ABSTRACT FUZZY ECONOMIES AND FUZZY EQUILIBRIUM PAIRS

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We propose the concept of fuzzy equilibrium pair for an abstract fuzzy economy and prove several theorems of equilibrium existence for abstract fuzzy economies with different types of correspondences.

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1. INTRODUCTION

Most essential results concerning the existence of equilibrium of an abstract economy extend earlier Debreu's works [7], [1]. Shafer and Sonnenschein [16] proved the existence of equilibrium of an economy with finite dimensional commodity space and irreflexive preferences represented as correspondences with open graph. Yannelis and Prahbakar's main result [21] concerns the existence of equilibrium when the constraint and preference correspondences have open lower sections. They work within different framework (countable infinite number of agents, infinite dimensional strategy spaces) and developed new techniques based on selection theorems and fixed-point theorems.

The concepts of K.F.-correspondences and KF-majorized correspondences were used by Borglin and Keiding [2] for their existence results. The second notion was extended by Yannelis and Prabhakar [21] to L-majorized correspondences. Ding, Kim and Tan [6], Tan, Yu and Yuan [17], Tulcea [18], Yannelis [20], Yannelis and Prabhakar [21] introduced different types of majorized correspondences. Yuan and Taradfar [23] proposed the notion of U-majorized correspondences and proved several equilibrium theorems. Liu and Cai [15] introduced the notion of Q-majorized correspondences and gave a new fixed point theorem. As its applications, they obtained some new existence theorems of an abstract economy.

Zadeh [24] initiated the theory of fuzzy sets as a framework for phenomena which cannot be characterized precisely. In Kim and Lee [12] the authors introduced the concept of a fuzzy game and proved the existence of equilibrium for 1-person fuzzy game. Also the existence of equilibrium points of fuzzy games was studied in [4], [5], [12],[13], [14]. Fixed point theorems for fuzzy mappings were proved in [3], [9].

In this paper we define a model of fuzzy equilibrium pair for an abstract fuzzy economy. We prove the existence a equilibrium pair of abstract fuzzy economies in several cases (economies with U-majorized preference correspondences and upper semicontinuous constraint correspondences, or economies with Q-majorized preference correspondences and upper semicontinuous constraint correspondences).

The paper is organized in the following way: Section 2 contains preliminaries and notation. The equilibrium theorems are stated in Section 3.

2. PRELIMINARIES AND NOTATION

Throughout this paper, we shall use the following notation and definitions:

Let A be a subset of a topological space X.

1. (A) denotes the family of all non-empty finite subset of A.
2. 2^A denotes the family of all subsets of A.
3. $\operatorname{cl} A$ denotes the closure of A in X.
4. If A is a subset of a vector space, co$A$ denotes the convex hull of A.
5. If $F, T : A \to 2^X$ are correspondences, then $\operatorname{co} T$, $\operatorname{cl} T$, $T \cap F : A \to 2^X$ are correspondences defined by $(\operatorname{co} T)(x) = \operatorname{co} T(x)$, $(\operatorname{cl} T)(x) = \operatorname{cl} T(x)$ and $(T \cap F)(x) = T(x) \cap F(x)$ for each $x \in A$, respectively.
6. The graph of $T : X \to 2^Y$ is the set $\operatorname{Gr}(T) = \{(x, y) \in X \times Y \mid y \in T(x)\}$.
7. The correspondence $\overline{T}$ is defined by $\overline{T}(x) = \{y \in Y : (x, y) \in \operatorname{cl}_{X\times Y} \operatorname{Gr}(T)\}$ (the set $\operatorname{cl}_{X\times Y} \operatorname{Gr}(T)$ is called the adherence of the graph of T).

It is easy to see that $\operatorname{cl} T(x) \subset \overline{T}(x)$ for each $x \in X$.

Definition 1. Let X, Y be topological spaces and $T : X \to 2^Y$ be a correspondence.

1. T is said to be *upper semicontinuous* if, for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(x) \subset V$ for each $y \in U$.
2. T is said to be *lower semicontinuous* (l.s.c.) if, for each x$\in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \emptyset$ for each $y \in U$.
3. T is said to have *open lower sections* if $T^{-1}(y) := \{x \in X : y \in T(x)\}$ is open in X for each $y \in Y$.

Remark 1 ([21]). If for each $y \in Y$, $T^{-1}(y)$ is open in X, then T is l.s.c.

Lemma 1 ([22]). *Let X and Y be two topological spaces and let A be a closed (resp. open) subset of X. Suppose $F_1 : X \to 2^Y$, $F_2 : X \to 2^Y$ are lower semicontinuous (resp. upper semicontinuous) such that $F_2(x) \subset F_1(x)$ for all $x \in A$. Then the correspondence $F : X \to 2^Y$ defined by*

$$F(x) = \begin{cases} F_1(x) & \text{if } x \notin A, \\ F_2(x) & \text{if } x \in A \end{cases}$$

is also lower semicontinuous (resp. upper semicontinuous).

Theorem 1 ([19]). *Let I be an index set. For each $i \in I$, let X_i be a nonempty convex subset of a Hausdorff locally convex space topological space E_i, D_i a nonempty compact metrizable subset of X_i and $S_i, T_i : X = \prod_{i \in I} X_i \to 2^{D_i}$ be correspondences such that:*

1. *$S_i(x)$ is nonempty and $\operatorname{cl}\operatorname{co} S_i(x) \subset T_i(x)$ for each $x \in X$.*
2. *S_i is lower semicontinuous.*

Then there exists $\overline{x} \in D := \prod_{i \in I} D_i$ such that $\overline{x}_i \in T_i(\overline{x})$ for each $i \in I$.

Definition 2 ([15]). Let X be a topological space and Y be a non-empty subset of a vector space E, $\theta : X \to E$ be a mapping and $T : X \to 2^Y$ be a correspondence.

1. T is said to be *of class Q_θ* (or Q) if
(a) for each $x \in X$, $\theta(x) \notin \operatorname{cl} T(x)$, and
(b) T is lower semicontinuous with open and convex values in Y.

2. A correspondence T_x is said to be a *Q_θ-majorant of T at x* if there exists an open neighborhood $N(x)$ of x such that $T_x : N(x) \to 2^Y$ and
(a) for each $z \in N(x)$, $T(z) \subset T_x(z)$ and $\theta(z) \notin \operatorname{cl} T_x(z)$;
(b) T_x is lower semicontinuous with open and convex values.

3. T is said to be *Q_θ-majorized* if for each $x \in X$ with $T(x) \neq \emptyset$ there exists a Q_θ-majorant T_x of T at x.

We need the following result to prove the existence theorems in the next section.

Lemma 2 ([15]). *Let X be a regular paracompact topological vector space and Y be a non-empty subset of a vector space E. Let $\theta : X \to E$ be a single-valued function and $P : X \to 2^Y \setminus \{\emptyset\}$ be Q_θ-majorized. Then there exists a correspondence $S : X \to 2^Y$ of class Q_θ such that $P(x) \subset S(x)$ for each $x \in X$.*

Definition 3 ([23]). Let X be a topological space and Y be a nonempty subset of a vector space E, $\theta : X \to E$ be a mapping and $T : X \to 2^Y$ be a correspondence.

1. T is said to be *of class U_θ* (or U) if
(a) for each $x \in X$, $\theta(x) \notin T(x)$ and

(b) T is upper semicontinuous with closed and convex values in Y.

2. A correspondence T_x is said to be a *U_θ-majorant of T at x* if there exists an open neighborhood $N(x)$ of x such that $T_x : N(x) \to 2^Y$ such that

(a) for each $z \in N(x)$, $T(z) \subset T_x(z)$ and $\theta(z) \notin T_x(z)$;

(b) T_x is upper semicontinuous with closed and convex values.

3. T is said to be *U_θ-majorized* if for each $x \in X$ with $T(x) \neq \emptyset$ there exists an U-majorant T_x of T at x.

We need the following result to prove the existence theorems in the next section.

LEMMA 3 ([23]). *Let X be a paracompact space and Y be a nonempty normal subset of a vector space E. Let $\theta : X \to E$ be a single-valued function and $P : X \to 2^Y$ be U_θ-majorized. Then there exists a correspondence $S : X \to 2^Y$ of class U_θ such that $P(x) \subset S(x)$ for each $x \in X$.*

Definition 4. Let X, Y be topological spaces and $T : X \to 2^Y$ be a correspondence. An element $x \in X$ is called *maximal element* for T if $T(x) = \emptyset$.

For each $i \in I$, let X_i be a nonempty subset of a topological space E_i and $T_i : X := \prod_{i \in I} X_i \to 2^{Y_i}$ a correspondence. Then a point $x \in X$ is called a maximal element for the family of correspondences $\{T_i\}_{i \in I}$ if $T_i(x) = \emptyset$ for all $i \in I$.

Notation. Let E and F be two Hausdorff topological vector spaces and $X \subset E$, $Y \subset F$ be two nonempty convex subsets. We denote by $\mathcal{F}(Y)$ the collection of fuzzy sets on Y. A mapping from X into $\mathcal{F}(Y)$ is called a fuzzy mapping. If $F : X \to \mathcal{F}(Y)$ is a fuzzy mapping, then for each $x \in X$, $F(x)$ (denoted by F_x in this sequel) is a fuzzy set in $\mathcal{F}(Y)$ and $F_x(y)$ is the degree of membership of point y in F_x. A fuzzy mapping $F : X \to \mathcal{F}(Y)$ is called convex, if for each $x \in X$, the fuzzy set F_x on Y is a fuzzy convex set, i.e., for any $y_1, y_2 \in Y$, $t \in [0,1]$, $F_x(ty_1 + (1-t)y_2) \geq \min\{F_x(y_1), F_x(y_2)\}$. In the sequel, we denote

$$(A)_q = \{y \in Y : A(y) \geq q\}, \quad q \in [0,1] \text{ the } q\text{-cut set of } A \in \mathcal{F}(Y).$$

3. EXISTENCE OF EQUILIBRIUM PAIRS FOR ABSTRACT ECONOMIES

In this section we describe the fuzzy equilibrium pair for an abstract fuzzy economy. We prove the existence of fuzzy equilibrium of abstract fuzzy economies in several cases.

Let I be a nonempty set (the set of agents). For each $i \in I$, let X_i be a nonempty topological vector space representing the set of actions and define $X := \prod_{i \in I} X_i$; let $A_i, B_i : X \to \mathcal{F}(X_i)$ be the constraint fuzzy correspondences and $P_i : X \to \mathcal{F}(X_i)$ the preference fuzzy correspondence, $a_i, b_i : X \to (0,1]$ fuzzy constraint functions and $p_i : X \to (0,1]$ fuzzy preference function.

Definition 4. An *abstract fuzzy economy* is defined as an ordered family $\Gamma = (X_i, A_i, B_i, P_i, a_i, b_i, p_i)_{i\in I}$.

If $A_i, B_i, P_i : X \to 2^{Y_i}$ are classical correspondences, then the previous definition can be reduced to the standard definition of abstract economy due to Yuan [22].

Definition 5. A *fuzzy equilibrium pair* for Γ is defined as a pair of points $(\overline{x}, \overline{y}) \in X \times X$ such that, for each $i \in I$, $\overline{x_i} \in (B_{i_{\overline{x}}})_{b_i(\overline{x})}$, $\overline{y_i} \in \operatorname{cl}(P_{i_{\overline{x}}})_{p_i(\overline{x})}$ and $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap (P_{i_{\overline{y}}})_{p_i(\overline{y})} = \emptyset$, where $(A_{i_{\overline{x}}})_{a_i(\overline{x})} = \{z \in Y_i : A_{i_{\overline{x}}}(z) \geq a_i(\overline{x})\}$, $(B_{i_{\overline{x}}})_{b_i(\overline{x})} = \{z \in Y_i : B_{i_{\overline{x}}}(z) \geq b_i(\overline{x})\}$, $(P_{i_{\overline{y}}})_{p_i(\overline{y})} = \{z \in Y_i : P_{i_{\overline{y}}}(z) \geq p_i(\overline{y})\}$.

We state some new equilibrium existence theorems for abstract fuzzy economies.

Theorem 3 is an existence theorem of fuzzy equilibrium pair for an abstract fuzzy economy with U-majorized correspondences $x \to (P_{i_x})_{p_i(x)}$ and upper semicontinuous correspondences $x \to (B_{i_x})_{b_i(x)}$. To prove this theorem we need the following result that is Theorem 4.2 in [23].

THEOREM 2 ([23]). *Let X be a nonempty convex subset of a Hausdorff locally convex topological vector space E and let D be a nonempty compact subset of X. Let $P : X \to 2^D$ be U_θ-majorized. Then there exists a point $\overline{x} \in \operatorname{co} D$ such that $P(\overline{x}) = \emptyset$.*

THEOREM 3. *Let $\Gamma = (X_i, A_i, B_i, P_i, a_i, b_i, p_i)_{i\in I}$ be an abstract fuzzy economy such that for each $i \in I$,*

(1) X_i is a nonempty compact and convex subset of a locally convex Hausdorff topological vector space E;

(2) P_i is such that $x \to (P_{i_x})_{p_i(x)} : X \to 2^{X_i}$ has nonempty values and is U_{π_i}-majorized on X;

(3) A_i, B_i are such that $x \to (B_{i_x})_{b_i(x)} : X \to 2^{X_i}$ is upper semicontinuous, each $(B_{i_x})_{b_i(x)}$ is a closed convex subset of X_i, $x \to (A_{i_x})_{a_i(x)} : X \to 2^{X_i}$ has nonempty closed convex values and $(A_{i_x})_{a_i(x)} \subset (B_{i_x})_{b_i(x)}$ for each $x \in X$.

Then there exists an equilibrium pair $(\overline{x}, \overline{y}) \in X \times X$ such that $\overline{x_i} \in (B_{i_{\overline{x}}})_{b_i(\overline{x})}$, $\overline{y_i} \in (P_{i_{\overline{x}}})_{p_i(\overline{x})}$ and $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap (P_{i_{\overline{y}}})_{p_i(\overline{y})} = \emptyset$ for each $i \in I$.

Proof. For each $i \in I$, $x \to (B_{i_x})_{b_i(x)}$ is upper semicontinuous and has non-empty, convex and closed values. We define $B : X \to 2^X$ by $B(x) = \prod_{i\in I}(B_{i_x})_{b_i(x)}$. Then B is upper semicontinuous with nonempty, convex and closed values. By Fan's fixed-point theorem [8], there exists a fixed point $\overline{x} \in X$ for B such that $\overline{x} \in B(\overline{x})$, i.e., $\overline{x}_i \in (B_{i_{\overline{x}}})_{b_i(\overline{x})}$ for each $i \in I$. It remains to show that there exists a point $\overline{y} \in X$ such that $\overline{y_i} \in (P_{i_{\overline{x}}})_{p_i(\overline{x})}$ and $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap (P_{i_{\overline{y}}})_{p_i(\overline{y})} = \emptyset$ for each $i \in I$.

Since X is paracompact and $x \to (P_{i_x})_{p_i(x)}$ is U_{π_i}-majorized, by Lemma 3 there exists a correspondence $\varphi_i : X \to 2^{X_i}$ of class U_{π_i} such that $(P_{i_x})_{p_i(x)} \subset \varphi_i(x)$ for each $x \in X$. Then, φ_i is upper semicontinuous with nonempty closed, convex values and $x_i \notin \varphi_i(x)$ for each $x \in X$.

Define $T_i : X \to 2^{X_i}$ by

$$T_i(y) = \begin{cases} (A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap \varphi_i(y) & \text{if } y_i \in \operatorname{int}(P_{i_{\overline{x}}})_{p_i(\overline{x})}, \\ \varphi_i(y) & \text{if } y_i \notin \operatorname{int}(P_{i_{\overline{x}}})_{p_i(\overline{x})}. \end{cases}$$

By Lemma 1, T_i is upper semicontinuous on X, has convex closed values and $y_i \notin T_i(y)$. Define $T : X \to 2^X$, $T(y) = \prod_{i\in I} T_i(y)$. T is upper semicontinuous on X, has convex closed values, and $y \notin T(y)$. Therefore, it is U-majorized.

By Theorem 2, there exists $\overline{y} \in X$ such that $T(\overline{y}) = \emptyset$, i.e., $T_i(\overline{y}) = \emptyset$ for each $i \in I$.

For each $y \in X$, $\varphi_i(y)$ is a nonempty subset of X_i. We have $\overline{y_i} \in \operatorname{int}(P_{i_{\overline{x}}})_{p_i(\overline{x})} \subset (P_{i_{\overline{x}}})_{p_i(\overline{x})}$ and $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap \varphi_i(\overline{y}) = \emptyset$. Since $(P_{i_{\overline{y}}})_{p_i(\overline{y})} \subset \varphi_i(\overline{y})$, we have that $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap (P_{i_{\overline{y}}})_{p_i(\overline{y})} = \emptyset$. Hence, $\overline{x_i} \in (B_{i_{\overline{x}}})_{b_i(\overline{x})}$, $\overline{y_i} \in (P_{i_{\overline{x}}})_{p_i(\overline{x})}$ and $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap (P_{i_{\overline{y}}})_{p_i(\overline{y})} = \emptyset$ for each $i \in I$, and then $(\overline{x}, \overline{y})$ is an equilibrium pair for Γ. □

Since an upper semicontinuous, irreflexive correspondence, with nonempty closed convex values is U-majorized, we get.

Corollary 1. *Let $\Gamma = (X_i, A_i, B_i, P_i, a_i, b_i, p_i)_{i\in I}$ be an abstract fuzzy economy such that for each $i \in I$ conditions* (1) *and* (3) *are the same as in* Theorem 3 *and condition* (2) *is:*

(2) *P_i is such that $x \to (P_{i_x})_{p_i(x)} : X \to 2^{X_i}$ is upper semicontinuous on X, has nonempty closed convex values, and $x_i \notin (P_{i_x})_{p_i(x)}$ for each $x \in X$.*

Then there exists an equilibrium pair $(\overline{x}, \overline{y}) \in X \times X$.

Theorem 5 is an existence theorem of fuzzy equilibrium pair for an abstract fuzzy economy with Q_{π_i}-majorized correspondences $x \to (P_{i_x})_{p_i(x)}$ and lower semicontinuous correspondences $x \to (B_{i_x})_{b_i(x)}$. To prove this theorem we need the following result that is Theorem 7 in [19].

Theorem 4 ([19]). *Let $\Gamma = (X_i, P_i)_{i\in I}$ be a qualitative game where I is an index set such that for each $i \in I$,*

1) *X_i is a nonempty convex compact metrizable subset of a Hausdorff locally convex topological vector space E and $X := \prod_{i\in I} X_i$;*

2) *$P_i : X \to 2^{X_i}$ is lower semi-continuous;*

3) *for each $x \in X$, $x_i \notin \operatorname{cl}\operatorname{co} P_i(x)$.*

Then there exists a point $\overline{x} \in X$ *such that* $P_i(\overline{x}) = \emptyset$ *for all* $i \in I$, *i.e.,* $\overline{x}$ *is a maximal element of* Γ.

THEOREM 5. *Let* $\Gamma = (X_i, A_i, B_i, P_i, a_i, b_i, p_i)_{i \in I}$ *be an abstract fuzzy economy such that, for each* $i \in I$,

(1) X_i *be a nonempty compact convex metrizable subset of a locally convex Hausdorff topological vector space* E;

(2) P_i *is such that* $x \to (P_{i_x})_{p_i(x)} : X \to 2^{X_i}$ *is* Q_{π_i}*-majorized on* X *and has nonempty values;*

(3) A_i, B_i *are such that* $x \to (B_{i_x})_{b_i(x)} : X \to 2^{X_i}$ *is lower semicontinuous, each* $(B_{i_x})_{b_i(x)}$ *is a closed convex subset of* X_i, $(A_{i_x})_{a_i(x)}$ *is nonempty convex and* $(A_{i_x})_{a_i(x)} \subset (B_{i_x})_{b_i(x)}$ *for each* $x \in X$.

Then there exists an equilibrium pair $(\overline{x}, \overline{y}) \in X \times X$ *such that* $\overline{x_i} \in (B_{i_{\overline{x}}})_{b_i(\overline{x})}$, $\overline{y_i} \in \mathrm{cl}(P_{i_{\overline{x}}})_{p_i(\overline{x})}$ *and* $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap (P_{i_{\overline{y}}})_{p_i(\overline{y})} = \emptyset$ *for each* $i \in I$.

Proof. For each $i \in I$, $x \to (B_{i_x})_{b_i(x)}$ is lower semicontinuous and has non-empty, convex and closed values. By Theorem 1, there exists $\overline{x} \in X$ such that $\overline{x}_i \in (B_{i_{\overline{x}}})_{b_i(\overline{x})}$ for each $i \in I$. It remains to show that there exists a point $\overline{y} \in X$ such that $\overline{y}_i \in \mathrm{cl}(P_{i_{\overline{x}}})_{p_i(\overline{x})}$ and $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap (P_{i_{\overline{y}}})_{p_i(\overline{y})} = \emptyset$ for each $i \in I$.

Since X is paracompact and $x \to (P_{i_x})_{p_i(x)}$ is Q_{π_i}-majorized, by Lemma 2 there exists a correspondence $\varphi_i : X \to 2^{X_i}$ of class Q_{π_i} such that $(P_{i_x})_{p_i(x)} \subset \varphi_i(x)$ for each $x \in X$. Then φ_i is lower semicontinuous with nonempty open convex values and $x_i \notin \mathrm{cl}\,\varphi_i(x)$ for each $x \in X$.

Define $T_i : X \to 2^{X_i}$ by

$$T_i(y) = \begin{cases} (A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap \varphi_i(y) & \text{if } y_i \in \mathrm{cl}(P_{i_{\overline{x}}})_{p_i(\overline{x})}, \\ \varphi_i(y) & \text{if } y_i \notin \mathrm{cl}(P_{i_{\overline{x}}})_{f_{pi}(\overline{x})}. \end{cases}$$

By Lemma 1, T_i is lower semicontinuous on X. Then $\mathrm{cl}\,T_i$ is lower semicontinuous, it has convex values and $x_i \notin \mathrm{cl}\,T_i(x)$.

By Theorem 4, there exists $\overline{y} \in X$ such that $\mathrm{cl}\,T_i(\overline{y}) = \emptyset$ for each $i \in I$.

For each $y \in X$, $\varphi_i(y)$ is a nonempty subset of X_i. We have $\overline{y_i} \in \mathrm{cl}(P_{i_{\overline{x}}})_{p_i(\overline{x})}$ and $\mathrm{cl}((A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap \varphi_i(\overline{y})) = \emptyset$. It follows that $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap \varphi_i(\overline{y}) = \emptyset$. Since $(P_{i_{\overline{y}}})_{p_i(\overline{y})} \subset \varphi_i(\overline{y})$, we have $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap (P_{i_{\overline{y}}})_{p_i(\overline{y})} = \emptyset$. Hence, $\overline{x_i} \in (B_{i_{\overline{x}}})_{b_i(\overline{x})}$, $\overline{y_i} \in (P_{i_{\overline{x}}})_{p_i(\overline{x})}$ and $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap (P_{i_{\overline{y}}})_{p_i(\overline{y})} = \emptyset$ for each $i \in I$, and then $(\overline{x}, \overline{y})$ is an equilibrium pair for Γ. □

Since a lower semicontinuous, irreflexive correspondence, with nonempty open convex values is Q-majorized, we get

COROLLARY 2. *Let* $\Gamma = (X_i, A_i, B_i, P_i, a_i, b_i, p_i)_{i \in I}$ *be an abstract fuzzy economy such that for each* $i \in I$ *conditions* (1) *and* (3) *are the same as in Theorem* 5 *and condition* (2) *is*

(2) P_i *is such that* $x \to (P_{i_x})_{p_i(x)} : X \to 2^{X_i}$ *is lower semicontinuous on* X, *has nonempty open convex values and* $x_i \notin \mathrm{cl}(P_{i_x})_{p_i(x)}$ *for each* $x \in X$.

Then there exists an equilibrium pair $(\overline{x}, \overline{y}) \in X \times X$.

By Remark 1, a correspondence with open lower sections is lower semicontinuous and then we obtain.

Corollary 3. *Let* $\Gamma = (X_i, A_i, B_i, P_i, a_i, b_i, p_i)_{i \in I}$ *be an abstract fuzzy economy such that for each* $i \in I$ *conditions* (1) *and* (3) *are the same as in Theorem* 5 *and condition* (2) *is*

(2) P_i *is such that* $x \to (P_{i_x})_{p_i(x)} : X \to 2^{X_i}$ *has open lower sections on* X, *has nonempty open convex values and* $x_i \notin \mathrm{cl}(P_{i_x})_{p_i(x)}$ *for each* $x \in X$.

Then there exists an equilibrium pair $(\overline{x}, \overline{y}) \in X \times X$.

Theorem 6 is an existence theorem of fuzzy equilibrium pair for an abstract fuzzy economy with Q_{π_i}-majorized correspondences $x \to (P_{i_x})_{p_i(x)}$ and upper semicontinuous correspondences $x \to (B_{i_x})_{b_i(x)}$.

Theorem 6. *Let* $\Gamma = (X_i, A_i, B_i, P_i, a_i, b_i, p_i)_{i \in I}$ *be an abstract fuzzy economy such that, for each* $i \in I$,

(1) X_i *be a nonempty compact convex metrizable subset of a locally convex Hausdorff topological vector space E;*

(2) P_i *is such that* $x \to (P_{i_x})_{p_i(x)} : X \to 2^{X_i}$ *is* Q_{π_i}*-majorized on* X *and has nonempty values;*

(3) A_i, B_i *are such that* $x \to (B_{i_x})_{b_i(x)} : X \to 2^{X_i}$ *is upper semicontinuous, each* $(B_{i_x})_{b_i(x)}$ *is a closed convex subset of* X_i, $(A_{i_x})_{a_i(x)}$ *is nonempty closed convex and* $(A_{i_x})_{a_i(x)} \subset (B_{i_x})_{b_i(x)}$ *for each* $x \in X$.

Then there exists an equilibrium pair $(\overline{x}, \overline{y}) \in X \times X$ *such that* $\overline{x_i} \in (B_{i_{\overline{x}}})_{b_i(\overline{x})}$, $\overline{y_i} \in (P_{i_{\overline{x}}})_{p_i(\overline{x})}$ *and* $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap (P_{i_{\overline{y}}})_{p_i(\overline{y})} = \emptyset$ *for each* $i \in I$.

Proof. For each $i \in I$, $x \to (B_{i_x})_{b_i(x)}$ is upper semicontinuous and has non-empty, convex and closed values. We define $B : X \to 2^X$ by $B(x) = \prod_{i \in I} (B_{i_x})_{b_i(x)}$. Then B is upper semicontinuous with nonempty, convex and closed values. By Fan's fixed-point theorem [8], there exists a fixed point $\overline{x} \in X$ for B such that $\overline{x} \in B(\overline{x})$, i.e., $\overline{x}_i \in (B_{i_{\overline{x}}})_{b_i(\overline{x})}$ for each $i \in I$. It remains to show that there exists a point $\overline{y} \in X$ such that $\overline{y_i} \in (P_{i_{\overline{x}}})_{p_i(\overline{x})}$ and $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap (P_{i_{\overline{y}}})_{p_i(\overline{y})} = \emptyset$ for each $i \in I$.

Since X is paracompact and $x \to (P_{i_x})_{p_i(x)}$ is Q_{π_i}-majorized, by Lemma 2 there exists a correspondence $\varphi_i : X \to 2^{X_i}$ of class Q_{π_i} such that $(P_{i_x})_{p_i(x)} \subset \varphi_i(x)$ for each $x \in X$. Then φ_i is lower semicontinuous with nonempty open convex values and $x_i \notin \mathrm{cl}\, \varphi_i(x)$ for each $x \in X$.

Define $T_i : X \to 2^{X_i}$ by

$$T_i(y) = \begin{cases} (A_{i\overline{x}})_{a_i(\overline{x})} \cap \varphi_i(y) & \text{if } y_i \in \mathrm{cl}(P_{i\overline{x}})_{p_i(\overline{x})}; \\ \varphi_i(y) & \text{if } y_i \notin \mathrm{cl}(P_{i\overline{x}})_{p_i(\overline{x})}. \end{cases}$$

By Lemma 1, T_i is lower semicontinuous on X. Then $\mathrm{cl}\, T_i$ is lower semicontinuous, has convex values, and $x_i \notin \mathrm{cl}\, T_i(x)$.

By Theorem 4, there exists $\overline{y} \in X$ such that $\mathrm{cl}\, T_i(\overline{y}) = \emptyset$ for each $i \in I$.

For each $y \in X$, $\varphi_i(y)$ is a nonempty subset of X_i. We have $\overline{y_i} \in \mathrm{cl}(P_{i\overline{x}})_{p_i(\overline{x})}$ and $\mathrm{cl}((A_{i\overline{x}})_{a_i(\overline{x})} \cap \varphi_i(\overline{y})) = \emptyset$. It follows that $(A_{i\overline{x}})_{a_i(\overline{x})} \cap \varphi_i(\overline{y}) = \emptyset$. Since $(P_{i\overline{y}})_{p_i(\overline{y})} \subset \varphi_i(\overline{y})$, we have $(A_{i\overline{x}})_{a_i(\overline{x})} \cap (P_{i\overline{y}})_{p_i(\overline{y})} = \emptyset$. Hence, $\overline{x_i} \in (B_{i\overline{x}})_{b_i(\overline{x})}$, $\overline{y_i} \in (P_{i\overline{x}})_{p_i(\overline{x})}$ and $(A_{i\overline{x}})_{a_i(\overline{x})} \cap (P_{i\overline{y}})_{p_i(\overline{y})} = \emptyset$ for each $i \in I$, and then $(\overline{x}, \overline{y})$ is an equilibrium pair for Γ. $\square$

A lower semicontinuous, irreflexive correspondence, with nonempty open convex values is Q-majorized and then we obtain

Corollary 4. *Let* $\Gamma = (X_i, A_i, B_i, P_i, a_i, b_i, p_i)_{i\in I}$ *be an abstract fuzzy economy such that for each* $i \in I$ *conditions* (1) *and* (3) *are the same as in Theorem* 6 *and condition* (2) *is*

(2) P_i *is such that* $x \to (P_{i_x})_{p_i(x)} : X \to 2^{X_i}$ *is lower semicontinuous on* X, *has nonempty open convex values and* $x_i \notin \mathrm{cl}(P_{i_x})_{p_i(x)}$ *for each* $x \in X$.

Then there exists an equilibrium pair $(\overline{x}, \overline{y}) \in X \times X$.

By Remark 1, a correspondence with open lower sections is lower semicontinuous and then we obtain

Corollary 5. *Let* $\Gamma = (X_i, A_i, B_i, P_i, a_i, b_i, p_i)_{i\in I}$ *be an abstract fuzzy economy such that for each* $i \in I$ *conditions* (1) *and* (3) *are the same as in Theorem* 6 *and condition* (2) *is*

(2) P_i *is such that* $x \to (P_{i_x})_{p_i(x)} : X \to 2^{X_i}$ *has open lower sections on* X, *has nonempty open convex values, and* $x_i \notin \mathrm{cl}(P_{i_x})_{p_i(x)}$ *for each* $x \in X$.

Then there exists an equilibrium pair $(\overline{x}, \overline{y}) \in X \times X$.

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