y_{jk}\) for all \((j, k) \in \mathbb{N} \times \mathbb{N}\). For given \(\varepsilon > 0\), \(t > 0\) and \((j, k) \in K^C\), we have

\[
\nu y_{jk} - \xi(t) = 1 > 1 - \varepsilon.
\]

Using (5) we have \(\nu\lim y = \xi\). Since \(\{(j, k) : z_{jk} \neq \theta\} \subset K^C\), we have \(\{(j, k) : z_{jk} \neq \theta\} \in I_2\).

Let the condition (ii) hold. Then \(K = \{(j, k) : z_{jk} = \theta\} \in F(I_2)\) is an infinite set. Obviously, \(K \in F(I_2)\) is an infinite set. Let \(K = \{(j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots \}\). Since \(x_{j_mk_m} = y_{j_mk_m}\) and \(\nu\lim_m y_{j_mk_m} = \xi\), \(\nu\lim_m x_{j_mk_m} = \xi\). Hence \(I_2^{\ast}\nu\)-\(\lim_{j,k} x_{jk} = \xi\). This completes the proof of the theorem. \(\square\)

4. \(I_2\)-LIMIT POINTS AND \(I_2\)-CLUSTER POINTS IN PNS

In this section we define \(I_2\)-limit points and \(I_2\)-cluster points in probabilistic normed space analogous to the statistical limit points and statistical cluster points due to Fridy [8].

Definition 4.1. Let \((X, \nu, \ast)\) be a PNS, and \(x = (x_{jk}) \in X\). An element \(\xi \in X\) is said to be a limit point of the sequence \(x = (x_{jk})\) with respect to the probabilistic norm \(\nu\) (or a \(\nu\)-limit point) if there is subsequence of the sequence \(x\) which converges to \(\xi\) with respect to the probabilistic norm \(\nu\).

By \(L^\nu_(x)\), we denote the set of all limit points of the double sequence \(x = (x_{jk})\) with respect to the probabilistic norm \(\nu\).

Definition 4.2. Let \((X, \nu, \ast)\) be a PNS, and \(x = (x_{jk}) \in X\). An element \(\xi \in X\) is said to be an \(I_2\)-limit point of the sequence \(x\) with respect to the
probabilistic norm $\nu$ (or $I_2^\nu$-limit point) if there is a subset $K = \{(j_m, k_m) : j_1 < j_2 < \cdots; k_1 < k_2 < \cdots\}$ of $\mathbb{N} \times \mathbb{N}$ such that $K \notin I_2$ and $\nu_{m \to \infty} x_{j_m k_m} = \xi$.

We denote by $\Lambda^I_2(x)$, the set of all $I_2^\nu$-limit points of the sequence $x = (x_{jk})$.

**Definition 4.3.** Let $(X, \nu, *)$ be a PNS, and $x = (x_{jk}) \in X$. An element $\xi \in X$ is said to be an $I_2$-cluster point of $x$ with respect to the probabilistic norm $\nu$ (or $I_2^\nu$-cluster point) if for each $\epsilon > 0$ and $t > 0$

$$K = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk}-\xi}(t) > 1 - \epsilon\} \notin I_2.$$

By $\Gamma_2^I(x)$, we denote the set of all $I_2^\nu$-cluster points of the sequence $x = (x_{jk})$.

**Theorem 4.1.** Let $(X, \nu, *)$ be a PNS. Then for every sequence $x = (x_{jk})$ in $X$ we have $\Lambda^I_2(x) \subset \Gamma_2^I(x) \subset \mathcal{L}_2^\nu(x)$.

**Proof.** Let $\xi \in \mathcal{L}_2^\nu(x)$. Then there exists a set $K = \{(j_m, k_m) : j_1 < j_2 < \cdots; k_1 < k_2 < \cdots\}$ of $\mathbb{N} \times \mathbb{N}$ such that $K \notin I_2$ and $\nu_{m \to \infty} x_{j_m k_m} = \xi$.

For each $\epsilon > 0$ and $t > 0$ there exists $N \in \mathbb{N}$ such that for $j, k > N$ we have $\nu_{x_{jk}-\xi}(t) > 1 - \epsilon$. Hence

$$\{(j, k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk}-\xi}(t) > 1 - \epsilon\} \supset \{(j_{N+1}, j_{N+2}, \ldots; k_{N+1}, k_{N+2}, \ldots)\}$$

and so

$$\{(j, k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk}-\xi}(t) > 1 - \epsilon\} \notin I_2,$$

which means that $\xi \in \mathcal{L}_2^\nu(x)$. Hence $\mathcal{L}_2^\nu(x) \subset \mathcal{L}_2^\nu(x)$.

Let $\xi \in \mathcal{L}_2^\nu(x)$. Then for given $\epsilon > 0$ and $t > 0$, we have

$$\{(j, k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk}-\xi}(t) > 1 - \epsilon\} \notin I_2.$$

Let $K = \{(j_m, k_m) : j_1 < j_2 < \cdots; k_1 < k_2 < \cdots\}$. Then there is a subsequence $(x_{jk})_{(j,k) \in K}$ of $(x_{jk})$ that converges to $\xi$ with respect to the probabilistic norm $\nu$. Therefore $\xi$ is an ordinary limit point of $(x_{jk})$, that is, $\xi \in \mathcal{L}_2^\nu(x)$ and hence $\mathcal{L}_2^\nu(x) \subset \mathcal{L}_2^\nu(x)$. This completes the proof of the theorem. □

**Theorem 4.2.** Let $x = (x_{jk})$ be a sequence in a PN-space $(X, \nu, *)$. Then $\Lambda^I_2(x) = \Gamma_2^I(x) = \{\xi\}$, provided $I_2^\nu\lim_{j,k} x_{jk} = \xi$.

**Proof.** Let $\eta \in \mathcal{L}_2^\nu(x)$, where $\xi \neq \eta$. Then there exist two subsets $K$ and $K'$, that is, $K = \{(j_m, k_m) : j_1 < j_2 < \cdots; k_1 < k_2 < \cdots\}$ and $K' = \{(j_{m+1}, k_{m+1}) : j_{m+1} < j_{m+2} < \cdots; k_{m+1} < k_{m+2} < \cdots\}$.
\{ (p_m, q_m) : p_1 < p_2 < \cdots ; q_1 < q_2 < \cdots \} \text{ of } \mathbb{N} \times \mathbb{N} \text{ such that}

(6) \quad K \not\in I_2 \quad \text{and} \quad \nu\text{-} \lim_{m \to \infty} x_{jmkm} = \xi,

(7) \quad K' \not\in I_2 \quad \text{and} \quad \nu\text{-} \lim_{m \to \infty} x_{pmqm} = \eta.

By (7), given \( \epsilon > 0 \) and \( t > 0 \), there exists \( N \in \mathbb{N} \) such that for \( m > N \) we have \( \nu_{pmqm-\eta}(t) > 1 - \epsilon \). Therefore,

\[
A = \{ (p_m, q_m) \in K' : \nu_{pmqm-\eta}(t) \leq 1 - \epsilon \} \subset \{ (p_m, q_m) : p_1 < p_2 < \cdots < pn; q_1 < q_2 < \cdots < qn \}.
\]

As \( I_2 \) is an admissible ideal we have \( A \in I_2 \). If we take

\[
B = \{ (p_m, q_m) \in K' : \nu_{pmqm-\eta}(t) > 1 - \epsilon \} \not\in I_2.
\]

Otherwise, if \( B \in I_2 \), then \( A \cup B = K' \in I_2 \), which contradicts (7). Since

\[
I_2^\nu\text{-} \lim x_{jk} = \xi,\]

we have that for each \( \epsilon > 0 \) and \( t > 0 \),

\[
C = \{ (j, k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk}-\xi}(t) \leq 1 - \epsilon \} \in I_2.
\]

Therefore,

\[
C^C = \{ (j, k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk}-\xi}(t) > 1 - \epsilon \} \in F(I_2).
\]

Since for every \( \xi \neq \eta \), we have \( B \cap C^C = \emptyset, B \subset C \). Since \( C \in I_2 \) implies \( B \in I_2 \), this contradicts the fact that \( B \not\in I_2 \). Hence \( \Lambda^\nu_2(x) = \{ \xi \} \).

On the other hand, suppose that \( \eta \in \Gamma^\nu_2(x) \), where \( \xi \neq \eta \). By definition, for each \( \epsilon > 0 \) and \( t > 0 \),

\[
A = \{ (j, k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk}-\xi}(t) > 1 - \epsilon \} \not\in I_2,
\]

\[
B = \{ (j, k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk}-\eta}(t) > 1 - \epsilon \} \not\in I_2.
\]

For \( \xi \neq \eta \), we have \( A \cap B = \emptyset \) and therefore \( B \subset C^C \). Also, \( I_2^\nu\text{-} \lim x_{jk} = \xi \) implies that

\[
C^C = \{ (j, k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk}-\xi}(t) \leq 1 - \epsilon \} \in I_2.
\]

Hence \( B \in I_2 \), which is a contradiction to \( B \not\in I_2 \). Therefore, \( \Gamma^\nu_2(x) = \{ \xi \} \).

This completes the proof of the theorem. \( \square \)

**Theorem 4.3.** Let \((X, \nu, \ast)\) be a PNS and for any two sequences \( x = (x_{jk}), y = (y_{jk}) \) in \( X \), the set \( A = \{ (j, k) \in \mathbb{N} \times \mathbb{N} : x_{jk} \neq y_{jk} \} \in I_2 \). Then \( \Lambda^\nu_2(x) = \Lambda^\nu_2(y) \) and \( \Gamma^\nu_2(x) = \Gamma^\nu_2(y) \).

**Proof.** Let \( \xi \in \Lambda^\nu_2(x) \). Then there exists a subset \( K = \{ (j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots \} \) of \( \mathbb{N} \times \mathbb{N} \) such that \( K \not\in I_2 \) and \( \nu\text{-} \lim_{m \to \infty} x_{jmkm} = \xi \).

Given \( \epsilon > 0 \) and \( t > 0 \), there exists \( N \in \mathbb{N} \) such that \( \nu_{x_{jmkm}-\xi}(t) > 1 - \epsilon \) for \( m > N \). Define \( K_1 = K \cap A \) and \( K_2 = K \setminus A \). Since \( A \in I_2 \) we have
Now, we have to show that \( C \) and hence \( \Gamma \). Given \( \epsilon > 0 \) and \( \epsilon > 0 \), therefore \( B \). By hypothesis, \( y \) the subsequence \((n)\setminus 2\) of the sequence \( y = (y_{jk}) \) is convergent to \( x \) with respect to the probabilistic norm \( \nu \) we have. This implies that \( \xi \in \Lambda_2^{I,\nu}(y) \) and hence \( \Lambda_2^{I,\nu}(x) \subset \Lambda_2^{I,\nu}(y) \).

Similarly, we can show that \( \Lambda_2^{I,\nu}(y) \subset \Lambda_2^{I,\nu}(x) \). Hence \( \Lambda_2^{I,\nu}(x) = \Lambda_2^{I,\nu}(y) \).

Now, we need to show that \( \Gamma_2^{I,\nu}(x) = \Gamma_2^{I,\nu}(y) \). Let \( \xi \in \Gamma_2^{I,\nu}(x) \). For each \( \epsilon > 0 \) and \( t > 0 \) the set

\[ B = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk} - \xi}(t) > 1 - \epsilon\} \notin I_2. \]

Given \( \epsilon > 0 \) and \( t > 0 \), define

\[ C = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu_{y_{jk} - \xi}(t) > 1 - \epsilon\}. \]

Now, we have to show that \( C \notin I_2 \). Let \( C \in I_2 \). Then

\[ C^C = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu_{y_{jk} - \xi}(t) \leq 1 - \epsilon\} \in F(I_2). \]

By hypothesis, \( A^C = \{(j,k) \in \mathbb{N} \times \mathbb{N} : x_{jk} = y_{jk}\} \in F(I_2) \). Since \( F(I_2) \) is a filter in \( \mathbb{N} \times \mathbb{N} \), \( C^C \cap A^C \in F(I_2) \). Since \( C^C \cap A^C \subset B^C \), \( B^C \in F(I_2) \). This implies that \( B \in I_2 \) which is a contradiction, since \( B \notin I_2 \). So that \( C \notin I_2 \) and therefore \( \Gamma_2^{I,\nu}(x) \subset \Gamma_2^{I,\nu}(y) \). Similarly, we can show that \( \Gamma_2^{I,\nu}(y) \subset \Gamma_2^{I,\nu}(x) \) and hence \( \Gamma_2^{I,\nu}(x) \cap \Gamma_2^{I,\nu}(y) \). This completes the proof of the theorem. □

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