ON IDEAL CONVERGENCE OF DOUBLE SEQUENCES IN PROBABILISTIC NORMED SPACES

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One of the generalizations of statistical convergence is I-convergence which was introduced by Kostyrko et al. [12]. In this paper, we define and study the concept of I-convergence, I^*-convergence, I-limit points and I-cluster points of double sequences in probabilistic normed space. We discuss the relationship between I_2-convergence and I_2^*-convergence, i.e., we show that I_2^*-convergence implies I_2-convergence in probabilistic normed space. Furthermore, we have also demonstrated through an example that, in general, I_2-convergence does not imply I_2^*-convergence in probabilistic normed space.

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1. INTRODUCTION AND PRELIMINARIES

The concept of statistical convergence for sequences of real numbers was introduced by Fast [4] and Steinhaus [22] independently in the same year 1951 and since then several generalizations and applications of this notion have been investigated by various authors, namely [3], [7], [15], [16], [17], [18], [19]. One of its interesting generalization is I-convergence which was given by Kostyrko et al. [12]. Recently I-convergence for sequences of functions has been studied by Balcerzak et al. [2] and by Komisarski [13].

The theory of probabilistic normed spaces [5] originated from the concept of statistical metric spaces which was introduced by Menger [14] and further studied by Schweizer and Sklar [20, 21]. It provides an important method of generalizing the deterministic results of normed linear spaces. It has also very useful applications in various fields, e.g., continuity properties [1], topological spaces [5], linear operators [8], study of boundedness [9], convergence of random variables [10] etc.

In this paper we study the concept of I-convergence and I^*-convergence in a more general setting, i.e., in probabilistic normed spaces. We also define

I-limit points and I-cluster points in probabilistic normed space and prove some interesting results.

We recall some notations and basic definitions used in this paper.

Definition 1.1 ([21]). A *triangular norm* (*t-norm*) is a continuous mapping $* : [0,1] \times [0,1] \to [0,1]$ such that $([0,1], *)$ is an abelian monoid with unit one and $c * d \geq a * b$ if $c \geq a$ and $d \geq b$ for all $a, b, c, d \in [0,1]$.

Definition 1.2 ([5]). A function $f : \mathbb{R} \to \mathbb{R}_0^+$ is called a distribution function if it is non-decreasing and left-continuous with $\inf_{t \in \mathbb{R}} f(t) = 0$ and $\sup_{t \in \mathbb{R}} f(t) = 1$.

By D, we denote the set of all distribution functions.

Definition 1.3 ([5]). Let X be a real linear space and $\nu : X \to D$. A probabilistic norm or *ν-norm* is a t-norm satisfying the following conditions:

(i) $\nu_x(0) = 0$,
(ii) $\nu_x(t) = 1$ for all $t > 0$ iff $x = 0$,
(iii) $\nu_{\alpha x}(t) = \nu_x(\frac{t}{|\alpha|})$ for all $\alpha \in \mathbb{R} \setminus \{0\}$ and for all $t > 0$,
(iv) $\nu_{x+y}(s+t) \geq \nu_x(s) * \nu_y(t)$ for all $x, y \in X$ and $s, t \in \mathbb{R}_0^+$,

where ν_x means $\nu(x)$ and $\nu_x(t)$ is the value of ν_x at $t \in \mathbb{R}$.

The $(X, \nu, *)$ is called a probabilistic normed space (for short, PNS).

Definition 1.4 ([11]). Let $(X, \nu, *)$ be an PNS. A sequence $x = (x_k)$ is said to be convergent to $\xi \in X$ with respect to the probabilistic norm ν, that is, $x_k \xrightarrow{\nu} \xi$ if for every $t > 0$ and $\epsilon \in (0,1)$, there is a positive integer k_0 such that $\nu_{x_k - \xi}(t) > 1 - \epsilon$ whenever $k \geq k_0$. In this case we write $\nu\text{-}\lim x = \xi$.

Remark 1.1. Let $(X, \|\cdot\|)$ be a real normed linear space, and

$$\nu(x,t) := \frac{t}{t + \|x\|}$$

for all $x \in X$ and $t > 0$. Then $x_n \xrightarrow{\|\cdot\|} x$ if and only if $x_n \xrightarrow{\nu} x$.

Definition 1.5 ([6]). Let K be a subset of $\mathbb{N}$, the set of natural numbers. The asymptotic density of K denoted by $\delta(K)$, is defined as

$$\delta(K) = \lim_n \frac{1}{n} |\{k \leq n : k \in K\}|,$$

where the vertical bars denote the cardinality of the enclosed set.

Definition 1.6 ([4, 22]). A number sequence $x = (x_k)$ is said to be statistically convergent to the number ℓ if for each $\epsilon > 0$, the set $K(\epsilon) = \{k \leq n :$

$|x_k - \ell| > \epsilon\}$ has asymptotic density zero, i.e.,

$$\lim_n \frac{1}{n}|\{k \leq n : |x_k - \ell| > \epsilon\}| = 0.$$

In this case we write $st\text{-}\lim x = \ell$.

Definition 1.7 ([11]). Let $(X, \nu, *)$ be an PNS. A sequence $x = (x_k)$ is said to be statistically convergent to $\xi \in X$ with respect to the probabilistic norm ν provided that, for every $t > 0$ and $\epsilon > 0$,

$$\delta(\{k \leq n : \nu_{x_k - \xi}(t) \leq 1 - \epsilon\}) = 0$$

or, equivalently,

$$\lim_n \frac{1}{n}|\{k \leq n : \nu_{x_k - \xi}(t) \leq 1 - \epsilon\}| = 0.$$

In this case we write $st_\nu\text{-}\lim x = \xi$.

Definition 1.8 ([12]). If X is a non-empty set, a family $I \subset 2^X$ of subsets of X is called an ideal in X if

(a) $\emptyset \in I$,
(b) $A, B \in I$ implies $A \cup B \in I$,
(c) for each $A \in I$ and $B \subset A$ we have $B \in I$.

An ideal I is called nontrivial if $X \notin I$.

Definition 1.9 ([12]). Let X be a non-empty set. A non-empty family of sets $F \subset P(X)$, the power set of X, is called a filter on X if and only if

(a) $\emptyset \notin F$,
(b) $A, B \in F$ implies $A \cap B \in F$,
(c) for each $A \in F$ and $B \supset A$ we have $B \in F$.

Definition 1.10 ([12]). A non-trivial ideal I in X is called an admissible ideal if it is different from $P(\mathbb{N})$ and it contains all singletons, i.e., $\{x\} \in I$ for each $x \in X$.

Let $I \subset P(X)$ be a non-trivial ideal. A class $F(I) = \{M \subset X : M = X \setminus A,$ for some $A \in I\}$ is a filter on X, called the filter associated with the ideal I.

Definition 1.11 ([12]). An admissible ideal $I \subset P(\mathbb{N})$ is said to satisfy the condition (AP) if for every sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets from I there are sets $B_n \subset \mathbb{N}$, $n \in \mathbb{N}$, such that the symmetric difference $A_n \triangle B_n$ is a finite set for every n and $\bigcup_{n\in\mathbb{N}} B_n \in I$.

Definition 1.12 ([12]). Let $I \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. A sequence $x = (x_k)$ is said to be I-convergent to L if for every $\epsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I$. In this case we write $I\text{-}\lim x = L$.

2. I_2-CONVERGENCE IN PNS

In this section, we study the concept of ideal convergence of double sequences in probabilistic normed space. Throughout the paper we take I_2 as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

We define

Definition 2.1. Let I be a non trivial ideal of $\mathbb{N} \times \mathbb{N}$ and $(X, \nu, *)$ be a probabilistic normed space. A double sequence $x = (x_{jk})$ of elements of X is said to be I_2-convergent to $\xi \in X$ with respect to the probabilistic norm ν (or I_2^ν*-convergent* to ξ) if for each $\epsilon > 0$ and $t > 0$,

$$\big\{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk}-\xi}(t) \le 1 - \epsilon\big\} \in I_2. \tag{1}$$

In this case we write $I_2^\nu\text{-}\lim x = \xi$.

THEOREM 2.1. *Let $(X, \nu, *)$ be a PNS. Then, the following statements are equivalent:*

(i) $I_2^\nu\text{-}\lim x = \xi$;
(ii) $\{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk}-\xi}(t) \le 1 - \epsilon\} \in I_2^\nu$ *for every* $\epsilon > 0$ *and* $t > 0$;
(iii) $\{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk}-\xi}(t) > 1 - \epsilon\} \in F(I_2^\nu)$ *for every* $\epsilon > 0$ *and* $t > 0$;
(iv) $I_2\text{-}\lim \nu_{x_{jk}-\xi}(t) = 1$.

The *proof* is standard. □

THEOREM 2.2. *Let $(X, \nu, *)$ be a PNS. If a double sequence $x = (x_{jk})$ is I_2^ν-convergent then I_2^ν-limit is unique.*

Proof. Suppose that $I_2^\nu\text{-}\lim x = \xi_1$ and $I_2^\nu\text{-}\lim x = \xi_2$. Given $\epsilon > 0$ and $t > 0$, choose $r > 0$ such that $(1-r) * (1-r) \ge 1 - \epsilon$. Then, we define the following sets as

$$K_{\nu,1}(r,t) = \big\{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk}-\xi_1}(t) \le 1 - r\big\},$$
$$K_{\nu,2}(r,t) = \big\{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk}-\xi_2}(t) \le 1 - r\big\}.$$

Since $I_2^\nu\text{-}\lim x = \xi_1$, we have $K_{\nu,1}(r,t) \in I_2$. Furthermore, using $I_2^\nu\text{-}\lim x = \xi_2$, we get $K_{\nu,2}(r,t) \in I_2$. Now, let $K_\nu(r,t) = K_{\nu,1}(r,t) \cup K_{\nu,2}(r,t) \in I_2$. Then we see that $K_\nu(r,t) \in I_2$. This implies that its complement $K_\nu^C(r,t)$ is non empty in $F(I_2)$. If $(j,k) \in K_\nu^C(r,t)$, then we have $(j,k) \in K_{\nu,1}^C(r,t) \cap K_{\nu,2}^C(r,t)$, and so

$$\nu_{\xi_1-\xi_2}(t) \ge \nu_{x_{jk}-\xi_1}(t/2) * \nu_{x_{jk}-\xi_2}(t/2) > (1-r) * (1-r). \tag{2}$$

Since $(1-r) * (1-r) \ge 1 - \epsilon$, we have $\nu_{\xi_1-\xi_2}(t) > 1 - \epsilon$. Since $\epsilon > 0$ was arbitrary, we get $\nu(\xi_1 - \xi_2, t) = 1$ for all $t > 0$, which yields $\xi_1 = \xi_2$.

This completes the proof of the theorem. □

THEOREM 2.3. *Let $(X, \nu, *)$ be a PNS.*

(i) *If* ν-$\lim x_{jk} = \xi$ *then* I_2^{ν}-$\lim x_{jk} = \xi$.
(ii) *If* I_2^{ν}-$\lim x_{jk} = \xi_1$ *and* I_2^{ν}-$\lim y_{jk} = \xi_2$ *then* I_2^{ν}-$\lim(x_{jk}+y_{jk}) = (\xi_1+\xi_2)$.
(iii) *If* I_2^{ν}-$\lim x_{jk} = \xi$ *then* I_2^{ν}-$\lim \alpha x_{jk} = \alpha\xi$.

Proof. (i) Suppose that ν-$\lim x_{jk} = \xi$. Then for each $\epsilon > 0$ and $t > 0$ there exists a positive integer N such that

$$\nu_{x_{jk}-\xi}(t) > 1 - \epsilon$$

for each $j, k > N$. Since the set

$$A(t) = \big\{(j,k) \in \mathbb{N}\times\mathbb{N} : \nu_{x_{jk}-\xi}(t) \leq 1-\epsilon\big\}$$

is contained in $\{1, 2, 3, \ldots, N-1\}$ and the ideal I_2 is admissible we have $A(t) \in I_2$. Hence I_2^{ν}-$\lim x_{jk} = \xi$.

(ii) Let I_2^{ν}-$\lim x_{jk} = \xi_1$ and I_2^{ν}-$\lim y_{jk} = \xi_2$. For given $\epsilon > 0$ and $t > 0$, choose $r > 0$ such that $(1-r) * (1-r) > 1-\epsilon$. Define the sets

$$K_{\nu,1}(r,t) = \big\{(j,k) \in \mathbb{N}\times\mathbb{N} : \nu_{x_{jk}-\xi_1}(t) \leq 1-r\big\},$$
$$K_{\nu,2}(r,t) = \big\{(j,k) \in \mathbb{N}\times\mathbb{N} : \nu_{y_{jk}-\xi_2}(t) \leq 1-r\big\}.$$

Since I_2^{ν}-$\lim x_{jk} = \xi_1$, we have

$$K_{\nu,1}(r,t) \in I_2.$$

Furthermore, using I_2^{ν}-$\lim x = \xi_2$, we get

$$K_{\nu,2}(r,t) \in I_2.$$

Now, let $K_{\nu}(r,t) = K_{\nu,1}(r,t) \cup K_{\nu,2}(r,t)$. Then $K_{\nu}(r,t) \in I_2$ which implies that $K_{\nu}^{C}(r,t)$ is non empty in $F(I_2)$. Now, we have to show that $K_{\nu}^{C}(r,t) \subset \{(j,k) \in \mathbb{N}\times\mathbb{N} : \nu_{(x_{jk}+y_{jk})-(\xi_1+\xi_2)}(t) > 1-\epsilon\}$. If $(j,k) \in K_{\nu}^{C}(r,t)$, then we have $\nu_{x_{jk}-\xi_1}(\frac{t}{2}) > 1-r$ and $\nu_{y_{jk}-\xi_2}(\frac{t}{2}) > 1-r$. Therefore,

$$\nu_{(x_{jk}+y_{jk})-(\xi_1+\xi_2)}(t) \geq \nu_{x_{jk}-\xi_1}\left(\frac{t}{2}\right) * \nu_{y_{jk}-\xi_2}\left(\frac{t}{2}\right) > (1-r) * (1-r) > 1-\epsilon.$$

This shows that

$$K_{\nu}^{C}(r,t) \subset \big\{(j,k) \in \mathbb{N}\times\mathbb{N} : \nu_{(x_{jk}+y_{jk})-(\xi_1+\xi_2)}(t) > 1-\epsilon\big\}.$$

Since $K_{\nu}^{C}(r,t) \in F(I_2)$, we have I_2^{ν}-$\lim(x_{jk}+y_{jk}) = (\xi_1+\xi_2)$.

(iii) It is trivial for $\alpha = 0$. Now let $\alpha \neq 0$. Then for given $\epsilon > 0$ and $t > 0$,

$$B(t) = \big\{(j,k) \in \mathbb{N}\times\mathbb{N} : \nu_{x_{jk}-\xi}(t) > 1-\epsilon\big\} \in F(I_2). \tag{3}$$

It is sufficient to prove that for each $\epsilon > 0$ and $t > 0$,

$$B(t) \subset \big\{(j,k) \in \mathbb{N}\times\mathbb{N} : \nu_{\alpha x_{jk}-\alpha\xi}(t) > 1-\epsilon\big\}.$$

Let $(j,k) \in B(t)$. Then we have $\nu_{x_{jk}-\xi}(t) > 1-\epsilon$. Now,

$$\nu_{\alpha x_{jk}-\alpha\xi}(t) = \nu_{x_{jk}-\xi}\left(\frac{t}{|\alpha|}\right) \geq \nu_{x_{jk}-\xi}(t) * \nu_0\left(\frac{t}{|\alpha|} - t\right) =$$
$$= \nu_{x_{jk}-\xi}(t) * 1 = \nu_{x_{jk}-\xi}(t) > 1-\epsilon.$$

Hence

$$B(t) \subset \left\{(j,k) \in \mathbb{N}\mathbb{N} : \nu_{\alpha x_{jk}-\alpha\xi}(t) > 1-\epsilon\right\}$$

and from (3), we conclude that $I_2^\nu\text{-}\lim \alpha x_{jk} = \alpha\xi$.

This completes the proof of the theorem. □

3. I_2^*-CONVERGENCE IN PNS

In this section, we introduce the concept of I_2^*-convergence of double sequences in probabilistic normed space and show that I_2^*-convergence implies I_2-convergence but not conversely.

Definition 3.1. Let $(X,\nu,*)$ be a probabilistic normed space. We say that a sequence $x = (x_{jk})$ of elements in X is I_2^*-convergent to $\xi \in X$ with respect to the probabilistic norm ν if there exists a subset $K = \{(j_m,k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\}$ of $\mathbb{N}\times\mathbb{N}$ such that $K \in F(I_2)$ (i.e., $\mathbb{N}\times\mathbb{N}\backslash K \in I_2$) and $\nu\text{-}\lim_m x_{j_m k_m} = \xi$.

In this case we write $I_2^{*,\nu}\text{-}\lim x = \xi$ and ξ is called the I_2^*-limit of the double sequence $x = (x_{jk})$.

Theorem 3.1. *Let $(X,\nu,*)$ be a PNS and I_2 be an admissible ideal. If $I_2^{*,\nu}\text{-}\lim x = \xi$ then $I_2^\nu\text{-}\lim x = \xi$.*

Proof. Suppose that $I_2^{*,\nu}\text{-}\lim x = \xi$. Then $K = \{(j_m,k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\} \in F(I_2)$ (i.e., $\mathbb{N}\times\mathbb{N}\setminus K = H(\text{say}) \in I_2$) such that $\nu\text{-}\lim_m x_{j_m k_m} = \xi$. But then for any $\epsilon > 0$ and $t > 0$ there exists a positive integer N such that $\nu_{x_{j_m k_m}-\xi}(t) > 1-\epsilon$ for all $m > N$. Since $\{(j_m,k_m) \in K : \nu_{x_{j_m k_m}-\xi}(t) \leq 1-\epsilon\}$ is contained in $\{j_1 < j_2 < \cdots < j_{N-1}; k_1 < k_2 < \cdots < k_{N-1}\}$ and the ideal I_2 is admissible, we have

$$\left\{(j_m,k_m) \in K : \nu_{x_{j_m k_m}-\xi}(t) \leq 1-\epsilon\right\} \in I_2.$$

Hence

$$\left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \nu_{x_{jk}-\xi}(t) \leq 1-\epsilon\right\} \subseteq$$
$$\subseteq H \cup \{j_1 < j_2 < \cdots < j_{N-1}; k_1 < k_2 < \cdots < k_{N-1}\} \in I_2$$

for all $\epsilon > 0$ and $t > 0$. Therefore, we conclude that $I_2^\nu\text{-}\lim x = \xi$.

Remark 3.1. The following example shows that the converse of Theorem 3.1 need not be true.

Example 3.1. Let $(\mathbb{R}, |\cdot|)$ denote the space of all real numbers with the usual norm, and let $a * b = ab$ for all $a, b \in [0,1]$. For all $x \in \mathbb{R}$ and every $t > 0$, consider

$$\nu_x(t) := \frac{t}{t + |x|}.$$

Then $(\mathbb{R}, \nu, *)$ is a PNS.

Let $\mathbb{N} \times \mathbb{N} = \bigcup\limits_{i,j} \triangle_{ij}$ be a decomposition of $\mathbb{N} \times \mathbb{N}$ such that for any $(m,n) \in \mathbb{N} \times \mathbb{N}$ each $\triangle_{ij}$ contains infinitely many $(i,j)'s$ where $i \geq m, j \geq n$ and $\triangle_{ij} \cap \triangle_{mn} = \emptyset$ for $(i,j) \neq (m,n)$. Let I_2 be the class of all subsets of $\mathbb{N} \times \mathbb{N}$ which intersect atmost a finite number of $\triangle_{ij}$'s. Then I_2 is an admissible ideal. Now, we define a double sequence $x_{mn} = \frac{1}{ij}$ if $(m,n) \in \triangle_{ij}$. Then

$$\nu_{x_{mn}}(t) = \frac{t}{t + |x_{mn}|} \rightarrow 1$$

as $m, n \rightarrow \infty$. Hence $I_2^{\nu}\text{-}\lim\limits_{m,n} x_{mn} = 0$.

Now, suppose that $I_2^{*,\nu}\text{-}\lim\limits_{m,n} x_{mn} = 0$. Then there exists a subset $K = \{(m_j, n_j) : m_1 < m_2 < \cdots ; n_1 < n_2 < \cdots\}$ of $\mathbb{N} \times \mathbb{N}$ such that $K \in F(I_2)$ and $\nu\text{-}\lim\limits_{j} x_{m_j n_j} = 0$. Since $K \in F(I_2)$, there is a set $H \in I_2$ such that $K = \mathbb{N} \times \mathbb{N} \setminus H$. Now, from the definition of I_2, there exist, say $p, q \in \mathbb{N}$, such that

$$H \subset \Big(\bigcup_{m=1}^{p} \Big(\bigcup_{n=1}^{\infty} \triangle_{mn} \Big) \Big) \cup \Big(\bigcup_{n=1}^{q} \Big(\bigcup_{m=1}^{\infty} \triangle_{mn} \Big) \Big).$$

But then $\triangle_{p+1,q+1} \subset K$, and therefore

$$x_{m_j n_j} = \frac{1}{(p+1)(q+1)} > 0$$

for infinitely many $(m_j, n_j)'s$ from K which contradicts $\nu\text{-}\lim_j x_{m_j n_j} = 0$. Therefore, the assumption $I_2^{*,\nu}\text{-}\lim\limits_{m,n} x_{mn} = 0$ leads to the contradiction.

Hence the converse of the theorem need not be true.

Remark 3.2. From the above result we have seen that I_2^*-convergence implies I_2-convergence but not conversely. Now the question arises under what condition the converse may hold. The following theorem shows that the converse holds if the ideal I_2 satisfies condition (AP).

Definition 3.2. An admissible ideal $I_2 \subset P(\mathbb{N} \times \mathbb{N})$ is said to satisfy the condition (AP) if for every sequence $(A_n)_{n \in \mathbb{N}}$ of pairwise disjoint sets from I_2

there are sets $B_n \subset \mathbb{N}$, $n \in \mathbb{N}$, such that the symmetric difference $A_n \triangle B_n$ is a finite set for every n and $\bigcup_{n\in\mathbb{N}} B_n \in I_2$.

Theorem 3.2. *Let $(X, \nu, *)$ be a PNS and the ideal I_2 satisfy the condition (AP). If $x = (x_{jk})$ is a double sequence in X such that I_2^{ν}-$\lim x = \xi$, then $I_2^{*,\nu}$-$\lim x = \xi$.*

Proof. Suppose I_2 satisfies condition (AP) and I_2^{ν}-$\lim x = \xi$. Then for each $\epsilon > 0$ and $t > 0$,

$$\big\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu_{x_{jk}-\xi}(t) \le 1-\epsilon\big\} \in I_2.$$

We define the set A_p for $p \in \mathbb{N}$ and $t > 0$ as

$$A_p = \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : 1-\frac{1}{p} \le \nu_{x_{jk}-\xi}(t) < 1-\frac{1}{p+1}\right\}.$$

Obviously, $\{A_1, A_2, \ldots\}$ is countable and belongs to I_2, and $A_i \cap A_j = \emptyset$ for $i \ne j$. By condition (AP), there is a countable family of sets $\{B_1, B_2, \ldots\} \in I_2$ such that the symmetric difference $A_i \triangle B_i$ is a finite set for each $i \in \mathbb{N}$ and $B = \bigcup_{i=1}^{\infty} B_i \in I_2$. From the definition of associate filter $F(I_2)$ there is a set $K \in F(I_2)$ such that $K = \mathbb{N}\times\mathbb{N} \setminus B$. To prove the theorem it is sufficient to show that the subsequence $(x_{jk})_{(j,k)\in K}$ is convergent to ξ with respect to the probabilistic norm ν. Let $\eta > 0$ and $t > 0$. Choose $q \in \mathbb{N}$ such that $\frac{1}{q} < \eta$. Then

$$\big\{(j,k) \in \mathbb{N}\times\mathbb{N} : \nu_{x_{jk}-\xi}(t) \le 1-\eta\big\} \subset$$
$$\subset \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \nu_{x_{jk}-\xi}(t) \le 1-\frac{1}{q}\right\} \subset \bigcup_{i=1}^{q+1} A_i.$$

Since $A_i \triangle B_i$, $i = 1, 2, \ldots, q+1$ are finite, there exists $(j_0, k_0) \in \mathbb{N}\times\mathbb{N}$ such that

$$\left(\bigcup_{i=1}^{q+1} B_i\right) \cap \{(j,k) : j \ge j_0 \text{ and } k \ge k_0\} = \tag{4}$$
$$= \left(\bigcup_{i=1}^{q+1} A_i\right) \cap \{(j,k) : j \ge j_0 \text{ and } k \ge k_0\}.$$

If $j \ge j_0, k \ge k_0$ and $(j,k) \in K$ then $(j,k) \notin \bigcup_{i=1}^{q+1} B_i$. Therefore by (4), we have $(j,k) \notin \bigcup_{i=1}^{q+1} A_i$. Hence for every $j \ge j_0$, $k \ge k_0$ and $(j,k) \in K$ we have

$$\nu_{x_{jk}-\xi}(t) > 1-\eta.$$

Since $\eta > 0$ was arbitrary, we have $I_2^{*,\nu}$-$\lim x = \xi$. This completes the proof of the theorem. $\square$

THEOREM 3.3. *Let $(X, \nu, *)$ be a PNS. Then the following conditions are equivalent:*

(i) $I_2^{*,\nu}\text{-}\lim x = \xi$.

(ii) *There exist two sequences $y = (y_{jk})$ and $z = (z_{jk})$ in X such that $x = y + z$, $\nu\text{-}\lim y = \xi$ and the set $\{(j,k) : z_{jk} \neq \theta\} \in I_2$, where θ denotes the zero element of X.*

Proof. Suppose that the condition (i) holds. Then there exists a subset $K = \{(j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\}$ of $\mathbb{N} \times \mathbb{N}$ such that

$$K \in F(I_2) \quad \text{and} \quad \nu\text{-}\lim_m x_{j_m k_m} = \xi. \tag{5}$$

We define the sequences $y = (y_{jk})$ and $z = (z_{jk})$ as

$$y_{jk} = \begin{cases} x_{jk} & \text{if } (j,k) \in K, \\ \xi & \text{if } (j,k) \in K^C; \end{cases}$$

and $z_{jk} = x_{jk} - y_{jk}$ for all $(j,k) \in \mathbb{N} \times \mathbb{N}$. For given $\epsilon > 0$, $t > 0$ and $(j,k) \in K^C$, we have

$$\nu_{y_{jk} - \xi}(t) = 1 > 1 - \epsilon.$$

Using (5) we have $\nu\text{-}\lim y = \xi$. Since $\{(j,k) : z_{jk} \neq \theta\} \subset K^C$, we have $\{(j,k) : z_{jk} \neq \theta\} \in I_2$.

Let the condition (ii) hold. Then $K = \{(j,k) : z_{jk} = \theta\} \in F(I_2)$ is an infinite set. Obviously, $K \in F(I_2)$ is an infinite set. Let $K = \{(j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\}$. Since $x_{j_m k_m} = y_{j_m k_m}$ and $\nu\text{-}\lim_m y_{j_m k_m} = \xi$, $\nu\text{-}\lim_m x_{j_m k_m} = \xi$. Hence $I_2^{*,\nu}\text{-}\lim_{j,k} x_{jk} = \xi$. This completes the proof of the theorem. $\square$

4. I_2-LIMIT POINTS AND I_2-CLUSTER POINTS IN PNS

In this section we define I_2-limit points and I_2-cluster points in probabilistic normed space analogous to the statistical limit points and statistical cluster points due to Fridy [8].

Definition 4.1. Let $(X, \nu, *)$ be a PNS, and $x = (x_{jk}) \in X$. An element $\xi \in X$ is said to be a limit point of the sequence $x = (x_{jk})$ with respect to the probabilistic norm ν (or a ν-limit point) if there is subsequence of the sequence x which converges to ξ with respect to the probabilistic norm ν.

By $\mathcal{L}_2^{\nu}(x)$, we denote the set of all limit points of the double sequence $x = (x_{jk})$ with respect to the probabilistic norm ν.

Definition 4.2. Let $(X, \nu, *)$ be a PNS, and $x = (x_{jk}) \in X$. An element $\xi \in X$ is said to be an I_2-limit point of the sequence x with respect to the probabilistic norm ν (or I_2^ν-limit point) if there is a subset $K=\{(j_m,k_m): j_1<j_2<\cdots; k_1<k_2<\cdots\}$ of $\mathbb{N}\times\mathbb{N}$ such that $K\notin I_2$ and $\nu\text{-}\lim_{m\to\infty} x_{j_m k_m}=\xi$.

We denote by $\Lambda_2^{I,\nu}(x)$, the set of all I_2^ν-limit points of the sequence $x=(x_{jk})$.

Definition 4.3. Let $(X,\nu,*)$ be a PNS, and $x=(x_{jk})\in X$. An element $\xi\in X$ is said to be an I_2-cluster point of x with respect to the probabilistic norm ν (or I_2^ν-cluster point) if for each $\epsilon>0$ and $t>0$

$$K=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\nu_{x_{jk}-\xi}(t)>1-\epsilon\right\}\notin I_2.$$

By $\Gamma_2^{I,\nu}(x)$, we denote the set of all I_2^ν-cluster points of the sequence $x=(x_{jk})$.

THEOREM 4.1. *Let $(X,\nu,*)$ be a PNS. Then for every sequence $x=(x_{jk})$ in X we have $\Lambda_2^{I,\nu}(x)\subset\Gamma_2^{I,\nu}(x)\subset\mathcal{L}_2^\nu(x)$.*

Proof. Let $\xi\in\Lambda_2^{I,\nu}(x)$. Then there exists a set $K=\{(j_m,k_m): j_1<j_2<\cdots; k_1<k_2<\cdots\}$ of $\mathbb{N}\times\mathbb{N}$ such that $K\notin I_2$ and $\nu\text{-}\lim_{m\to\infty} x_{j_m k_m}=\xi$. For each $\epsilon>0$ and $t>0$ there exists $N\in\mathbb{N}$ such that for $j,k>N$ we have $\nu_{x_{jk}-\xi}(t)>1-\epsilon$. Hence

$$\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\nu_{x_{jk}-\xi}(t)>1-\epsilon\right\}\supset\left\{j_{N+1},j_{N+2},\ldots;k_{N+1},k_{N+2},\ldots\right\}$$

and so

$$\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\nu_{x_{jk}-\xi}(t)>1-\epsilon\right\}\notin I_2,$$

which means that $\xi\in\Gamma_2^{I,\nu}(x)$. Hence $\Lambda_2^{I,\nu}(x)\subset\Gamma_2^{I,\nu}(x)$.

Let $\xi\in\Gamma_2^{I,\nu}(x)$. Then for given $\epsilon>0$ and $t>0$, we have

$$\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\nu_{x_{jk}-\xi}(t)>1-\epsilon\right\}\notin I_2.$$

Let $K=\{(j_m,k_m): j_1<j_2<\cdots; k_1<k_2<\cdots\}$. Then there is a subsequence $(x_{jk})_{(j,k)\in K}$ of (x_{jk}) that converges to ξ with respect to the probabilistic norm ν. Therefore ξ is an ordinary limit point of (x_{jk}), that is $\xi\in\mathcal{L}_2^\nu(x)$ and hence $\Gamma_2^{I,\nu}(x)\subset\mathcal{L}_2\nu(x)$. This completes the proof of the theorem. □

THEOREM 4.2. *Let $x=(x_{jk})$ be a sequence in a PN-space $(X,\nu,*)$. Then $\Lambda_2^{I,\nu}(x)=\Gamma_2^{I,\nu}(x)=\{\xi\}$, provided $I_2^\nu\text{-}\lim_{j,k} x_{jk}=\xi$.*

Proof. Let $\eta\in\Lambda_2^{I,\nu}(x)$, where $\xi\neq\eta$. Then there exist two subsets K and K', that is, $K=\{(j_m,k_m): j_1<j_2<\cdots; k_1<k_2<\cdots\}$ and $K'=$

$\{(p_m, q_m) : p_1 < p_2 < \cdots ; q_1 < q_2 < \cdots\}$ of $\mathbb{N} \times \mathbb{N}$ such that

$$K \notin I_2 \quad \text{and} \quad \nu\text{-} \lim_{m\to\infty} x_{j_m k_m} = \xi, \tag{6}$$

$$K' \notin I_2 \quad \text{and} \quad \nu\text{-} \lim_{m\to\infty} x_{p_m q_m} = \eta. \tag{7}$$

By (7), given $\epsilon > 0$ and $t > 0$, there exists $N \in \mathbb{N}$ such that for $m > N$ we have $\nu_{p_m q_m - \eta}(t) > 1 - \epsilon$. Therefore,

$$A = \{(p_m, q_m) \in K' : \nu_{p_m q_m - \eta}(t) \leq 1 - \epsilon\} \subset$$
$$\subset \{(p_m, q_m) : p_1 < p_2 < \cdots < p_N; q_1 < q_2 < \cdots < q_N\}.$$

As I_2 is an admissible ideal we have $A \in I_2$. If we take

$$B = \{(p_m, q_m) \in K' : \nu_{p_m q_m - \eta}(t) > 1 - \epsilon\} \notin I_2.$$

Otherwise, if $B \in I_2$, then $A \cup B = K' \in I_2$, which contradicts (7). Since $I_2^\nu\text{-}\lim_{j,k} x_{jk} = \xi$, we have that for each $\epsilon > 0$ and $t > 0$,

$$C = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk} - \xi}(t) \leq 1 - \epsilon\} \in I_2.$$

Therefore,

$$C^C = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk} - \xi}(t) > 1 - \epsilon\} \in F(I_2).$$

Since for every $\xi \neq \eta$, we have $B \cap C^C = \emptyset$, $B \subset C$. Since $C \in I_2$ implies $B \in I_2$, this contradicts the fact that $B \notin I_2$. Hence $\Lambda_2^{I,\nu}(x) = \{\xi\}$.

On the other hand, suppose that $\eta \in \Gamma_2^{I,\nu}(x)$, where $\xi \neq \eta$. By definition, for each $\epsilon > 0$ and $t > 0$,

$$A = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk} - \xi}(t) > 1 - \epsilon\} \notin I_2,$$
$$B = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk} - \eta}(t) > 1 - \epsilon\} \notin I_2.$$

For $\xi \neq \eta$, we have $A \cap B = \emptyset$ and therefore $B \subset A^C$. Also, $I_2^\nu\text{-}\lim_{j,k} x_{jk} = \xi$ implies that

$$A^C = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk} - \xi}(t) \leq 1 - \epsilon\} \in I_2.$$

Hence $B \in I_2$, which is a contradiction to $B \notin I_2$. Therefore, $\Gamma_2^{I,\nu}(x) = \{\xi\}$. This completes the proof of the theorem. $\square$

THEOREM 4.3. *Let $(X, \nu, *)$ be a PNS and for any two sequences $x = (x_{jk})$, $y = (y_{jk})$ in X, the set $A = \{(j,k) \in \mathbb{N} \times \mathbb{N} : x_{jk} \neq y_{jk}\} \in I_2$. Then $\Lambda_2^{I,\nu}(x) = \Lambda_2^{I,\nu}(y)$ and $\Gamma_2^{I,\nu}(x) = \Gamma_2^{I,\nu}(y)$.*

Proof. Let $\xi \in \Lambda_2^{I,\nu}(x)$. Then there exists a subset $K = \{(j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\}$ of $\mathbb{N} \times \mathbb{N}$ such that $K \notin I_2$ and $\nu\text{-}\lim_{m\to\infty} x_{j_m k_m} = \xi$. Given $\epsilon > 0$ and $t > 0$, there exists $N \in \mathbb{N}$ such that $\nu_{x_{j_m k_m} - \xi}(t) > 1 - \epsilon$ for $m > N$. Define $K_1 = K \cap A$ and $K_2 = K \setminus A$. Since $A \in I_2$ we have

$K_1 \in I_2$. As $K = K_1 \cup K_2$ and $K \notin I_2$ we have $K_2 \notin I_2$. It is clear that the subsequence $(y_{jk})_{(j,k)\in K_2}$ of the sequence $y = (y_{jk})$ is convergent to ξ with respect to the probabilistic norm ν we have. This implies that $\xi \in \Lambda_2^{I,\nu}(y)$ and hence $\Lambda_2^{I,\nu}(x) \subset \Lambda_2^{I,\nu}(y)$.

Similarly, we can show that $\Lambda_2^{I,\nu}(y) \subset \Lambda_2^{I,\nu}(x)$. Hence $\Lambda_2^{I,\nu}(x) = \Lambda_2^{I,\nu}(y)$.

Now, we need to show that $\Gamma_2^{I,\nu}(x) = \Gamma_2^{I,\nu}(y)$. Let $\xi \in \Gamma_2^{I,\nu}(x)$. For each $\epsilon > 0$ and $t > 0$ the set

$$B = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu_{x_{jk}-\xi}(t) > 1 - \epsilon\} \notin I_2.$$

Given $\epsilon > 0$ and $t > 0$, define

$$C = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu_{y_{jk}-\xi}(t) > 1 - \epsilon\}.$$

Now, we have to show that $C \notin I_2$. Let $C \in I_2$. Then

$$C^C = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu_{y_{jk}-\xi}(t) \leq 1 - \epsilon\} \in F(I_2).$$

By hypothesis, $A^C = \{(j,k) \in \mathbb{N} \times \mathbb{N} : x_{jk} = y_{jk}\} \in F(I_2)$. Since $F(I_2)$ is a filter in $\mathbb{N} \times \mathbb{N}$, $C^C \cap A^C \in F(I_2)$. Since $C^C \cap A^C \subset B^C$, $B^C \in F(I_2)$. This implies that $B \in I_2$ which is a contradiction, since $B \notin I_2$. So that $C \notin I_2$ and therefore $\Gamma_2^{I,\nu}(x) \subset \Gamma_2^{I,\nu}(y)$. Similarly, we can show that $\Gamma_2^{I,\nu}(y) \subset \Gamma_2^{I,\nu}(x)$ and hence $\Gamma_2^{I,\nu}(x) = \Gamma_2^{I,\nu}(y)$. This completes the proof of the theorem. □

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