We extend some properties of the fractal interpolation from the finite case to the case of countable set of data. The main result is that, given an countable system of data $\Delta$ in $[a, b] \times Y$, where $[a, b]$ is a real interval and $Y$ a compact metric space and $\psi$ a proper continuous function which transforms $\Delta$ in some other countable system of data, then $\psi(A)$ is the attractor corresponding to $\psi(\Delta)$, where $A$ is the attractor associated with $\Delta$. The particular case $\Delta \subset \mathbb{R}^2$ space is described.

AMS 2010 Subject Classification: Primary 28A80, Secondary 65D15.

Key words: countable iterated function system, interpolatory scheme, attractor, countable system of data, fractal interpolation function

1. INTRODUCTION

In his famous paper [4], J.E. Hutchinson proves that, given a set of contractions (IFS) $(\omega_n)_{n=1}^{N}$ in a complete metric space $X$, there exists a unique nonempty compact set $A \subset X$, named the attractor of IFS which is its fixed point. This attractor is, generally, a fractal set. These ideas have been extended to infinitely many contractions, such a generalization can be found in [5] for Countable Iterated Function Systems (CIFS) on a compact metric space.

Barnsley ([1], [2]) shows that, for a given finite system of data $\Delta_N = (x_n, y_n)_{n=0}^{N}$, there exists a unique fractal interpolation function $f$, i.e., a continuous function such that $f(x_n) = y_n$ for all $0 \leq n \leq N$ and its graph is the attractor of a proper IFS.

In [7] the problem of fractal interpolation has been extended to the case of a countable system of data $\Delta = (x_n, y_n)_{n \geq 0}$ in the compact metric space $X = [a, b] \times Y$, $Y$ being a compact metric space. One constructs a CIFS on $X$ whose attractor is the graph of a proper countable fractal interpolation function $f$ (namely, a continuous function $f$ with $f(x_n) = y_n$ for every $n = 0, 1, \ldots$).
2. PRELIMINARIES

2.1. Iterated function systems and countable iterated function systems

In this subsection we recall some well known aspects on Fractal Theory used in the sequel (more complete treatments may be found in [4], [2], [5], [7]).

Let \((X, d)\) be a complete metric space and \(\mathcal{K}(X)\) the class of all compact non-empty subsets of \(X\).

The function \(h : \mathcal{K}(X) \times \mathcal{K}(X) \to \mathbb{R}_+\), \(h(A, B) = \max\{d(A, B), d(B, A)\}\), where \(d(A, B) = \sup_{x \in A} \left\{ \inf_{y \in B} d(x, y) \right\}\), for all \(A, B \in \mathcal{K}(X)\), is a metric, called the Hausdorff metric. The set \(\mathcal{K}(X)\) is a complete metric space with respect to this metric \(h\).

Suppose that \((X, d)\) is a complete metric space. A set of contractions \(\omega_n : X \to X, 1 \leq n \leq N\), is called an iterated function system (IFS). Such a system of maps induces a set function \(S_N : \mathcal{K}(X) \to \mathcal{K}(X)\), \(S_N(E) = \bigcup_{n=1}^N \omega_n(E)\), which is a contraction on \(\mathcal{K}(X)\) with contraction ratio \(r \leq \max_{1 \leq n \leq N} r_n\), \(r_n\) being the contraction ratio of \(\omega_n\), \(n = 1, \ldots, N\). According to the Banach contraction principle, there exists a unique set \(A_N \in \mathcal{K}(X)\) which is invariant with respect to \(S_N\), that is, \(A_N = S_N(A_N) = \bigcup_{n=1}^N \omega_n(A_N)\). We say that the set \(A_N \in \mathcal{K}(X)\) is the attractor of the IFS \((\omega_n)_{n=1}^N\).

Now, we suppose further that \((X, d)\) is a compact metric space.

A sequence of contractions \((\omega_n)_{n \geq 1}\) on \(X\) whose contraction ratios are, respectively, \(r_n > 0\), such that \(\sup_{n} r_n < 1\) is called a countable iterated function system, for short CIFS. For each \(N \geq 1\), \((\omega_n)_{n=1}^N\) will be a partial IFS of the considered CIFS and \(S_N\) defined as below is the associated set function.

If we consider the CIFS \((\omega_n)_{n \geq 1}\), then the set function \(S : \mathcal{K}(X) \to \mathcal{K}(X)\), given by

\[
S(E) = \bigcup_{n \geq 1} \overline{\omega_n(E)}
\]

(the bar means the closure of the respective set), is a contraction map on \((\mathcal{K}(X), h)\) with contraction ratio \(r \leq \sup_{n} r_n\). Thus, there exists a unique non-empty compact set \(A \subset X\) invariant for the family \((\omega_n)_{n \geq 1}\), that is, \(A = S(A) = \bigcup_{n \geq 1} \omega_n(A)\). The set \(A\) is called the attractor of the CIFS \((\omega_n)_{n \geq 1}\) and it can be obtained as limit in the Hausdorff metric of the sequence of attractors \((A_N)_{N \geq 1}\) of the partial IFS \((\omega_n)_{n=1}^N, N = 1, 2, \ldots\) (see [5]).
Theorem 1. [8] Assume that \((B_k)_k\) is a convergent sequence of nonempty compact subsets of \(X\). Then \(A\) can be approximated by the sequence of compact nonempty sets \(\bigcap_{k=1}^\infty S_k(B_k)\). More precisely, we have \(\lim_{p \to \infty} \lim_{k \to \infty} S_p^k(B_k) = A\), the limiting processes being taken in the Hausdorff metric, and \(S_p^k := S_k \circ \cdots \circ S_k\), \(p\) times.

2.2. Countable fractal interpolation

A. Now, we describe an extension of the fractal interpolation to the case of the countable system of data (more details can be found in [7]).

Let \((Y,d_Y)\) be a compact and arcwise connected metric space. A countable system of data (abbreviated CSD) is a set of points of the form

\[
\Delta = \{(x_n,y_n) : n \geq 0, n = 0,1,\ldots\},
\]

where the sequence \((x_n)_{n\geq0}\) is strictly increasing and bounded and \((y_n)_{n\geq0}\) is convergent.

In the sequel \(\Delta = (x_n,y_n)_{n\geq0}\) will be a CSD and we put \(a = x_0, b = \lim_n x_n, M = \lim_n y_n\) and \(X = [a,b] \times Y\). Clearly, \((b,M) \in X\).

An interpolation function corresponding to this CSD is a continuous map \(f : [a,b] \to Y\) such that \(f(x_n) = y_n\) for \(n = 0,1,\ldots\). The points \((x_n,y_n) \in X, n \geq 0\), are called the interpolation points. Note that \(f(b) = M\), since \(f\) is continuous. Define a metric \(\delta\) on \(X\) by

\[
\delta((x,y),(x',y')) = |x-x'| + \theta d_Y(y,y'),
\]

for all \((x,y),(x',y') \in X\), where \(\theta\) is a positive real number to be specified below. The metric space \((X,\delta)\) is compact.

Let \(c\) and \(s\) be real numbers with \(0 \leq s < 1\) and \(c > 0\). For each \(n = 1,2,\ldots\), let \(\varphi_n : X \to Y\) be a function which satisfies the inequalities

\[
d_Y(\varphi_n(\alpha,y),\varphi_n(\beta,y)) \leq c |\alpha - \beta|, \quad \forall \alpha, \beta \in [a,b], \forall y \in Y,
\]

and

\[
d_Y(\varphi_n(\alpha,y),\varphi_n(\alpha,z)) \leq s d_Y(y,z), \quad \forall \alpha \in [a,b], \forall y, z \in Y.
\]

Define a transformation \(\omega_n : X \to X\) by

\[
\omega_n(x,y) = (l_n(x),\varphi_n(x,y)) \quad \text{for all} \quad (x,y) \in X, \quad n = 1,2,\ldots,
\]

where \(l_n : [a,b] \to [x_{n-1},x_n]\) is the invertible transformation \(l_n(x) = a_n x + e_n\), with

\[
a_n = \frac{x_n - x_{n-1}}{b-a} \quad \text{and} \quad e_n = \frac{bx_{n-1} - ax_n}{b-a}.
\]
Notice that

\[ a_n > 0 \text{ for all } n \geq 1 \quad \text{and} \quad \sup_{n \geq 1} a_n < 1. \]

**Theorem 2.** [7] Let the family \((\omega_n)_{n \geq 1}\) be defined as above. Assume that there exist real constants \(c\) and \(s\) such that \(c > 0, 0 \leq s < 1,\) and conditions (3) and (4) are fulfilled. Let \(\theta\) be the constant in the definition of the metric \(\delta\) in equation (2) given by \(\theta = \inf (1 - a_n)/2c.\) Then \((\omega_n)_{n \geq 1}\) is a CIFS in the metric \(\delta.\) Consequently, there exists a unique nonempty compact set \(A \subset X\) such that\( A = \bigcup_{n \geq 1} \omega_n(A).\)

We refer to the CIFS \((\omega_n)_{n}\) from the preceding theorem associated to \(\Delta\) and \(\Phi := (\varphi_n)_{n}\) as a *countable interpolatory scheme* \(\sigma(\Delta, \Phi).\)

We now constrain the CIFS \((\omega_n)_{n \geq 1}\) on \(X\) defined above to ensure that the attractor includes the CSD. Namely, we assume that

\[ \varphi_n(x_0, y_0) = y_{n-1} \quad \text{and} \quad \varphi_n(b, M) = y_n \quad \text{for } n = 1, 2, \ldots. \]

Then it follows that

\[ \omega_n(x_0, y_0) = (x_{n-1}, y_{n-1}) \quad \text{and} \quad \omega_n(b, M) = (x_n, y_n) \quad \text{for } n = 1, 2, \ldots. \]

**Theorem 3.** [7] Let \(\sigma(\Delta, \Phi)\) be a countable interpolatory scheme. Assume also that there exist real constants \(c\) and \(s\) such that \(0 \leq s < 1,\) \(c > 0,\) and conditions (3), (4) and (7) are fulfilled. Then there exists an interpolation function \(f\) corresponding to the CSD such that the graph \(A\) of \(f\) is the attractor of the CIFS \((\omega_n)_{n \geq 1}\). That is, \(A = \{(x, f(x)) \mid x \in [a, b]\}.\)

**B.** We consider now the particular case when \(\Delta = \{(x_n, y_n) \in \mathbb{R}^2 : n = 0, 1, 2, \ldots\},\) \(a = x_0 < x_1 < \cdots,\) \(b = \lim x_n,\) \((y_n)_{n \geq 0}\) is a convergent sequence, \(M = \lim y_n,\) \(Y\) is a compact interval and \(X := [a, b] \times Y.\) We will obtain a generalization for countable sets of data of Barnsley’s construction [2].

The following lemma proves that there exists an interpolation function for such CSD.

**Lemma 1.** There exists an *infinite polygonal line* which is an interpolation function corresponding to the CSD considered above.

**Proof.** For each \(n = 1, 2, \ldots,\) let be the affine function \(f_n : [a, b] \to Y\) given by

\[ f_n(x) = y_{n-1} + \frac{y_n - y_{n-1}}{x_n - x_{n-1}} (x - x_{n-1}) \]

and \(f : [a, b] \to Y\) given by

\[ f(x) = \begin{cases} f_n(x) & \text{for } x \in [x_{n-1}, x_n], \ n = 1, 2, \ldots, \\ M & \text{for } x = b. \end{cases} \]
The function \( f \) is obviously continuous on \([a, b)\). To prove that \( f \) is continuous at \( b \), let be \( \varepsilon > 0 \). Then there exist \( n_1^2 \geq 1 \) and \( n_2^2 \geq 1 \) such that

\[
n \geq n_1^2 \Rightarrow |M - y_n| < \frac{\varepsilon}{2}
\]

and, respectively,

\[
n, m \geq n_2^2 \Rightarrow |y_m - y_n| < \frac{\varepsilon}{2}.
\]

Let \( N_\varepsilon = \max\{n_1^2, n_2^2\} \). For any \( x \in (x_{N_\varepsilon}, b) \), there exists a unique \( n > N_\varepsilon \) such that \( x_{n-1} \leq x \leq x_n \), hence \( f(x) = f_n(x) \) and therefore

\[
|M - f(x)| = \left| M - y_{n-1} - \frac{y_n - y_{n-1}}{x_n - x_{n-1}} \cdot (x - x_{n-1}) \right| \leq
\]

\[
\leq |M - y_{n-1}| + |y_n - y_{n-1}| \cdot \frac{x - x_{n-1}}{x_n - x_{n-1}} \leq
\]

\[
\leq |M - y_{n-1}| + |y_n - y_{n-1}| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Thus \( f \) is continuous and, clearly, \( f(x_n) = y_n \) for all \( n = 0, 1, \ldots \), therefore \( f \) is an interpolation function for the CSD \((x_n, y_n)_{n \geq 0}\). □

The maps \( \varphi_n \), \( n \geq 1 \), in the construction of \((\omega_n)_{n \geq 1}\) can be the affine transformations

\[
\varphi_n(x, y) = c_n x + d_n y + g_n, \quad c_n, d_n, g_n \in \mathbb{R}, \quad n = 1, 2, \ldots.
\]

If \( d_n \) is any real number, we obtain, using the above notations \( \omega_n : X \to \mathbb{R}^2 \),

\[
\omega_n(x, y) = (a_n x + e_n, c_n x + d_n y + g_n) \quad \text{for any } n = 1, 2, \ldots,
\]

where the constants used will be

\[
an = \frac{x_n - x_{n-1}}{b - a}, \quad e_n = \frac{bx_n - ax_n}{b - a},
\]

\[
c_n = \frac{y_n - y_{n-1}}{b - a} - \frac{d_n (M - y_0)}{b - a}, \quad g_n = \frac{by_n - ay_n}{b - a} - \frac{d_n by_0 - aM}{b - a}.
\]

We deduce immediately

\[
e_n = x_n - a_n b, \quad g_n = y_n - c_n b - d_n M, \quad \forall n = 1, 2, \ldots.
\]

Standard calculus proves the existence, for any \( n \geq 1 \), of the vertical scaling factor \( d_n \) satisfying the conditions \( d_n \geq 0 \), \( \sup_{n \geq 1} d_n < 1 \) and \( \omega_n(X) \subset X \)

for all \( n \). The metric \( \delta \) on \( \mathbb{R}^2 \) from (2) is equivalent to the Euclidean metric and \((\omega_n)_{n \geq 1}\) is a CIFS on the compact metric space \((X, \delta)\) which is associated
to the considered CSD. We refer to the CIFS \((\omega_n)_n\) associated to CSD \(\Delta\) with \(d = (d_n)_{n \geq 1}\) as a countable interpolatory scheme \(\sigma(\Delta, d)\).

From Theorem 3 we obviously obtain

**Theorem 4.** Under the above conditions, there exists an interpolation function corresponding to the countable set of data such that its graph is the attractor of the associated CIFS \((\omega_n)_{n \geq 1}\).

The interpolation function whose graph is the attractor of CIFS described above is called the countable fractal interpolation function.

### 3. AFFINE INVARIANCE OF COUNTABLE INTERPOLATORY SCHEME

We now extend some aspects concerning the affine invariance of an interpolatory scheme described by Kocić and Simoncelli [3] from the finite case to the case of countable data. The particular case when the countable system of data is contained in the \(\mathbb{R}^2\) Euclidean space is treated. Also we will give some results concerning the approximation in the Hausdorff metric of the attractor corresponding to a transformed countable system of data under a feasible mapping.

Let \(\Delta = (x_n, y_n)_{n \geq 0}\) be a CSD and \(X = [a, b] \times Y\) defined in Section 2.2 and \(\psi : X \to X\) be a continuous function.

**Definition 1.** We say that the mapping \(\psi\) on \(X\) is outstanding for a given CSD \(\Delta = (x_n, y_n)_{n \geq 0}\) on \(X\) if \(\psi(\Delta) = (x'_n, y'_n)_{n \geq 0}\) is also a CSD.

Let \(\sigma(\Delta, \Phi)\) be a countable interpolatory scheme and \(\psi\) an affine outstanding mapping for it. There exists a new CIFS \((\omega'_n)_{n \geq 1}\) corresponding to the transformed CSD \(\psi(\Delta)\) on the compact space \(\psi(X) \subseteq X\).

**Lemma 2.** Let \(X, Y\) be two metric spaces and \(f : X \to Y\) a continuous function. Then

\[ f(\overline{A}) = \overline{f(A)}, \]

for any relatively compact set \(A \subseteq X\).

**Proof.** Because \(f\) is continuous, we have \(f(\overline{A}) \subseteq \overline{f(A)}\). Conversely,

\[ y \in \overline{f(A)} \Rightarrow \exists (y_n)_n \subset f(A),\ y_n \to y \Rightarrow \exists (x_n)_n \subset A,\ f(x_n) = y_n \to y. \]

Next, \(\overline{A}\) being compact, by Weierstrass-Bolzano Theorem there exists a convergent subsequence \((x_{n_k})_k\) of \((x_n)_n\). Hence \(x_{n_k} \to x \in \overline{A}\).

Thus \(f(x_{n_k}) \to f(x) = y\), consequently \(y \in f(\overline{A})\). \(\Box\)

**Remark.** If in the above lemma the metric space \(X\) is compact, then the equality from statement holds for any subset \(A\) of \(X\).
LEMMA 3. Assume that
\[ \omega'_n \circ \psi = \psi \circ \omega_n, \quad \forall n \geq 1. \]
Then \( \psi(A) \) is the attractor of the associated CIFS of CSD \( \psi(\Delta) \) (\( A \) being the attractor of \( (\omega_n)_n \)).

Proof. We must show that \( \psi(A) = S'(\psi(A)) = \bigcup_{n=1}^\infty \omega'_n(\psi(A)) \). Since \( A = S(A) \), it remain to prove the equality \( \psi(S(A)) = S'(\psi(A)) \). By Lemma 2 and the hypothesis, we have
\[
\psi(S(A)) = \psi\left( \bigcup_{n=1}^\infty \omega_n(A) \right) = \psi\left( \bigcup_{n=1}^\infty \omega_n(A) \right) = \bigcup_{n=1}^\infty (\psi \circ \omega_n)(A) = \bigcup_{n=1}^\infty (\omega'_n \circ \psi)(A) = S'(\psi(A)). \]

We consider now the particular case described in Section 2.2 A (using the notations from there). Let a nonsingular affine mapping \( \psi : \mathbb{R}^2 \to \mathbb{R}^2 \) given by
\[ \psi(x, y) = (px + qy + \alpha, rx + sy + \beta), \quad p, q, r, s, \alpha, \beta \in \mathbb{R}, \quad ps \neq qr. \]
Recall that a mapping is affine if and only if it is a composition of a regular linear transformation \( (\alpha = \beta = 0) \) and a translation \( (q = r = 0, p = s = 1, |\alpha| + |\beta| > 0) \).

Let us consider a countable interpolatory scheme \( \sigma(\Delta, \mathbf{d}) \) and suppose that \( \psi \) is an outstanding mapping for it. The CIFS corresponding to the new interpolatory scheme \( \sigma'(\Delta' = \psi(\Delta), \mathbf{d}) \) on the space \( \psi(X) \subset \mathbb{R}^2 \) will be
\[ \omega'_n(x, y) = (a'_n x + e'_n, c'_n x + d_n y + g'_n), \quad n = 1, 2, \ldots, \]
with
\[
a'_n = \frac{p(x_n - x_{n-1}) + q(y_n - y_{n-1})}{p(b - a) + q(M - y_0)}, \quad c'_n = \frac{r(x_n - x_{n-1}) + s(y_n - y_{n-1})}{p(b - a) + q(M - y_0)} - d_n \frac{r(b - a) + s(b - a)}{p(b - a) + q(M - y_0)}, \quad e'_n = \frac{pb + qM + \alpha)(px_{n-1} + qy_{n-1} + \alpha) - (pa + qy_0 + \alpha)(px_n + qy_n + \alpha)}{p(b - a) + q(M - y_0)}, \]
\[ g'_n = r x_n + s y_n + \beta - c'_n(px_n + qy_n + \beta) - d_n(r x_n + s y_n + \beta). \]
We get obviously
\[ e'_n = px_n + qy_n + \alpha - a'_n(pb + qM + \alpha), \quad \forall n = 1, 2, \ldots. \]
Notice that \((\omega'_n)_n\) is indeed a CIFS because the denominators in (16) cannot vanish because of the regularity properties of \(\psi\).

The next theorem establishes the relationship between the attractors of the two CIFSs \((\omega_n)_n\) and \((\omega'_n)_n\).

**Lemma 4.** Under the hypothesis of the present section the condition (13) is fulfilled if and only if \(q = 0\).

**Proof.** I: The regular linear case \((\alpha = \beta = 0)\)

By using (10), (14), (12), one has

\[
(\psi \circ \omega_n)(x, y) = (p_n x + q_n y + r_n x + s_n y + t_n) = \left( \begin{array}{c} p_n x + q_n y + r_n x + s_n y + t_n \\ p_n x + q_n y + r_n x + s_n y + t_n \\ \vdots \\ p_n x + q_n y + r_n x + s_n y + t_n \end{array} \right) = \left( \begin{array}{c} \alpha_n x + \beta_n y + \gamma_n x + \delta_n y + \epsilon_n \end{array} \right)
\]

\[
\text{for every } (x, y) \in X \text{ and } n \geq 1. \text{ Next, by the formulas (14), (15), (17), we have}
\]

\[
\omega_n'(\psi(x, y)) = (a'_n x + b'_n y + c'_n x + d'_n y + e'_n) = (pa'_n x + qa'_n y + ra'_n x + sa'_n y + tc'_n)
\]

\[
\text{for all } (x, y) \in X \text{ and } n = 1, 2, \ldots. \text{ It follows from the last two relations that}
\]

\[
(\psi \circ \omega_n)(x, y) - (\omega'_n \circ \psi)(x, y) = (0, 0), \quad \forall (x, y) \in X,
\]

if and only if

\[
(p(a_n - a'_n) + q c_n) (b - x) + q (d_n - a'_n) (M - y) = 0
\]

and

\[
(r(a_n - d_n) + s c_n - p c'_n) (b - x) - q c'_n (M - y) = 0
\]

for any \((x, y) \in X\) and any \(n = 1, 2, \ldots\). The last two equalities hold if and only if

\[
(p(a_n - a'_n) + q c_n) = 0, \quad (d_n - a'_n) = 0, \quad \forall n
\]

and

\[
(r(a_n - d_n) + s c_n - p c'_n) = 0, \quad (c'_n) = 0, \quad \forall n.
\]

Since, in (16), \(q = 0\) implies \(a'_n = a_n\) and \(c'_n = \frac{r(a_n - d_n) + s c_n}{p}\), and, consequently, the equalities (18) and (19) are valid.

We need to show that \(q\) cannot be null. By the second equations in (18) and (19), \(c'_n = 0, a'_n = d_n\) and hence the first equations by (18) and (19) will be

\[
p(a_n - d_n) + q c_n = 0, \quad r(a_n - d_n) + s c_n = 0, \quad \forall n.
\]

The above linear equations system has nontrivial solution iff \(ps - rq = 0\) which, by assumption in (14), never occurs. So, one must have \(q = 0\).
II: The translation case \((q = r = 0, p = s = 1, |\alpha| + |\beta| > 0)\)

We have \(\psi(x, y) = (x + \alpha, y + \beta), |\alpha| + |\beta| > 0, \alpha'_n = a_n, c'_n = c_n\) hence, by (17) and (12),
\[
e'_n = e_n + (1 - a_n)\alpha, \quad g'_n = g_n + (1 - d_n)\beta - c_n\alpha,
\]
for any \(n = 1, 2, \ldots\).

In view of the preceding facts, it is simple to verify that, for each \(n = 1, 2, \ldots\), and any \((x, y) \in X\), the equality \((\psi \circ \omega_n)(x, y) = (\omega'_n \circ \psi)(x, y)\) holds, thus completing the proof. \(\square\)

**THEOREM 5.** If \(q = 0\) and \(A\) is the attractor of the associated CIFS of CSD \(\Delta\), then \(\psi(A)\) represents the attractor of the associated CIFS of CSD \(\psi(\Delta)\).

**Proof.** The assertion follows from Lemmas 3 and 4. \(\square\)

**Remark.** We notice that, if \(\psi\) is nonsingular, then \(q = 0\) implies \(p \neq 0\), assuring that \(\psi\) is outstanding.

In \(K(\mathbb{R}^2)\), define the **scalar multiplication** by
\[
\lambda M = \{(x, \lambda y); (x, y) \in M\}, \quad M \in K(\mathbb{R}^2), \lambda \in \mathbb{R}.
\]

Taking \(p = 1\), \(q = r = \alpha = \beta = 0\) and \(s \neq 0\) in (14), we obtain an outstanding affine transformation for a given CSD \(\Delta = (x_n, y_n)_{n \geq 0}\).

Applying Theorem 5 we obtain

**COROLLARY 1.** (Homogeneity) Let \(\Delta\) be a CSD and \(\lambda \in \mathbb{R}\). Assume that the corresponding countable interpolatory schemes of \(\Delta\) and \(\lambda\Delta\) have the same vertical scaling factor \(d\). Then \(\lambda A\) is the attractor corresponding to the CSD \(\lambda\Delta\) (\(A\) being the attractor of the associated CIFS of \(\Delta\)). In other words, the countable interpolating scheme given by \(\sigma(\Delta, d)\) is homogeneous.

In the sequel we give some methods for the approximation of the attractor of \(\psi(\Delta)\), \(\psi\) being defined in (14) with \(q = 0\). First, we need

**LEMMA 5.** Let \((X, d), (Y, \delta)\) be two metric spaces and \(f : X \to Y\) a uniformly continuous map. If \((A_n)_{n}, A \in K(X)\) are such that \(A_n \to A\), then \(f(A_n) \to f(A)\), the limits being taken in the Hausdorff metrics.

**Proof.** Assume by reductio ad absurdum that \(A_n \to A\) while \((f(A_n))_{n}\) does not converge to \(f(A)\). Then there exists \(\varepsilon_0 > 0\) such that
\[
\forall n \in \mathbb{N}, \exists k_n \geq n \text{ such that } h(f(A_{k_n}), f(A)) \geq \varepsilon_0
\]
(we will denote by \(h\) the Hausdorff metric in both spaces \(X\) and \(Y\)). That is, for each \(n = 1, 2, \ldots\), one has
\[
\sup_{y' \in f(A)} \left( \inf_{y \in f(A_{k_n})} \delta(y', y) \right) \geq \varepsilon_0 \quad \text{or} \quad \sup_{y \in f(A_{k_n})} \left( \inf_{y' \in f(A)} \delta(y, y') \right) \geq \varepsilon_0.
\]
Case I: By considering, if necessary, a subsequence, we can suppose that
\[
\sup_{y' \in f(A)} \left( \inf_{y \in f(A_{n_k})} \delta(y', y) \right) \geq \varepsilon_0 \text{ for any } n \geq 1, \text{ hence, for each } n = 1, 2, \ldots,
\]
there exists \( y_n \in f(A_{n_k}) \) such that for any \( y' \in f(A) \) we have \( \delta(y_n, y') \geq \varepsilon_0 \).
So, there exists \( x_n \in A_n \) such that
\[
\delta(f(x_n), f(x')) = \delta(y_n, y') \geq \varepsilon_0,
\]
for every \( x' \in A \). Now, let \( \varepsilon > 0, \varepsilon < \varepsilon_0 \). By uniform continuity of \( f \), there exists \( \eta > 0 \) such that
\[
\forall x, x' \in X \text{ with } d(x, x') < \eta \Rightarrow \delta(f(x), f(x')) < \varepsilon.
\]
Next, \( A_n \to A \) implies \( A_{n_k} \to A \), hence there exists \( n_\eta \geq 1 \) such that
\[
h(A_{n_k}, A) < \eta \text{ for all } n \geq n_\eta.
\]
It follows that
\[
\sup_{x \in A_{n_k}} \left( \inf_{x' \in A} d(x, x') \right) < \eta \text{ and therefore, for any } n \geq n_\eta \text{ and any } x \in A_{n_k}, \text{ there exists } x' \in A \text{ with } d(x, x') < \eta.
\]
Consequently, \( d(x_n, x'_n) < \eta \) and hence \( \delta(f(x_n), f(x'_n)) < \varepsilon < \varepsilon_0 \), contradicting (20).

Case II: We proceed in a similar way as in the preceding case. Suppose that
\[
\sup_{y \in f(A_{n_k})} \left( \inf_{y' \in f(A)} \delta(y, y') \right) \geq \varepsilon_0 \text{ for all } n \geq 1.
\]
Then, for every \( n = 1, 2, \ldots \), there exists \( x'_n \in A \) such that, for any \( x \in A_{n_k} \), one has
\[
\delta(f(x'_n), f(x)) \geq \varepsilon_0.
\]
At the same time, for an arbitrary \( \varepsilon > 0, \varepsilon < \varepsilon_0 \), there exists \( \eta > 0 \) such that
\[
\forall x', x \in X \text{ with } d(x', x) < \eta \Rightarrow \delta(f(x'), f(x)) < \varepsilon.
\]
Next,
\[
A_n \to A \Rightarrow A_{n_k} \to A \Rightarrow \exists n_\eta \in \mathbb{N} \text{ such that}
\]
\[
\sup_{x' \in A} \left( \inf_{x \in A_{n_k}} d(x', x) \right) < \eta, \forall n \geq n_\eta,
\]
namely, for any \( x' \in A \), there exists \( x_{n_k} \in A_{n_k} \), such that \( d(x', x_{n_k}) < \eta \).
In particular, taking \( x' = x'_0 \in A \) and \( x = x_{k_n} \in A_{n_k} \), we have, by (22), \( d(x', x) < \eta \) and hence \( \delta(f(x'_0), f(x)) < \varepsilon \), which contradicts relation (21).

Remark. If \( f \) is an uniformly continuous map between two metric spaces \( X \) and \( Y \), then the corresponding set function \( \mathcal{F} : \mathcal{K}(X) \to \mathcal{K}(Y) \), \( \mathcal{F}(A) = f(A) \) for any \( A \in \mathcal{K}(X) \), is continuous in the Hausdorff metrics. In particular, if \( X \) is compact, it is sufficient for \( f \) to be continuous.

For the rest of paper we suppose that \( \psi \) is an outstanding mapping with respect to the CSD \( \Delta \). The associated CIFS of \( \Delta \) will be \( (\omega_n)_{n \geq 1} \) whose attractor is denoted by \( A \) and \( (\omega'_n)_{n \geq 1} \) will be the CIFS corresponding to the
CSD $\psi(\Delta)$. Assume that $\omega'_n \circ \psi = \psi \circ \omega_n$ for any $n = 1, 2, \ldots$. In the case of $\Delta \subset \mathbb{R}^2$, we consider $q = 0$.

In the sequel some results concerning the approximation of the attractor of the CIFS associated to the transformed CSD $\psi(\Delta)$ will be formulated.

**Proposition 1.** The sequence of sets $(\psi(\Delta_p))_p$ converges in the Hausdorff metric to the attractor of $\psi(\Delta)$, where $\Delta_1 := S(\Delta), \Delta_p := S(\Delta_{p-1})$, for $p \geq 2$.

*Proof.* For any $B \in \mathcal{K}(X)$, $S^p(B) = S \circ \cdots \circ S(B)$ converges (in the Hausdorff metric) to the fixed point $A$ of $S$, namely the attractor of the CIFS associated to CSD $\Delta$. Next, by the uniform continuity of $\psi$ it follows from Lemma 5 that $\psi(S^p) \to \psi(A)$. The assertion comes now by applying Lemma 3. $\square$

**Proposition 2.** For each $N \geq 1$, let $A_N$ be the attractor of the partial IFS $(\omega_n)^N_{n=1}$ of CIFS associated to the countable interpolatory scheme $\sigma(\Delta, \Phi)$. Then $(\psi(A_N))_N$ converges in the Hausdorff metric to the attractor corresponding to the transformed by $\psi$ of the considered countable interpolatory scheme.

*Proof.* From Section 2.1 we deduce that $A_N \to A$, $A$ being the attractor of $(\omega_n)_{n \geq 1}$. Then, using Lemma 5, one has $\psi(A_N) \to \psi(A)$. The assertion comes clearly from Lemma 3. $\square$

As a consequence of Theorems 5 and Proposition 1, we get obviously

**Proposition 3.** Let $\Delta$ be a CSD on $\mathbb{R}^2$ and $\psi$ an outstanding affine transformation on $X$ given by (14) with $q = 0$. If $A$ is the attractor of the associated CIFS of $\sigma(\Delta, d)$, then $(\psi(B_p))_p$ converges to the attractor of the associated CIFS of $\sigma(\psi(\Delta), d)$, where, for $p \geq 1$, $B_p := S(B_{p-1})$, $B_0$ being the compact set $\Delta$ or $B_0 := L_0$, the infinite polygonal line defined in (9).

The next proposition is a simple consequence of Theorem 1 and Lemma 3 (respectively 5). It gives a sequence of finite nonempty sets which approximate “from inside” the attractor of the transformed CSD. This is very useful for computer graphical representation.

We will use the notation $\Delta_N := \{(x_n, y_n); \ n = 0, 1, \ldots, N\}, N \geq 1$, for the partial CSD. For each $N = 1, 2, \ldots$, we define $S'_N(E) := \bigcup_{n=1}^N \omega'_n(E)$, $S'^p_N(E) := S'_N(S'^{p-1}_N(E))$.

**Proposition 4.** The sequence of sets $(S'^p_N(\psi(\Delta_N)))_{N,p}$ converges with respect to the Hausdorff metric to the attractor $\psi(A)$ corresponding to CSD $\psi(\Delta)$. 

We shall conclude with an example of application of Theorem 5.

**Example.** Take \([a, b] = [0, 1] \) and \(\Delta = (x_n, y_n)_{n\geq 0} \subset \mathbb{R}^2\), where
\[
x_n := \frac{\sqrt{n}}{\sqrt{n} + 1}, \quad y_n := \frac{\sin \pi (\sqrt{n} + 1)}{\sqrt{n} + 1}, \quad n = 0, 1, \ldots.
\]
We apply Theorem 5 with the vertical scaling factor \(d_n = 0.4\) for any \(n \geq 1\) and the affine mapping
\[
\Psi(x, y) = (x + 1, 0.1x - 8.5y - 1.8).
\]
The attractor \(A\) corresponding to the countable interpolatory scheme \(\sigma = (\Delta, d = (d_n)_n)\) is represented in Figure 1 and the attractor \(\psi(A)\) associated with \(\sigma' = (\psi(\Delta), d = (d_n)_n)\) appears in Figure 2.

![Fig. 1](image1)

![Fig. 2](image2)

**REFERENCES**


Received 3 June 2010

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