NUMERICAL SOLUTION TO DIFFERENTIAL EQUATIONS VIA HYBRID OF BLOCK-PULSE AND RATIONALIZED HAAR FUNCTIONS

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In this article, we approximate the solution to differential equations using hybrid of Block-pulse and rationalized Haar functions. The operational matrices of integration and product are given to reduce the computation of differential equations to a system of algebraic equations. By testing the method on several examples we show that our estimation has a good degree of accuracy.

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1. INTRODUCTION

Many different bases functions have been used to estimate the solution to differential equations, such as orthogonal bases [3, 4, 14, 15], wavelets [7–8] and hybrid [2, 13, 16–17]. The various systems of orthogonal functions may be classified into two categories. The first is piecewise continuous function (PCBF) to which the orthogonal systems of Walsh functions [5], Block-pulse functions [4, 10] and Haar functions [1, 11, 12, 19] belong. The second group consists in continuous orthogonal functions such as orthogonal polynomials and sin-cos basis [3, 9]. We notice that, by approximating a discontinuous function by a continuous basis we can not properly model the discontinuities and therefore we must approximate such a function by PCBFs.

One of the main characteristics of the orthogonal function techniques for solving different problems is to reduce these problems to those of solving a system of algebraic equations. These techniques have been presented, among others by Hwang and Shih [9] and Lepik [11].

Refs. [14, 16–17] introduced the hybrid of Block-pulse and orthogonal polynomials to approximate different problems. Refs. [2, 13] used the hybrid of Block-pulse and rationalized Haar functions to approximate time-varying differential equations and nonlinear Volterra-Hammerstein equations. The aim of this paper is to reduce the variables and get higher accuracy for approximation.

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by PCBFs. Therefore, we use the idea of hybrid Block-pulse and rationalized Haar functions to approximate differential equations. By using this approach, the number of variables is of the form $M \times 2^k$, where $M$, $k$ are positive integer numbers. Also the accuracy will be improved [13–17].

Also, we use this method, to solve the population balance differential equation. Several numerical schemes developed to solve the population balance differential equation, such as Block-pulse method [10], weighted residual method [20], the method of orthogonal series expansion and wavelet Galerkin method [6] and rationalized Haar functions [1].

The organization of this paper is as follows: in Section 2, the rationalized Haar functions are presented. The operational matrices of integral and operational product matrix are also given in this section. In Section 3, the hybrid of Block-pulse and rationalized Haar function and the operational matrices of integration and product are given. We show that every function in $L^2[0,1]$ can be approximated by a hybrid Block-pulse and rationalized Haar functions. Also, in this section we state how to approximate differential equations and population balance differential equation by hybrid of Block-pulse and rationalized Haar functions. In Section 4, three examples are solved with this new method and the obtained results are compared with other methods.

2. PROPERTIES OF RATIONALIZED HAAR FUNCTIONS

2.1. Rationalized Haar functions

The RH functions $RH(n,t)$, $n = 1, 2, 3, \ldots$ are defined [1, 12] on the interval $[0,1)$ as

\[
RH(n,t) = \begin{cases} 
+1 & J_1 \leq t < J_{1.5}, \\
-1 & J_{1.5} \leq t < J_0, \\
0 & \text{otherwise,}
\end{cases}
\]

(2.1)

where $J_u = \frac{i-u}{2^j}$, $u = 0, \frac{1}{2}, 1$, and $n = 2^j + j - 1$, $i = 0, 1, 2, \ldots, j = 1, 2, \ldots, 2^i$. The value of $n = 0$ is defined for $i = j = 0$ and is given by

\[
RH(0,t) = 1, \quad 0 \leq t < 1.
\]

(2.2)

The orthogonality property of this basis is given by

\[
\int_0^1 RH(r,t)RH(v,t)dt = \begin{cases} 
2^{-i} & r = v, \\
0 & r \neq v.
\end{cases}
\]

(2.3)
2.2. Operational matrix of integration

Now let that $\phi(z) = [RH(0, z), RH(1, z), \ldots, RH(M - 1, z)]^T$, then the integral of the $\phi(z)$ is given by

$$
\int_0^t \phi(z)dz = P\phi(t),
$$

where $P$ is the $M \times M$ matrix for integration and is given in [1, 11, 19] as

$$
P_{M \times M} = \frac{1}{2M} \begin{pmatrix}
2MP_{M \times M} & -\Psi_{M \times M} \\
\Psi_{M \times M}^{-1} & 0
\end{pmatrix},
$$

where $\Psi_{1 \times 1} = [1]$, $P_{1 \times 1} = [\frac{1}{2}]$ and $\Psi_{k \times k}$ is given by Eq. (2.7) and

$$
\Psi_{M \times M}^{-1} = \frac{1}{M} \Psi_{M \times M}^T \text{diag} \left[ 1, \frac{1}{2}, \frac{2}{2^2}, \ldots, \frac{2^{M-1}}{2^2}, \frac{2^{M-2}}{2^3}, \ldots, \frac{2}{2^M}, \ldots, \frac{1}{2^M} \right].
$$

The matrix $\Psi_{M \times M}$ can be expressed as

$$
\Psi_{M \times M} = \begin{bmatrix}
\phi \left( \frac{1}{2M} \right) & \phi \left( \frac{3}{2M} \right) & \ldots & \phi \left( \frac{2M - 1}{2M} \right)
\end{bmatrix}.
$$

If each waveform is divided into intervals, the magnitude of the waveform can be represented as

$$
\Psi_{8 \times 8} = \begin{pmatrix}
RH(0, \frac{1}{16}) \\
RH(1, \frac{1}{16}) \\
\vdots \\
RH(7, \frac{1}{16})
\end{pmatrix} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{pmatrix}.
$$

In Eq. (2.7), the row denotes the order of the Haar function. Also, the integral of the production of two $\phi$ vector function is given by

$$
D = \int_0^1 \phi(t)\phi^T(t)dt,
$$

where $D$ is a diagonal matrix given by

$$
D = \text{diag} \left[ 1, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \ldots, \frac{1}{2}, \ldots, \frac{1}{2}, \ldots, \frac{1}{2}, \ldots, \frac{1}{2}, \ldots, \frac{1}{2} \right].
$$
3. PROPERTIES OF HYBRID FUNCTIONS

3.1. Hybrid functions of Block-pulse and rationalized Haar functions

We divided the domain $[0, 1)$, into $N$ subintervals $\left[\frac{n-1}{N}, \frac{n}{N}\right)$, $n = 1, 2, \ldots, N$. In every subinterval $\left[\frac{n-1}{N}, \frac{n}{N}\right)$, we can approximate function $f \in L^2\left[\frac{n-1}{N}, \frac{n}{N}\right)$ by $M$ rationalized Haar functions. Therefore, in every of these subintervals, define new functions $\theta(n, m, t)$, $m = 0, 1, \ldots, M - 1$, which is called hybrid of Block-pulse and rationalized Haar function, as

$$\theta(n, m, t) = \begin{cases} RH(m, N(t - t_{n-1})) & t \in \left[\frac{n-1}{N}, \frac{n}{N}\right), \\ 0 & \text{otherwise.} \end{cases}$$

Here $t_n = \frac{n}{N}$, $n = 1, 2, \ldots, N$, $m = 0, 1, \ldots, M - 1$ and also we note that $M = 2^k$.

In Ref. [14] has been shown that the functions given by Eq. (3.1) are orthogonal and complete, therefore we can express function $f \in L^2[0, 1)$ as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \theta(n, m, t),$$

where

$$c_{n,m} = \frac{(f(t), \theta(n, m, t))}{(\theta(n, m, t), \theta(n, m, t))},$$

in which $(\cdot, \cdot)$ denote the the inner product.

3.2. Operational matrices of Hybrid functions

The series in Eq. (3.2) can be written as

$$f(t) \simeq \sum_{n=1}^{N} \sum_{m=0}^{2^k - 1} c_{n,m} \theta(n, m, t),$$

or

$$f(t) \simeq C^T \Theta(t),$$

where

$$C = [c_{1,0}, \ldots, c_{1,2^k-1}; c_{2,0}, \ldots, c_{2,2^k-1}; \ldots; c_{N,0}, \ldots, c_{N,2^k-1}]^T$$

and $\Theta(t)$ is a vector as

$$\Theta_{n\times m}(t) = \theta(n, m, t), \quad n = 1, 2, \ldots, N, \quad m = 0, 1, \ldots, 2^k - 1.$$ 

The integration of the vector $\Theta(t)$ defined in Eq. (3.4) can be approximated by

$$\int_0^t \Theta(z)dz \simeq R\Theta(t),$$
where $R$ is the $NM \times NM$ operational matrix for integration and is given by [14, 16]

$$
R = \frac{1}{N} \begin{pmatrix}
P & H & H & \cdots & H \\
0 & P & H & \cdots & H \\
0 & 0 & P & \cdots & H \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & P
\end{pmatrix},
$$

(3.6)

where

$$
H = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}_{N \times M},
$$

(3.7)

and $P$ is the operational matrix of integration for rationalized Haar functions on the interval $\left[\frac{n-1}{N}, \frac{n}{N}\right]$ given by Eq. (2.5).

The integration of product of two hybrid vectors can be obtained as

$$
W = \int_0^1 \Theta(t)\Theta^T(t)dt,
$$

where $W$ is a diagonal matrix, given by

$$
W = \begin{pmatrix}
D & 0 & 0 & \cdots & 0 \\
0 & D & 0 & \cdots & 0 \\
0 & 0 & D & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & D
\end{pmatrix}.
$$

(3.8)

This matrix plays an important role for solving differential equations and $D$ be given in Eq. (2.8).

### 3.3. Approximate differential equations of the second order

In this section, we show how to approximate the differential equations by hybrid of Block-pulse and rationalized Haar functions, of the form

$$
g_1(t) \frac{d^2y(t)}{dt^2} + g_2(t) \frac{dy(t)}{dt} + g_3(t)y(t) = f(t), \quad t \in [0, 1]
$$

(3.9)

with initial conditions

$$
y'(0) = a, \quad y(0) = b,
$$

where $g_1, g_2, g_3, f$ are bounded and continuous functions.
where \( g_1(t), g_2(t) \) and \( g_3(t) \) are given functions on the interval \([0, 1]\). To solve Eq. (3.9) with initial conditions, we let

\[
(3.10) \quad \frac{d^2 y(t)}{dt^2} \simeq \sum_{n=1}^{N} \sum_{m=0}^{2^k-1} c(n, m) \theta(n, m, t) = C^T \Theta(t).
\]

Using Eqs. (3.5) and (3.10) we have that

\[
(3.11) \quad \frac{dy(t)}{dt} \simeq C^T R \Theta(t) + a,
\]

\[
(3.12) \quad y(t) \simeq C^T R^2 \Theta(t) + at + b.
\]

Now using Eqs. (3.10)–(3.12) and replacing in Eq. (3.9) we have that

\[
(3.13) \quad C^T \left[ g_1(t) \Theta(t) + g_2(t) R \Theta(t) + g_3(t) R^2 \Theta(t) \right] \equiv h(t),
\]

where \( h(t) = f(t) - a (g_2(t) + tg_3(t)) - bg_3(t) \).

If we multiply both sides of Eq. (3.13) by \( \Theta^T(t) \) and integrate the equation from 0 to 1, we obtain

\[
(3.14) \quad C^T \left[ \int_0^1 g_1(t) \Theta(t) \Theta^T(t) dt + R \int_0^1 g_2(t) \Theta(t) \Theta^T(t) dt + \right. \\
+ \left. R^2 \int_0^1 g_3(t) \Theta(t) \Theta^T(t) dt \right] \equiv \int_0^1 h(t) \Theta^T(t) dt.
\]

In Eq. (3.14) the integrals can be approximated by numerical integration and if one of the functions \( g_i(t), i = 1, 2, 3 \) is constant then we can write

\[
\int_0^1 g_i(t) \Theta(t) \Theta^T(t) dt = g_i \int_0^1 \Theta(t) \Theta^T(t) dt = g_i W, \quad i = 1, 2, 3,
\]

and not need to be integrated.

### 3.4. Approximate population balance equation

In this section, we discuss about the population balance equations which are described by the following differential equation model [1, 6, 10, 20]

\[
\frac{dy(x)}{dx} + (1 + \kappa x^m) y(x) = 2^{m+1} \kappa x^m y(2x), \quad 0 \leq x \leq T,
\]

where \( \kappa, T \) and \( m \) are constant. If the binary equal breakage is assumed then the value of \( m \) is assumed to be 4 and the initial condition \( y(x) \) is

\[
y(0) = 1.
\]
As we know, there is no exact solution for this equation \([6, 10, 20]\), so we have to solve it by approximation methods. Let that \(x = Tt\) then we have

\[
\frac{1}{T} \frac{dy(t)}{dt} + (1 + \kappa T^m t^m) y(t) = 2^{m+1} \kappa T^m t^m y(2t), \quad 0 \leq t \leq 1,
\]

To solve Eq. (3.15), with initial condition \(y(0) = 1\), we let

\[
\frac{dy(t)}{dt} \approx \sum_{n=1}^{N} \sum_{m=0}^{2^k-1} c(n, m) \theta(n, m, t) = C^T \Theta(t),
\]

\[
y(t) \approx C^T R \Theta(t) + 1.
\]

Now, by replacing Eqs. (3.16)–(3.17) in Eq. (3.15) we have that

\[
C^T \left[ \frac{1}{T} \Theta(t) + (1 + \kappa T^m t^m) R \Theta(t) - 2^{m+1} \kappa T^m t^m R \Theta(2t) \right] = h(t),
\]

where \(h(t) = \kappa T^m t^m (2^{m+1} - 1) - 1\). If we assume that \(Q(t) = R \Theta(t)\), \(g(t) = (1 + \kappa T^m t^m)\) and multiply both sides of Eq. (3.18) by \(\Theta^T(t)\) and integrate equation from 0 to 1, we obtain

\[
C^T \int_0^1 \Theta^T(t) \left[ \frac{1}{T} \Theta(t) + g(t)Q(t) - 2^{m+1} \kappa T^m t^m Q(2t) \right] dt = F,
\]

where \(F\) is a vector given by

\[
F = \int_0^1 \Theta^T(t)h(t)dt.
\]

Now by using Eq. (3.8), we can simplify Eq. (3.19) as

\[
C^T \left[ \frac{1}{T} W + \int_0^1 \Theta^T(t) g(t)Q(t) dt - \kappa 2^{m+1} T^m \int_0^1 \Theta^T(t)t^m Q(2t) dt \right] = F.
\]

Integration in Eqs. (3.19)–(3.21) is simple and can be calculated by numerical integration techniques. By solving the linear system of equations (3.21) we can obtain the unknown vector \(C\).

4. NUMERICAL EXAMPLES

We use the presented method for solving three examples. The first two examples are differential equations of the second order and the third is population balance differential equation.

4.1. Example 1. Consider the differential equation \([12]\) as follows

\[
t \frac{d^2 y(t)}{dt^2} + \frac{dy(t)}{dt} + ty(t) = 0, \quad y'(0) = 0, \quad y(0) = 1.
\]
The exact solution is the Bessel function of the zero order

\[ y(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{2^n (n!)^2}. \]

Numerical results for different value of \( M = 2^k \) and \( N \) are given in Table 1.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( N = 6, M = 8 )</th>
<th>( N = 8, M = 16 )</th>
<th>( N = 8, M = 32 )</th>
<th>Exact</th>
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4.2. Example 2. Consider the differential equation of the error function [11],

\[ \frac{d^2 y(t)}{dt^2} + 2t \frac{dy(t)}{dt} = 0, \quad y(0) = 0, \quad y'(0) = \frac{2}{\sqrt{\pi}}, \]

where the exact solution is given by

\[ y(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-z^2} \, dz. \]

The computational results for different \( M = 2^k \) and \( N \) together with exact solutions are given in Table 2.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( N = 6, M = 8 )</th>
<th>( N = 8, M = 16 )</th>
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4.3. Example 3. Consider the population balance differential equation given in Eq. (3.15). We solve this differential equation for constants $m = 4$, $\kappa = \frac{0.5}{\sqrt{m}} = \frac{1}{16}$ and $T = 6$. The numerical results by hybrid of Block-pulse and rationalized Haar functions together with the results obtained by 120-term Block-pulse series [10], the method of weighted residuals [20] and wavelet Galerkin method [6] are given in Table 3.

**Table 3**

<table>
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<td>5.5</td>
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<td>0.994869e−5</td>
<td>0.101114e−4</td>
<td>0.1008e−4</td>
<td>0.978324e−5</td>
</tr>
<tr>
<td>6.0</td>
<td>0.19753e−6</td>
<td>0.20653e−6</td>
<td>0.207547e−6</td>
<td>0.1864e−6</td>
<td>0.190671e−6</td>
</tr>
</tbody>
</table>

5. CONCLUSION

In this paper we have investigated the application of hybrid of rationalized Haar and Block-pulse functions for solving the differential equations of second order and the population balance differential equation. The matrices $R$ and $W$ introduced in Eqs. (3.6) and (3.8) are sparse, therefore they make the presented method computationally very attractive. The numerical solution of population balance differential equation by MWR [20], the wavelet Galerkin method [6], the rationalized Haar functions [1] and the Block-pulse function method [10] are not satisfactory and the method consumes too much computing time.

The main characteristic of this scheme with respect to the rationalized Haar function is that, the number of basis function is of the form $N \times 2^k$, and by selecting appropriate $k$, $N$ we can get much accuracy, while in rationalized Haar function for get higher accuracy the number of basis must be multiplied by two.
REFERENCES