ON THE MOMENTS OF ITERATED TAIL

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The classical distribution in ruins theory has the property that the sequence of the first moment of the iterated tails is convergent. We give sufficient conditions under which this property holds and also, we construct a counterexample that shows that this property it is not generally true.

AMS 2000 Subject Classification: 60E05, 40A05.

Key words: moments, integrated tail.

1. BASIC DEFINITIONS AND STATEMENT OF THE PROBLEM

Let (Ω, K, P) be a probability space and $\mathbf{L} = \bigcap_{p\geq 1} L_+^p(\Omega, K, P)$. Thus, $X \in \mathbf{L}$ iff $X \geq 0$ (a.s.) and $\mathrm{E}X^p < \infty$ for every $1 \leq p < \infty$. Let $\mathbf{M}$ be the set of the distributions of the random variables $X \in \mathbf{L}$. In other words, $F \in \mathbf{M}$ iff $F([0,\infty)) = 1$ and $\int x^p \mathrm{d}F(x) < \infty$ $\forall 1 \leq p < \infty$. The integral $\int x^p \mathrm{d}F(x) = \mathrm{E}X^p$ will be denoted by $\mu_p(F)$. It is the pth moment of X. We shall denote by $F(x)$ the distribution function of F and by $\underline{F}(x)$ its right tail. Precisely, $F(x)$ will stand for $F([0,x])$ and $\underline{F}(x)$ for $F((x,\infty))$.

The set of absolutely continuous probability distribution from $\mathbf{M}$ will be denoted by $\mathbf{M}_{ac}$. The density of F (provided that it does exist) will be denoted by f_F and the hazard rate (see e.g. [2]) by λ_F. The hazard rate of an absolutely continuous distribution $F \in \mathbf{M}$ is defined by

$$(1.1)\qquad \lambda_F(x) = \frac{f_F(x)}{\underline{F}(x)}$$

or, alternatively, by

$$(1.2)\qquad \underline{F}(x) = \mathrm{e}^{-\int_0^x \lambda_F(y)\mathrm{d}y}.$$

The characteristic function of F will be denoted by φ_F and the moment generating function (abbreviated as mgf) by $\mathbf{m}_F$. Thus, for $t \in \mathbb{R}$,

$$(1.3)\qquad \varphi_F(t) = \int \mathrm{e}^{\mathrm{i}tx}\mathrm{d}F(x), \quad \mathbf{m}_F(t) = \int \mathrm{e}^{tx}\mathrm{d}F(x)$$

or, integrating by parts

$$(1.4)\qquad \varphi_F(t) = 1 + \mathrm{i}t\int_0^\infty \mathrm{e}^{\mathrm{i}tx}\underline{F}(x)\mathrm{d}x, \quad \mathbf{m}_F(t) = 1 + t\int_0^\infty \mathrm{e}^{tx}\underline{F}(x)\mathrm{d}x.$$

Let $t^\star(F)$ be defined by

$$(1.5)\qquad t^\star(F) = \sup\{t \in \mathbb{R};\ \mathbf{m}_F(t) < \infty\}.$$

If $t^\star(F) = \infty$ we say that F is **short tailed**; if $t^\star(F) = 0$ we call F **long tailed** and if $0 < t^\star(F) < \infty$, F is **medium tailed**. (See e.g. [4].)

In renewal and ruin theories the following distribution is of interest: it is called **the integrated tail** (see e.g. [1], [3] or [4]). It is defined for $F \in \mathbf{M}$ by

$$(1.6)\qquad F_I(x) = 0, \text{ for } x < 0 \text{ and } F_I(x) = \frac{\int_0^x \underline{F}(y)\mathrm{d}y}{\int_0^\infty \underline{F}(y)\mathrm{d}y}, \text{ for } x \geq 0.$$

The main result in [13] was the following (Theorem 3.5, Corollary 3.7).

Theorem A. *Consider the operator* $T : \mathbf{M}_{ac} \to \mathbf{M}_{ac}$ *defined by* $TF = F_I$. *Suppose that the limit* $\lambda_F(\infty)$ *does exist. Then*

(i) $\lambda_F(\infty) = t^\star(F)$, *where* $t^\star(F)$ *is defined by* (1.5);

(ii) *if* F *is medium tailed, then* $T^nF \to \mathrm{Exp}(\lambda_F(\infty))$, $n \to \infty$;
if F *is short tailed, then* $T^nF \to \delta_0$, $n \to \infty$;
if F *is long tailed, then* T^nF *does not converge at all.*

To be precise, in the case of long tails the result was that the mass of T^nF vanishes at infinity: $(\underline{T^nF})(x) \to 1$ as $n \to \infty$, for $x \geq 0$.

Recall that if $\lambda_F \leq \lambda_G$, we say that G is **HR-dominated** by F (and write $G \prec_{\mathrm{HR}} F$). Obviously, $G \prec_{\mathrm{HR}} F \Rightarrow G \prec_{\mathrm{st}} F$: the HR-domination implies the stochastic one. Further on, if λ_F is non-decreasing, then F is called a **IFR-distribution** and if λ_F is non-increasing, then F is a **DFR-distribution** (see e.g. [2], [8]).

The main tool used in [13] were the HR-monotonousness properties of T (Propositions 2.5, 2.6 and 3.1). We state them in a shortened way:

Lemma. *Let* $F, G \in \mathbf{M}_{ac}$. *Then*

(i) $\lambda_F \leq \lambda_G \Rightarrow \lambda_{TF} \leq \lambda_{TG}$;

(ii) *if* λ_F *is non-increasing* (*respectively non-decreasing*), *then* λ_{TF} *is non-increasing* (*respectively non-decreasing*) *too;*

(iii) *if* $F \in \mathrm{IFR}$ *then* $T(F) \prec_{\mathrm{HR}} F$, *and if* $F \in \mathrm{DFR}$ *then* $F \prec_{\mathrm{HR}} T(F)$.

In this paper we are concerned with studying the moments of T^nF in order to find another property of the moments of a random variable, besides the already known ones (see [9]). Our result is

Theorem B. *Let* $F \in \mathbf{M}_{ac}$ *and let* $\mu_k = \int x^k \mathrm{d}F(x)$ *be the moment of order* k. *Suppose that the limit* $\lambda_F(\infty)$ *of the hazard rate* (1.1) *does exist.*

Then

$$\lim_{n\to\infty} \frac{\mu_{n+1}}{n\mu_n} = \frac{1}{\lambda_F(\infty)} \quad (\textit{with the convention that } \frac{1}{0} = \infty.) \tag{1.7}$$

Moreover, if $\lambda_F(\infty)$ does not exist, it is possible that the limit (1.7) *does not exist, too.*

This result can be reformulated in terms of the first moment of $T^n F$.

2. MOMENTS OF $T^n(F)$

PROPOSITION 1 (Moments of F_I). *Let $F \in \mathbf{M}$, $\mu_k = \mu_k(F)$ be the moments of F, φ be the characteristic function of F_I. Then*

$$\int x^k \mathrm{d}F_I(x) := \mu_k(F) = \frac{\mu_{k+1}}{(k+1)\mu_1}, \quad \forall\, k \geq 0. \tag{2.1}$$

Proof. We know $\varphi_I(t) = \frac{\varphi(t)-1}{\mathrm{i}t\mu_1}$. Then

$$\mu_k(F_I) = \frac{(\varphi_I)^{(k)}(0)}{\mathrm{i}^k} = \lim_{t\to 0} \frac{\varphi_I(t)-1}{\mathrm{i}^k t^k/(k!)} = \lim_{t\to 0} \frac{k!(\varphi(t)-1-t\mu_1)}{\mathrm{i}^{k+1}t^{k+1}\mu_1}.$$

If we apply l'Hospital's rule $k+1$ times, the last limit is $\frac{\mu_{k+1}}{(k+1)\mu_1}$. □

PROPOSITION 2. *Let $F \in \mathbf{M}$. Let $F_n = T^n(F)$ and $\mu_{n,k} = \int x^k \mathrm{d}F_n(x)$ be the moments of F_n. Let also $\mu_m = \int x^m \mathrm{d}F(x)$ be the moments of F. Then*

$$\mu_{n,k} = \frac{\mu_{n+k}}{\binom{n+k}{k}\mu_n}. \tag{2.2}$$

Proof. Let φ be the characteristic function of F and φ_n be the characteristic function of $T^n F$. The recurrence between the characteristic functions φ_n is

$$\varphi_{n+1}(t) = \frac{\varphi_n(t)-1}{t(\varphi_n)'(0)} \tag{2.3}$$

which can be better written as

$$\varphi_n(t) = 1 + t(\varphi_n)'(0)\varphi_{n+1}(t). \tag{2.4}$$

Let $z_n = (\varphi_n)'(0)$. With our notation, one can easily see that $z_n = \mathrm{i}\mu_n$. Let $k \geq 1$. By differentiating (2.4) k times and applying Leibniz's formula one gets

$$(\varphi_n)^{(k+1)}(t) = z_n\Big(t(\varphi_{n+1})^{(k+1)}(t) + (k+1)(\varphi_{n+1})^{(k)}(t)\Big). \tag{2.5}$$

For $t = 0$ we find that $\mathrm{i}^{k+1}\mu_{n,k+1} = \mathrm{i}^{k+1}\mu_{n,1}\mu_{n+1,k}$. Hence the moments satisfy the recurrence

$$\mu_{n,k+1} = (k+1)\mu_{n,1}\mu_{n+1,k} \Leftrightarrow \mu_{n+1,k} = \frac{\mu_{n,k+1}}{(k+1)\mu_{n,1}}. \tag{2.6}$$

Then we prove by induction

$$\mu_{n+1,k} = \frac{\mu_{n-j+1,k+j}}{\binom{k+j}{j}\mu_{n-j+1,j}}, \quad 1 \le j \le n+1. \tag{2.6a}$$

Indeed, for $j = 1$, relation (2.6a) coincides with relation (2.6). If (2.6a) is true for $j \le n$, then applying again relation (2.6) we find

$$\mu_{n+1,k} = \frac{\mu_{n-j+1,k+j}}{\binom{k+j}{j}\mu_{n-j+1,j}} = \frac{\frac{\mu_{n-j,k+j+1}}{(k+j+1)\mu_{n-j,1}}}{\binom{k+j}{j}\frac{\mu_{n-j}j+1}{(j+1)\mu_{n-j,1}}} = \frac{\mu_{n-j,k+j+1}}{\binom{k+j+1}{j+1}\mu_{n-j,j+1}}.$$

For $j = n+1$ one gets $\mu_{n+1,k} = \frac{\mu_{n+k+1}}{\binom{n+k+1}{n+1}\mu_{n+1}}$. Replacing n with $n-1$ we get 2.2). $\square$

Corollary 3. *With the same notation as in Proposition* 2, *the first moment of F_n is*

$$\mu_{n,1} = \frac{\mu_{n+1}}{(n+1)\mu_n}. \tag{2.7}$$

Proof. Obvious. $\square$

Now we take into account the monotonicity properties stated in Lemma.

Proposition 4. *Let $F \in \mathbf{M}_{ac}$ and let λ be the hazard rate of F.*
(i) *If $F \in$*IFR *then* $\left(\frac{\mu_{n+1}}{(n+1)\mu_n}\right)_n \downarrow \frac{1}{\lambda(\infty)}$ *as $n \to \infty$.*
(ii) *If $F \in$*DFR *then* $\left(\frac{\mu_{n+1}}{(n+1)\mu_n}\right)_n \uparrow \frac{1}{\lambda(\infty)}$ *as $n \to \infty$.*

Proof. If $F \in$ IFR then the sequence of random variables $(F_n)_n$, $F_n = T^n(F)$, is HR-decreasing. This implies that $F_{n+1} \prec_{\mathrm{HR}} F_n$. Therefore, $\mu_1(F_{n+1}) \le \mu_1(F_n)$, (see e.g. [8]). Thus the sequence $(\mu_{n,1})_n$ is non-increasing. It must have a limit. But we know that $F_n \to \mathrm{Exp}(\lambda(\infty))$. By Beppo-Levi's theorem, the first moments of F_n should converge to the first moment of $\mathrm{Exp}(\lambda(\infty))$, that is, to $\frac{1}{\lambda(\infty)}$. If $F \in$ DFR and $\lambda(\infty) = 0$, then the sequence $\left(\frac{\mu_{n+1}}{(n+1)\mu_n}\right)_n$ cannot have other limit but ∞. For, if that limit would be $\lambda \in \mathbb{R}$ (let us say) then, by (2.2), $\mu_{n,k} = \frac{\mu_{n+k}}{\binom{n+k}{k}\mu_n}$ would converge to $k!\lambda^k$, $k \in \mathbb{N}$. Hence F_n would converge to $\mathrm{Exp}(\lambda)$, contradiction. $\square$

Proof of Theorem B – direct part. Let $\lambda = \lambda_F(\infty)$. Suppose first that $0 < \lambda < \infty$. Let

$$\lambda_\star(x) = \inf \lambda((x,\infty)) \quad \text{and} \quad \lambda^\star(x) = \sup \lambda((x,\infty)), \quad x \geq 0. \tag{2.8}$$

Then $\lambda_\star \leq \lambda \leq \lambda^\star$, $\lambda_\star$ is non-decreasing, $\lambda^\star$ is non-increasing. Moreover, as $\lambda(\infty) = \lambda$ does exist, $\lambda_\star(\infty) = \lambda^\star(\infty) = \lambda$.

Let $F_\star$ be the distribution with hazard rate $\lambda_\star$ and $F^\star$ be the distribution with hazard rate $\lambda^\star$. Then $F_\star \in$IFR, $F^\star \in$DFR and $F^\star \prec_{\mathrm{HR}} F \prec_{\mathrm{HR}} F_\star$. According to the above lemma – (iii), the sequence $(T^n F^\star)_n$ is HR-increasing (since $F^\star$ is a DFR distribution) and $(T^n F_\star)_n$ is HR-decreasing (since $F_\star$ is a IFR distribution). By the same lemma – (i) we have the inequalities

$$T^n(F^\star) \prec_{\mathrm{HR}} T^n(F) \prec_{\mathrm{HR}} T^n(F_\star). \tag{2.9}$$

It follows that $\mu_1(T^n(F^\star)) \leq \mu_1(T^n(F)) \leq \mu_1(T^n(F_\star))$. According to (2.7) we obtain the inequalities

$$\mu_1(T^n(F^\star)) \leq \frac{\mu_{n+1}}{(n+1)\mu_n} \leq \mu_1(T^n(F_\star)). \tag{2.10}$$

But the sequence $(\mu_1(T^n(F^\star)))_n$ is increasing and the sequence $(\mu_1(T^n(F_\star)))_n$ is decreasing. Moreover, $T^n(F^\star) \to \mathrm{Exp}(\lambda)$ and $T^n(F^\star) \to \mathrm{Exp}(\lambda)$ as $n \to \infty$, (Theorem A) and the convergence is monotonous. According to Beppo-Levi's theorem, $\mu_1(T^n(F^\star)) \to \mu_1(\mathrm{Exp}(\lambda))$ and $\mu_1(T^n(F_\star)) \to \mu_1(\mathrm{Exp}(\lambda))$. The proof is complete since $\mu_1(\mathrm{Exp}(\lambda)) = \frac{1}{\lambda}$.

Consider now the case $\lambda = \infty$. We use the inequality $T^n(F) \prec_{\mathrm{HR}} T^n(F_\star)$ from (2.9). We know (Theorem A) that $T^n(F_\star)$ is decreasing and converges to δ_0. Then $\mu_1(T^n(F_\star))$ must converge to 0. Hence $\frac{\mu_{n+1}}{(n+1)\mu_n}$ converges to 0, too.

The last case is when $\lambda = 0$. We use the inequality $T^n(F^\star) \prec_{\mathrm{HR}} T^n(F)$. According to Theorem A, the tails $\overline{(T^n(F^\star))}(x)$ converge to 1 and the convergence is monotonous. Thus $\mu_1(T^n(F^\star)) = \int_0^\infty \overline{T^n(F^\star)}(x)\mathrm{d}x$ is increasing and, again by Beppo-Levi's theorem, its limit is $\int_0^\infty 1\mathrm{d}x = \infty$. $\square$

Example. The Poisson distribution. Let $F = \mathrm{Poisson}(\lambda)$. It is true that Theorem B cannot be applied as it is stated, since F is not absolutely continuous. But if we replace F by $TF = F_I$, we obtain an absolutely continuous distribution with $\lambda(\infty) = \infty$. It means that for the Poisson distribution with $\lambda(\infty) = \infty$ we have $\lim\limits_{n\to\infty} \frac{\mu_{n+1}}{n\mu_n} = \lim\limits_{n\to\infty} \frac{\mu_{n+1}}{(n+1)\mu_n} = \frac{1}{\infty} = 0$.

Remark. The Poisson distribution $F = \mathrm{Poisson}(\lambda)$ has no simple formula for the moments. It is known that $\mu_n(F) = P_n(\lambda)$, where $P_n(\lambda)$ are the so called Touchard Polynomials in λ (see [5]). They are obtained by the recurrence $T_{n+1}(\lambda) = \lambda \sum\limits_{k=0}^{n} \binom{n}{k} P_k(\lambda)$. Our result says that

$$\lim_{n\to\infty} \frac{T_{n+1}(\lambda)}{nT_n(\lambda)} = 0. \tag{2.11}$$

In the particular case $\lambda = 1$, the values $T_n(1)$ are famous under the name of Bell numbers. They represent the number of possible partitions of a n-point set and are denoted by B_n [10]. A particular case of (2.11) points out that $\lim\limits_{n\to\infty} \frac{B_{n+1}}{nB_n} = 0$. It is funny that we were able to check the last limit using brute force, but not for (2.11). Indeed, it is known (see [7]) that

$$\lim_{n\to\infty} \frac{B_n\sqrt{n}\mathrm{e}^{n+1-\lambda(n)}}{[\lambda(n)]^{n+0.5}} = 1,$$

where this time $\lambda(n) = \frac{n}{W(n)}$ and W is the so called *Lambert function*: $W : [0,\infty) \to [0,\infty)$ is the inverse of the function $x \mapsto x\mathrm{e}^x$. Using these facts we can prove that for the distribution Poisson(1) we have $\frac{B_{n+1}}{nB_n} = \frac{C}{\sqrt{n}}$, where C is some constant.

The real problem is whether the limit of $T^n(F)$ does always exist if λ is bounded? Is it true that the sequence $\left(\frac{\mu_{n+1}}{(n+1)\mu_n}\right)_n$ has always a limit? If the second question has an answer in the affirmative, the same would hold for the first one. But the answer is NO: there exist medium tailed distributions F for which the sequence $\left(\frac{\mu_{n+1}}{(n+1)\mu_n}\right)_n$ is bounded and divergent. Of course in this case λ_F cannot have any limit to infinity.

Proof of the second part of Theorem B. We construct a distribution $F \in \mathbf{M}_{ac}$ for which the limit $\lim\limits_{n\to\infty} \frac{\mu_n}{(n+1)\mu_n}$ does not exist. For $\rho > 0$, $c > 0$, $\mu > 0$ and $k > 0$, denote

$$A(c,\rho) = \frac{3^c}{3} - \frac{8}{9} + \rho + \frac{1}{9\ln 3} \tag{2.12}$$

$$B(c,\rho,\nu,k) = -3^c + (1+3^{-k})\left(1+\rho+\frac{1}{3^{2k}\ln 3}\right) + \nu. \tag{2.13}$$

We can choose the numbers $0 < \rho < \frac{1}{2}$, $0 < c < 1$, $0 < \nu < \frac{1}{6}$ and $k_0 \in N$, such that for all $k \geq k_0$ to have $A(c,\rho) < 0$ and $B(c,\rho,\mu,k) < 0$. Indeed, choose two numbers q_1, q_2 such that $\frac{1}{9\ln 3} < q_1 < q_2 < \frac{5}{9}$. We can fix the number $0 < c < 1$ such that $3^c < 3(\frac{8}{9} - q_2)$. Then we can find a number $\eta > 0$ such that $B(c,\rho,\nu,k) < 0$ if $\rho, \nu, 3^{-k} \in (0,\eta)$. Fix $\nu \in (0,\eta)$ such that $\nu < \frac{1}{6}$ and fix $k_0 \in N$such that $3^{-k_0} \in (0,\eta)$. Finally, we can choose $\rho \in (0,\eta)$ such that $\rho < \frac{1}{2}$ and $\rho + \frac{1}{9\ln 3} < q_1$. It follows $A(c,\rho) < q_1 - q_2 < 0$ and $B(c,\rho,\nu,k) < 0$

for $k \geq k_0$. Denote

$$p^\star =: -A(c,\rho) \quad \text{and} \quad r^\star = -B(c,\rho,\nu,k_0). \tag{2.14}$$

For $k \in N$, define

$$a_k = \begin{cases} 3^{k+\rho} & k \text{ even} \\ 3^k & k \text{ odd} \end{cases} \qquad \text{and} \quad b_k = \prod_{i=1}^{k-1} 3^{3^{i+c}}.$$

Let $M = \sum\limits_{j=1}^{\infty} \frac{1}{3^j b_j} < \infty$. The function f, given by

$$f = \sum_{j=1} \frac{1}{Mb_j} 1_{[a_j, a_j+3^{-j}]}$$

is a density function since it is positive, integrable and $\int f(t)\mathrm{d}t = 1$. The corresponding distribution F given by $F(x) = 0$ for $x \leq 0$ and $F(x) = \int_0^x f(t)\mathrm{d}t$ for $x > 0$ is clear an absolute continuous function. Denote $S_n = \frac{\mu_{n+1}}{n\mu_n}$, $n \in N$, where the moments μ_n, $n \in N$, are defined as above. We show that the sequence $(S_n)_n$ has not limit. First note that

$$\mu_{3^k} = \sum_{j=1}^{\infty} \frac{1}{Mb_j} \int_{a_j}^{a_j+3^{-j}} t^{3^k} \mathrm{d}t \geq \frac{1}{Mb_k} \int_{a_k}^{a_k+3^{-k}} t^{3^k} \mathrm{d}t$$

and on the other hand

$$\mu_{3^k} \leq \left(\frac{1}{Mb_k} \int_{a_k}^{a_k+3^{-k}} t^{3^k} \mathrm{d}t \right) \left[1 + \sum_{j \neq k} 3^{k-j} \cdot \frac{b_k}{b_j} \left(\frac{a_j + 3^{-j}}{a_k} \right)^{3^k} \right].$$

In other words,

$$\mu_{3^k} = \left(\frac{1}{Mb_k} \int_{a_k}^{a_k+3^{-k}} t^{3^k} \mathrm{d}t \right) [1 + \gamma_k],$$

where

$$|\gamma_k| \leq \sum_{j \neq k} 3^{k-j} \cdot \frac{b_k}{b_j} \left(\frac{a_j + 3^{-j}}{a_k} \right)^{3^k}.$$

Similarly,

$$\mu_{3^k+1} = \left(\frac{1}{Mb_k} \int_{a_k}^{a_k+3^{-k}} t^{3^k+1} \mathrm{d}t \right) [1 + \gamma_k^\star],$$

where

$$|\gamma_k^\star| \leq \sum_{j \neq k} 3^{k-j} \cdot \frac{b_k}{b_j} \left(\frac{a_j + 3^{-j}}{a_k} \right)^{3^k+1}.$$

We shall prove that

$$\lim_{k\to\infty} \rho_k = 0, \quad \lim_{k\to\infty} \rho_k^\star = 0. \tag{2.15}$$

For this it is sufficient to show that

$$\lim_{k\to\infty} \max\{\gamma_k, \gamma_k^\star\} = 0. \tag{2.16}$$

Let $j, k \in N$. Using the inequality $0 < \rho < \frac{1}{2}$, the inequalities

$$\frac{a_j + 3^{-j}}{a_k} < 1,\ j < k \quad \text{and} \quad \frac{a_j + 3^{-j}}{a_k} > 1,\ j > k.$$

are immediate, independently if k and j are odd or even. It follows that

$$\begin{aligned}\max\{\gamma_k, \gamma_k^\star\} \le \sum_{j=1}^{k-1} 3^{k-j} \cdot \frac{b_k}{b_j}\left(\frac{a_j + 3^{-j}}{a_k}\right)^{3^k} + \sum_{j=k+1}^{\infty} 3^{k-j} \cdot \frac{b_k}{b_j}\left(\frac{a_j + 3^{-j}}{a_k}\right)^{3^k+1} \\ := T_1^k + T_2^k\end{aligned}$$

Limit of T_1^k. Let $1 \le j < k$. Hence $k \ge 2$. First we have

$$\frac{b_k}{b_j} = \prod_{i=j}^{k-1} 3^{3^{i+c}} = 3^{\sum\limits_{i=j}^{k-1} 3^{i+c}} = 3^{\frac{1}{2}\cdot 3^c (3^k - 3^j)}.$$

Also

$$\begin{aligned}\left(\frac{a_j + 3^{-j}}{a_k}\right)^{3^k} &\le \left(\frac{3^{j+\rho} + 3^{-j}}{3^k}\right)^{3^k} = 3^{(j-k+\rho)3^k}\left(1 + 3^{-2j-\rho}\right)^{3^k} \\ &\le 3^{(j-k+\rho)3^k} \mathrm{e}^{3^{k-2j-\rho}} = 3^{(j-k+\rho)3^k + \frac{1}{\ln 3}\cdot 3^{k-2j-\rho}}.\end{aligned}$$

So that we obtain $T_1 \le \sum\limits_{j=1}^{k-1} 3^{3^k G(j,k)}$, where

$$\begin{aligned}G(j,k) &= \frac{1}{2}\cdot 3^c(1 - 3^{j-k}) + j - k + \rho + \frac{1}{3^{2j+\rho}\ln 3} + (k-j)3^{-k} \\ &\le \frac{1}{2}\cdot 3^c(1 - 3^{j-k}) + j - k + \rho + \frac{1}{9\ln 3} + \frac{k-j}{9}.\end{aligned}$$

It is immediate that the function $\Psi(s) = \frac{1}{2}\cdot 3^c(1-3^{-s}) - \frac{8}{9}\cdot s + \rho + \frac{1}{9\ln 3}$, $s \ge 1$, is decreasing. By taking into account (2.16) we obtain $G(j,k) \le \Psi(1) = -p^\star$. Then $T_1^k \le (k-1)3^{-kp^\star}$ and $\lim\limits_{k\to\infty} T_1^k = 0$.

Limit of T_2^k. Let $1 \le k < j$. We have

$$\frac{b_k}{b_j} = \left(\prod_{i=k}^{j-1} 3^{3^{i+c}}\right)^{-1} = 3^{-\sum\limits_{i=k}^{j-1} 3^{i+c}} = 3^{\frac{1}{2}\cdot 3^c(3^k - 3^j)}.$$

Also, similarly as above we obtain

$$\left(\frac{a_j+3^{-j}}{a_k}\right)^{3^k} \le 3^{(j-k+\rho)(3^k+1)+\frac{1}{\ln 3}\cdot(3^k+1)3^{-2j-\rho}}.$$

Let ν be the number fixed at the beginning. It follows

$$T_2 \le \sum_{j=k+1}^{\infty} 3^{3^k(H(j,k)-\nu(j-k)}, $$

where

$$\begin{aligned} H(j,k) &= \frac{3^c}{2}(1-3^{j-k}) + \left(1+\frac{1}{3^k}\right)\left[j-k+\rho+\frac{1}{3^{2j+\rho}\ln 3}\right] + (k-j)\left(\frac{1}{3^k}-\nu\right) \\ &\le \frac{3^c}{2}(1-3^{j-k}) + \left(1+\frac{1}{3^k}\right)\left[j-k+\rho+\frac{1}{3^{2k}\ln 3}\right] + \nu(j-k). \end{aligned}$$

For a fixed number $k \ge 1$, consider the function $\Theta(s) = \frac{1}{2}\cdot 3^c(1-3^s) + (1+3^{-k})\left[s+\rho+\frac{1}{3^{2k}\ln 3}\right] + \nu s$ $s \ge 1$. We have $\Theta'(s) = -\frac{1}{2}\cdot 3^{c+s}\ln 3 + (1+3^{-k}) + \nu \le -\frac{3}{2}\cdot \ln 3 + \frac{4}{3} + \nu < 0$. Taking into account relation (2.16) we obtain, $H(j,k) \le \Theta(1) = -r^\star$ for $k \ge k_0$. It follows, for such integers k, that

$$T_2^k \le \sum_{j=k+1}^{\infty} 3^{3^k(-r^\star-\nu(j-k))} \le \sum_{p=1}^{\infty}\left(3^{-3^k\nu}\right)^p = \frac{3^{-3^k\nu}}{1-3^{-3^k\nu}}$$

and then $\lim\limits_{k\to\infty} T_2^k = 0$. Relations (2.15) are proved.

Now, we have $S_{3^k} = I_k\frac{1+\gamma_k^\star}{1+\gamma_k}$ where

$$I_k = \left(\int_{a_k}^{a_k+3^{-k}} t^{3^k+1}\mathrm{d}t \Big/ 3^k\cdot \int_{a_k}^{a_k+3^{-k}} t^{3^k}\mathrm{d}t\right).$$

But

$$\frac{a_k}{3^k}\left(\frac{a_k}{a_k+3^{-k}}\right)^{3^k} \le I_k \le \frac{a_k+3^{-k}}{3^k}\left(\frac{a_k+3^{-k}}{a_k}\right)^{3^k}.$$

From relations (2.11) and from $\lim\limits_{k\to\infty}\left(\frac{a_k}{a_k+3^{-k}}\right)^{3^k} = 1$, we find

$$\lim_{k\to\infty} S_{3^k} = \lim_{k\to\infty}\frac{a_k}{3^k}.$$

Finally, we deduce

$$\lim_{l\to\infty} S_{3^{2l}} = 3^\rho \quad \text{while} \quad \lim_{l\to\infty} S_{3^{2l+1}} = 1.$$

Since the sequence $(S_n)_n$ has two different limit points it has no limit.

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Received 17 June 2009

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