GENERAL Δ-ERGODIC THEORY, WITH SOME RESULTS ON SIMULATED ANNEALING

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We set forth a simplified version of the general Δ-ergodic theory given in [14] (see also [10–13] and [15–16]). Moreover, in the limit Δ-ergodic theory we consider four limit equivalence relations (two of them are new and for the other ones see also [10], [12], and [14]); thus, we have a more complete picture on the iterated limit behaviour of the matrix product $P_{m,n}$ of a finite Markov chain with transition matrices $(P_n)_{n\geq 1}$, where $P_{m,n} := P_{m+1}P_{m+2}\dots P_n$, $\forall m,n$, $0 \leq m < n$. Also, we give some results on the simulated annealing (chain) with transition matrices $(P_n)_{n\geq 1}$ in connection with R and T, the sets of recurrent and transient states of P, respectively, where $P = \lim\limits_{n\to\infty} P_n$.

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1. Δ-ERGODIC THEORY

In this section we set forth a simplified version of the Δ-ergodic theory given in [14]. Then some results are given: a few refer to the Δ-ergodic theory (see [7–16] and the references therein for others) and the other ones to the simulated annealing.

In this article, a vector x is a row vector and x' denotes its transpose.

Consider a finite Markov chain $(X_n)_{n\geq 0}$ with state space $S = \{1, 2, \dots, r\}$, initial (probability) distribution p_0, and transition matrices $(P_n)_{n\geq 1}$. We frequently shall refer to it as the (finite) Markov chain $(P_n)_{n\geq 1}$. For all integers $m \geq 0$, $n > m$, define

$$P_{m,n} = P_{m+1}P_{m+2}\dots P_n = ((P_{m,n})_{ij})_{i,j\in S}\,.$$

(The entries of a matrix Z will be denoted Z_{ij}.)

Set

$$\mathrm{Par}(E) = \{\Delta \mid \Delta \text{ is a partition of } E\}\,,$$

where E is a nonempty set. We shall agree that the partitions do not contain the empty set.

Definition 1.1. Let $\Delta_1, \Delta_2 \in \operatorname{Par}(E)$. We say that Δ_1 *is finer than* Δ_2 if $\forall V \in \Delta_1$, $\exists W \in \Delta_2$ such that $V \subseteq W$.

Write $\Delta_1 \preceq \Delta_2$ when Δ_1 is finer than Δ_2.

Let $\emptyset \neq A \subseteq S$ and $\emptyset \neq B \subseteq \mathbf{N}$. Let $\Sigma \in \operatorname{Par}(A)$. To show how we simplify the language of Δ-ergodic theory given in [14] consider, e.g., the next definition.

Definition 1.2 ([14]). Let $i, j \in S$. We say that i and j are in the same *weakly ergodic class on* $A \times B$ (or *on* $A \times B$ *with respect to* Σ, or *on* $(A \times B, \Sigma)$ when confusion can arise) if $\forall K \in \Sigma$, $\forall m \in B$ we have

$$\lim_{n \to \infty} \sum_{k \in K} [(P_{m,n})_{ik} - (P_{m,n})_{jk}] = 0.$$

Since the matrices are stochastic, we can use $\Sigma \cup \{A^c\} \in \operatorname{Par}(S)$ instead of $\Sigma \in \operatorname{Par}(A)$, where A^c is the complement of A. This implies that we can use $\Sigma \in \operatorname{Par}(S)$ and B instead of A, B, and $\Sigma \in \operatorname{Par}(A)$. Further, we consider $\Sigma \in \operatorname{Par}(S)$ and $\emptyset \neq B \subseteq \mathbf{N}$. (Equivalently, we can use a σ-algebra, say $\mathcal{F}$, on S and $\emptyset \neq B \subseteq \mathbf{N}$ instead of $\Sigma \in \operatorname{Par}(S)$ and $\emptyset \neq B \subseteq \mathbf{N}$.) Thus, in the Δ-ergodic theory the natural space becomes $\operatorname{Par}(S) \times \mathcal{P}^*(\mathbf{N})$, called *partition-time space*, in place of $S \times \mathbf{N}$ (see [14]), where $\mathcal{P}^*(\mathbf{N}) := \{B \mid \emptyset \neq B \subseteq \mathbf{N}\}$. Coming back to Σ, we suppose, moreover, that it is an ordered set (this condition is necessary for simplification when we use the operator $(\cdot)^+$ (see below)).

Under the above considerations Definition 1.2 becomes

Definition 1.3. Let $i, j \in S$. We say that i and j are in the same *weakly ergodic class on* $\Sigma \times B$ if $\forall K \in \Sigma$, $\forall m \in B$ we have

$$\lim_{n \to \infty} \sum_{k \in K} [(P_{m,n})_{ik} - (P_{m,n})_{jk}] = 0.$$

Write $i \overset{\Sigma \times B}{\sim} j$ when i and j are in the same weakly ergodic class on $\Sigma \times B$. Then $\overset{\Sigma \times B}{\sim}$ is an equivalence relation and determines a partition $\Delta = \Delta(\Sigma, B) = (C_1, C_2, \ldots, C_s)$ of S. The sets $C_1, C_2, \ldots, C_s$ are called *weakly ergodic classes on* $\Sigma \times B$.

The two definitions below are simplified versions of Definitions 1.3–4 in [14], respectively.

Definition 1.4. Let $\Delta = (C_1, C_2, \ldots, C_s)$ be the partition of weakly ergodic classes on $\Sigma \times B$ of a Markov chain. We say that the chain is *weakly* Δ-*ergodic on* $\Sigma \times B$. In particular, a weakly (S)-ergodic chain on $\Sigma \times B$ is called *weakly ergodic on* $\Sigma \times B$ for short.

Definition 1.5. Let $(C_1, C_2, \ldots, C_s)$ be the partition of weakly ergodic classes on $\Sigma \times B$ of a Markov chain with state space S and $\Delta \in \mathrm{Par}(S)$. We say that the chain is *weakly* $[\Delta]$*-ergodic on* $\Sigma \times B$ if $\Delta \preceq (C_1, C_2, \ldots, C_s)$.

In connection with the above notions and notation we mention some special cases.

1. $\Sigma \times B = (\{i\})_{i \in S} \times \mathbf{N}$ $((\{i\})_{i\in S} := (\{1\}, \{2\}, \ldots, \{r\}))$. In this case, we can write $\sim$ instead of $\overset{(\{i\})_{i\in S}\times\mathbf{N}}{\sim}$ and can omit '*on* $(\{i\})_{i\in S} \times \mathbf{N}$' in Definitions 1.3–5.

2. $\Sigma = (\{i\})_{i\in S}$. In this case, we can write $\overset{B}{\sim}$ instead of $\overset{(\{i\})_{i\in S}\times B}{\sim}$ and can replace '$(\{i\})_{i\in S} \times B$' by '(*time set*) B' in Definitions 1.3–5. A special subcase is $B = \{m\}$ $(m \geq 0)$; in this case we can write $\overset{m}{\sim}$ and can replace '*on* (*time set*) $\{m\}$' by '*at time* m' in Definitions 1.3–5.

3. $B = \mathbf{N}$. In this case, we can set $\overset{\Sigma}{\sim}$ instead of $\overset{\Sigma\times\mathbf{N}}{\sim}$ and can replace '$\Sigma \times \mathbf{N}$' by 'Σ' in Definitions 1.3–5.

Also, the special case $\Sigma \times B = (A, A^c) \times \{0\}$ $(A \neq \emptyset, S)$ is an important one because it is the natural framework for the study of limit behaviour of $P\,(X_n \in A)$ (see, e.g., [14, Example 2.25], [16], and Theorems 1.18 and 1.25 here).

The three definitions below are simplified versions of Definitions 1.7–9 in [14], respectively.

Definition 1.6. Let $i, j \in S$. We say that i and j are in the same *uniformly weakly ergodic class on* $\Sigma \times B$ if $\forall K \in \Sigma$ we have

$$\lim_{n\to\infty} \sum_{k\in K} [(P_{m,n})_{ik} - (P_{m,n})_{jk}] = 0$$

uniformly with respect to $m \in B$.

Write $i \overset{u,\Sigma\times B}{\sim} j$ when i and j are in the same uniformly weakly ergodic class on $\Sigma \times B$. Then $\overset{u,\Sigma\times B}{\sim}$ is an equivalence relation and determines a partition $\Delta = \Delta\,(\Sigma, B) = (U_1, U_2, \ldots, U_t)$ of S. The sets $U_1, U_2, \ldots, U_t$ are called *uniformly weakly ergodic classes on* $\Sigma \times B$.

Definition 1.7. Let $\Delta = (U_1, U_2, \ldots, U_t)$ be the partition of uniformly weakly ergodic classes on $\Sigma \times B$ of a Markov chain. We say that the chain is *uniformly weakly* Δ*-ergodic on* $\Sigma \times B$. In particular, a uniformly weakly (S)-ergodic chain on $\Sigma \times B$ is called *uniformly weakly ergodic on* $\Sigma \times B$ for short.

Definition 1.8. Let $(U_1, U_2, \ldots, U_t)$ be the partition of uniformly weakly ergodic classes on $\Sigma \times B$ of a Markov chain with state space S and $\Delta \in$

Par(S). We say that the chain is *uniformly weakly* $[\Delta]$-*ergodic on* $\Sigma \times B$ if $\Delta \preceq (U_1, U_2, \ldots, U_t)$.

A cyclic homogeneous (finite) Markov chain is a simple example of uniformly weakly Δ-ergodic Markov chain on $(\{i\})_{i\in S} \times \mathbf{N}$, where Δ is the partition of cyclic subclasses of the chain.

$\overset{\Sigma\times B}{\sim}$ is called the *simple equivalence relation on* $\Sigma \times B$ while $\overset{u,\Sigma\times B}{\sim}$ is called the *uniform equivalence relation on* $\Sigma \times B$.

As for uniform weak Δ-ergodicity we mention some special cases.

1. $\Sigma \times B = (\{i\})_{i\in S} \times \mathbf{N}$. In this case, we can write $\overset{u}{\sim}$ instead of $\overset{u,(\{i\})_{i\in S}\times\mathbf{N}}{\sim}$ and can omit '*on* $(\{i\})_{i\in S} \times \mathbf{N}$' in Definitions 1.6–8.

2. $\Sigma = (\{i\})_{i\in S}$. In this case, we can write $\overset{u,B}{\sim}$ instead of $\overset{u,(\{i\})_{i\in S}\times B}{\sim}$ and can replace '$(\{i\})_{i\in S} \times B$' by '(*time set*) B' in Definitions 1.6–8.

3. $B = \mathbf{N}$. In this case, we can write $\overset{u,\Sigma}{\sim}$ instead of $\overset{u,\Sigma\times\mathbf{N}}{\sim}$ and can replace '$\Sigma \times \mathbf{N}$' by 'Σ' in Definitions 1.6–8.

The two definitions below are simplified versions of Definitions 1.13–14 in [14], respectively.

Definition 1.9. Let C be a weakly ergodic class on $\Sigma\times B$. Let $\emptyset \neq \Sigma_0 \subseteq \Sigma$ and $\emptyset \neq B_0 \subseteq B$. We say that C is a *strongly ergodic class on* $\Sigma_0 \times B_0$ *with respect to* $\Sigma \times B$ if $\forall i \in C$, $\forall K \in \Sigma_0$, $\forall m \in B_0$ the limit

$$\lim_{n\to\infty} \sum_{j\in K} (P_{m,n})_{ij} := \sigma_{m,K} = \sigma_{m,K}(C)$$

exists and does not depend on i.

Definition 1.10. Let C be a uniformly weakly ergodic class on $\Sigma\times B$. Let $\emptyset \neq \Sigma_0 \subseteq \Sigma$ and $\emptyset \neq B_0 \subseteq B$. We say that C is a *uniformly strongly ergodic class on* $\Sigma_0 \times B_0$ *with respect to* $\Sigma \times B$ if $\forall i \in C$, $\forall K \in \Sigma_0$ the limit

$$\lim_{n\to\infty} \sum_{j\in K} (P_{m,n})_{ij} := \sigma_{m,K} = \sigma_{m,K}(C)$$

exists uniformly with respect to $m \in B_0$ and does not depend on i.

In connection with the last two definitions we mention some special cases.

1. $\Sigma_0 \times B_0 = \Sigma \times B$. In this case, we can say that C is a *strongly* (respectively, *uniformly strongly*) *ergodic class on* $\Sigma \times B$. A special subcase is $\Sigma_0 \times B_0 = \Sigma \times B = (\{i\})_{i\in S} \times \mathbf{N}$ and $C = S$ when we can say that the Markov chain itself is *strongly* (respectively, *uniformly strongly*) *ergodic.*

2. $\Sigma_0 = \Sigma = (\{i\})_{i\in S}$. In this case, we can say that C is a *strongly* (respectively, *uniformly strongly*) *ergodic class on* (*time set*) B_0 *with respect*

to (*time set*) B. If $B_0 = B$, then we can say that C is a *strongly* (respectively, *uniformly strongly*) *ergodic class on* (*time set*) B. A special subcase of the case $\Sigma_0 = \Sigma = (\{i\})_{i\in S}$ and $B_0 = B$ is $B_0 = B = \{m\}$ when we can say that C is a *strongly* (respectively, *uniformly strongly*) *ergodic class at time* m.

3. $B_0 = B = \mathbf{N}$. In this case, we can say that C is a *strongly* (respectively, *uniformly strongly*) *ergodic class on* Σ_0 *with respect to* Σ. If $\Sigma_0 = \Sigma$, then we can say that C is a *strongly* (respectively, *uniformly strongly*) *ergodic class on* Σ.

The two definitions below are simplified versions of Definitions 1.16–17 in [14], respectively.

Definition 1.11. Consider a weakly (respectively, uniformly weakly) Δ-ergodic chain on $\Sigma \times B$. We say that the chain is *strongly* (respectively, *uniformly strongly*) Δ-*ergodic on* $\Sigma \times B$ if any $C \in \Delta$ is a strongly (respectively, uniformly strongly) ergodic class on $\Sigma \times B$. In particular, a strongly (respectively, uniformly strongly) (S)-ergodic chain on $\Sigma\times B$ is called *strongly* (respectively, *uniformly strongly*) *ergodic on* $\Sigma \times B$ for short.

Definition 1.12. Consider a weakly (respectively, uniformly weakly) $[\Delta]$-ergodic chain on $\Sigma \times B$. We say that the chain is *strongly* (respectively, *uniformly strongly*) $[\Delta]$-*ergodic on* $\Sigma \times B$ if any $C \in \Delta$ is included in a strongly (respectively, uniformly strongly) ergodic class on $\Sigma \times B$.

Set

$$R_{m,n} = \{E \mid E \text{ is a real } m \times n \text{ matrix}\},$$
$$S_{m,n} = \{E \mid E \text{ is a stochastic } m \times n \text{ matrix}\},$$
$$R_n = R_{n,n} \quad \text{and} \quad S_n = S_{n,n}.$$

Let $E = (E_{ij}) \in R_{m,n}$, $\emptyset \neq U \subseteq \{1, 2, \ldots, m\}$, $\emptyset \neq V \subseteq \{1, 2, \ldots, n\}$, and $\Sigma = (K_1, K_2, \ldots, K_p) \in \mathrm{Par}(\{1, 2, \ldots, n\})$. Suppose that Σ is an ordered set. Define

$$E_U = (E_{ij})_{i\in U, j\in\{1,2,\ldots,n\}}, \quad E^V = (E_{ij})_{i\in\{1,2,\ldots,m\}, j\in V}, \quad E_U^V = (E_{ij})_{i\in U, j\in V},$$
$$|||E|||_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |E_{ij}|$$

(the ∞-*norm of* E),

$$\bar{\alpha}(E) = \frac{1}{2} \max_{1\le i,j\le m} \sum_{k=1}^{n} |E_{ik} - E_{jk}|$$

($\bar{\alpha}$ is a well-known coefficient (see, e.g., [9] or [19, p. 82]), and

$$E^+ = \left(E^+_{ij}\right), \quad E^+_{ij} = \sum_{k \in K_j} E_{ik}, \quad \forall i \in \{1, 2, \ldots, m\}, \ \forall j \in \{1, 2, \ldots, p\}.$$

We call E^+ the *column-reduced matrix of* E (*on* Σ; $E^+ = E^+(\Sigma)$, i.e., it depends on Σ (if confusion can arise we write $E^{+\Sigma}$ instead of E^+)) (see also [14]). In this article, when we use the operator $(\cdot)^+ = (\cdot)^+(\Sigma)$, we suppose that Σ is an ordered set, even if we omit to precise this.

Note that if $E \in S_{m,n}$, then $E^{+\Sigma} \in S_{m,p}$, $\forall \Sigma \in \operatorname{Par}(\{1, 2, \ldots, n\})$ with $|\Sigma| = p$, where $|\Sigma|$ is the cardinal of Σ. Also, note that if $E \in R_{m,n}$ and $F \in R_{n,q}$, then $(EF)^+ = EF^+$ (see [14, Proposition 2.17(v)]).

Definition 1.13 (see also [16, Definition 1.9]). Consider a strongly (respectively, uniformly strongly) $[\Delta]$- or Δ-ergodic Markov chain on $\Sigma \times B$. We say that the chain has *limits* Π_m, $m \in B$, (*on* $\Sigma \times B$) if

$$\lim_{n \to \infty} (P_{m,n})^+ = \Pi_m, \quad \forall m \in B.$$

($\lim_{n \to \infty} (P_{m,n})^+$ exists because of Definitions 1.9–12.) In particular, if there exists a matrix Π such that $\Pi_m = \Pi$, $\forall m \in B$, then we say that the chain has *limit* Π.

Set $e = e(n) = (1, 1, \ldots, 1) \in \mathbf{R}^n$ $(n \geq 1)$. Note that the collection of absorbing homogeneous Markov chains is included in the collection of Markov chains $(P_n)_{n \geq 1}$ for which $\exists (R, T) \in \operatorname{Par}(S)$, where S is the state space of $(P_n)_{n \geq 1}$, such that:

(i) the chain is strongly ergodic on (R, T) and has limit $(e', 0)$;

(ii) $(P_n)^T_R = 0$, $\forall n \geq 1$;

(iii) T is the largest set with the properties (i) and (ii).

Also, in Definitions 1.11–13 we can simplify the language when referring to Σ and B. These are left to the reader.

Definition 1.14 (see, e.g., [10] or [11]). Let $\Delta \in \operatorname{Par}(\{1, 2, \ldots, m\})$. We say that a matrix $E \in R_{m,n}$ is $[\Delta]$-*stable* if E_K is a stable matrix (i.e., a real matrix whose rows are identical), $\forall K \in \Delta$.

Definition 1.15 (see, e.g., [10] or [11]). Let $\Delta \in \operatorname{Par}(\{1, 2, \ldots, m\})$. We say that a matrix $E \in R_{m,n}$ is Δ-*stable* if Δ is the least fine partition for which E is a $[\Delta]$-stable matrix. In particular, a $(\{1, 2, \ldots, m\})$-stable matrix is called *stable* for short.

Concerning the behaviour of $P(X_n \in A)$ $(\emptyset \neq A \subset S)$ of a finite Markov chain $(X_n)_{n \geq 0}$ we distinguish two types: 1) finite-time behaviour (see among other things the Δ-ergodic theory); 2) limit behaviour (see the Δ-ergodic theory and, also, its connections with the limit Δ-ergodic theory) while concerning the behaviour of matrix product $P_{m,n}$ we distinguish three types: 1) finite-time behaviour (see among other things the Δ-ergodic theory); 2) limit behaviour (see the Δ-ergodic theory and, also, its connections with the limit Δ-ergodic theory); 3) iterated limit behaviour (see the limit Δ-ergodic theory and, also, its connections with the Δ-ergodic theory).

The result below is about the limit behaviour of matrix product $P_{m,n}$ of a Markov chain and is a generalization of Theorem 2.9, (i)⇔(iii), in [11]. (The generalization, if any, of Theorems 2.9, 2.11–12, and 2.17–18 in [11] is left to the reader (an important special case is $\Sigma = (\{i\})_{i\in S}$).)

THEOREM 1.16. *Let $(P_n)_{n\geq 1}$ be a Markov chain. Let $\Sigma=(K_1, K_2, \ldots, K_p) \in \mathrm{Par}(S)$ (Σ is an ordered set). Then the chain is weakly $[\Delta]$-ergodic on $\Sigma\times B$ if and only if $\forall m \in B$ there exist $[\Delta]$-stable $r \times p$ matrices $\Pi_{m,n}$, $m < n$, such that*

$$\lim_{n\to\infty} \left[(P_{m,n})^+ - \Pi_{m,n}\right] = 0.$$

Proof. "⇒" Let $C \in \Delta$. Set

$$\pi_{m,n,C}(k) = \frac{1}{|C|}\sum_{i\in C}(P_{m,n})^+_{ik}, \quad \forall m \in B,\ \forall n > m,\ \forall k \in \{1, 2, \ldots, p\},$$

and

$$(\Pi_{m,n})_C = e'\pi_{m,n,C}, \quad \forall m \in B,\ \forall n > m$$

($e = (1, 1, \ldots, 1) \in \mathbf{R}^{|C|}$ and e' is the transpose of e). Further,

$$(P_{m,n})^+_{jk} - (\Pi_{m,n})_{jk} = (P_{m,n})^+_{jk} - \pi_{m,n,C}(k) = (P_{m,n})^+_{jk} - \frac{1}{|C|}\sum_{i\in C}(P_{m,n})^+_{ik} =$$

$$= \frac{1}{|C|}\sum_{i\in C}\left[(P_{m,n})^+_{jk} - (P_{m,n})^+_{ik}\right] \to 0$$

as $n \to \infty$, $\forall m \in B$, $\forall j \in C$, $\forall k \in \{1, 2, \ldots, p\}$.

"⇐" Let $C \in \Delta$. Since $(\Pi_{m,n})_{ik} = (\Pi_{m,n})_{jk}$, $\forall m \in B$, $\forall n > m$, $\forall i, j \in C$, $\forall k \in \{1, 2, \ldots, p\}$, we have

$$(P_{m,n})^+_{ik} - (P_{m,n})^+_{jk} = \left[(P_{m,n})^+_{ik} - (\Pi_{m,n})_{ik}\right] + \left[(\Pi_{m,n})_{jk} - (P_{m,n})^+_{jk}\right] \to 0$$

as $n \to \infty$, $\forall m \in B$, $\forall i, j \in C$, $\forall k \in \{1, 2, \ldots, p\}$. □

Definition 1.17 ([13]). Let $\emptyset \neq K \subseteq S$. Let p be a probability distribution on S. Set $p^{\emptyset} = 0$. We say that p is *concentrated on* K if $p^{K^c} = 0$.

Set $P_{m,m} = I_r$, $\forall m \geq 0$ ($r = |S|$).

The result below is about the limit behaviour of $P(X_n \in A)$.

THEOREM 1.18. *Consider a weakly* $[\Delta]$*-ergodic Markov chain* $(X_n)_{n\geq 0}$ *on* (A, A^c) *at time* 0. *Let* $K \in \Delta$. *Suppose that the initial distribution* p_0 *of chain is concentrated on* K. *Then there exists a sequence* $(q_n)_{n\geq 0}$ *of real numbers depending on* K *but not on* p_0 *such that*

$$\lim_{n\to\infty} [P(X_n \in A) - q_n] = 0.$$

Proof. By Theorem 1.16, there exist $[\Delta]$-stable $r\times 2$ matrices $\Pi_{0,n}$, $0 < n$, such that

$$\lim_{n\to\infty} \left[(P_{0,n})^+ - \Pi_{0,n}\right] = 0.$$

We take the matrices $\Pi_{0,n}$, $0 < n$, as in the proof of Theorem 1.16, "$\Rightarrow$"; it follows that these do not depend on p_0. Suppose that

$$(\Pi_{0,n})_K = \begin{pmatrix} \pi_{n,1} & \pi_{n,2} \\ \pi_{n,1} & \pi_{n,2} \\ \vdots & \vdots \\ \pi_{n,1} & \pi_{n,2} \end{pmatrix}, \quad \forall n \geq 1$$

$(\pi_{n,1} + \pi_{n,2} = 1, \forall n \geq 1)$. Then

$$p_0\Pi_{0,n} = (p_0)^K (\Pi_{0,n})_K = (\pi_{n,1}, \pi_{n,2}), \quad \forall n \geq 1.$$

Therefore, $p_0\Pi_{0,n}$ does not depend on p_0 (on the other hand, it depend on K), $\forall n \geq 1$. Define $\pi_{0,1} = \left(p_0^+\right)_1$ (recall that (A, A^c) is an ordered set) and $q_n = \pi_{n,1}$, $\forall n \geq 0$. Finally, we have

$$\lim_{n\to\infty} [P(X_n \in A) - q_n] = \lim_{n\to\infty} [P(X_n \in A) - \pi_{n,1}] =$$

$$= \lim_{n\to\infty} \left[\left(p_0(P_{0,n})^+\right)^{\{1\}} - (p_0\Pi_{0,n})^{\{1\}}\right] = \lim_{n\to\infty} \left[p_0((P_{0,n})^+)^{\{1\}} - p_0(\Pi_{0,n})^{\{1\}}\right] =$$

$$= p_0 \lim_{n\to\infty} \left[\left((P_{0,n})^+\right)^{\{1\}} - (\Pi_{0,n})^{\{1\}}\right] = 0. \quad \square$$

Remark 1.19. (a) Weak Δ-ergodicity on $\Sigma\times B$ implies weak $[\Delta]$-ergodicity on $\Sigma \times B$. Strong Δ-ergodicity (respectively, $[\Delta]$ -ergodicity) on $\Sigma \times B$ implies weak Δ-ergodicity (respectively, $[\Delta]$-ergodicity) on $\Sigma \times B$. Further, see Theorem 1.18.

(b) If the chain is weakly ergodic on (A, A^c) at time 0 ($\Delta = (S)$), then the limit behaviour of $P(X_n \in A)$ is independent on p_0, where p_0 is the initial distribution of chain.

THEOREM 1.20. *Let* $(P_n)_{n\geq 1}$ *be a strongly* (*respectively, uniformly strongly*) *ergodic Markov chain on* Σ ($\Delta = (S)$ *and* $B = \mathbf{N}$) *with limits* Π_m, $m \geq 0$. *Then there exists a stable* (*stochastic*) *matrix* Π *such that* $\Pi_m = \Pi$, $\forall m \geq 0$ (*therefore, the chain has limit* Π).

Proof. We have

$$\Pi_{m-1} = \lim_{n\to\infty} (P_{m-1,n})^+ = \lim_{n\to\infty} P_m (P_{m,n})^+ =$$
$$= P_m \lim_{n\to\infty} (P_{m,n})^+ = P_m \Pi_m = \Pi_m, \quad \forall m \geq 1$$

(the last equation follows from the fact that Π_m is a stable stochastic matrix, $\forall m \geq 0$). Therefore, there exists a stable matrix Π such that $\Pi_m = \Pi$, $\forall m \geq 0$. □

Theorem 1.21. *Let $(P_n)_{n\geq 1}$ be a Markov chain. Let $\Pi \in S_r$ be a stable matrix. Then the chain is uniformly weakly ergodic and strongly ergodic with limit Π if and only if*

$$\limsup_{l\to\infty} \limsup_{n\to\infty} |||P_{n,n+l} - \Pi|||_\infty = 0.$$

Proof. See the proof of Theorem 2.6, (i)⇔(iii), in [12]. □

Theorem 1.22 ([8]). *Let $(P_n)_{n\geq 1}$ be a Markov chain. Then the chain is uniformly strongly ergodic if and only if it is uniformly weakly ergodic and strongly ergodic (in all cases $\Delta = (S)$, $\Sigma = (\{i\})_{i\in S}$, and $B = \mathbf{N}$).*

Proof. We give a full proof of this result here because in [8] was given an incorrect proof.

"⇒" Obvious.

"⇐" By Theorem 1.21 we have

$$\limsup_{l\to\infty} \limsup_{n\to\infty} |||P_{n,n+l} - \Pi|||_\infty = 0,$$

where $\Pi := \lim_{n\to\infty} P_{m,n}$, $\forall m \geq 0$. Let $\varepsilon > 0$. Then $\exists l_\varepsilon \geq 1$ such that

$$\limsup_{n\to\infty} |||P_{n,n+l} - \Pi|||_\infty < \varepsilon, \quad \forall l \geq l_\varepsilon.$$

This implies

$$\limsup_{n\to\infty} |||P_{n,n+l_\varepsilon} - \Pi|||_\infty < \varepsilon.$$

Further, $\exists n_\varepsilon \geq 1$ such that

$$|||P_{n,n+l_\varepsilon} - \Pi|||_\infty < \varepsilon, \quad \forall n \geq n_\varepsilon.$$

Let $m \geq 0$. Then

$$|||P_{m,m+n+n_\varepsilon+l_\varepsilon} - \Pi|||_\infty = |||P_{m,m+n+n_\varepsilon} P_{m+n+n_\varepsilon,m+n+n_\varepsilon+l_\varepsilon} - \Pi|||_\infty \leq$$

(by [12, Proposition 2.4(ii)])

$$\leq |||P_{m+n+n_\varepsilon,m+n+n_\varepsilon+l_\varepsilon} - \Pi|||_\infty < \varepsilon, \quad \forall n \geq 0.$$

Therefore, the chain is uniformly strongly ergodic. □

Note that Theorem 1.22 suggests the next problem. Let $(P_n)_{n\geq 1}$ be a Markov chain. Is the chain uniformly strongly ergodic on Σ if and only if is it uniformly weakly ergodic on Σ and strongly ergodic on Σ? Also, note that the proof of Theorem 1.22, "$\Leftarrow$", gives another proof of Theorem 2.6, (iii)$\Rightarrow$(i), in [12].

Theorem 2.7 in [13] suggests the next result ($e = e(|S|)$).

THEOREM 1.23. *Consider a Markov chain $(P_n)_{n\geq 1}$ with $P_n \to P$ as $n \to \infty$. Let R and T be the sets of recurrent and transient states of P, respectively. Suppose that $T \neq \emptyset$. Then the chain is uniformly strongly ergodic on (R, T) and has limit $(e', 0)$ (i.e., $\lim_{n\to\infty} (P_{m,n})^+$ exists, $\forall m \geq 0$, and $\lim_{n\to\infty} (P_{m,n})^+ = (e', 0)$ uniformly with respect to $m \geq 0$).*

Proof. It is known that $(P^n)^T \to 0$ as $n \to \infty$ (see, e.g., [1, p. 91]). It follows that $(P^n)^+ \to (e', 0)$ as $n \to \infty$. Let $\varepsilon > 0$. As $|S| < \infty$, $\exists n_0 \geq 1$ such that $(P^{n_0})^+_{i2} < \varepsilon$, $\forall i \in S$ (see the definition of the operator $(\cdot)^+$; (R, T) is an ordered set). Further, since $|S| < \infty$ and

$$\lim_{n\to\infty} (P_{n,n+n_0})^+ = (P^{n_0})^+,$$

$\exists n_1 \geq 0$ such that

$$(P_{n,n+n_0})^+_{i2} < \varepsilon, \quad \forall n \geq n_1, \ \forall i \in S.$$

From

$$(P_{m,m+n+n_1+n_0})^+ = P_{m,m+n+n_1} (P_{m+n+n_1,m+n+n_1+n_0})^+, \quad \forall m, n \geq 0,$$

we have

$$(P_{m,m+n+n_1+n_0})^+_{i2} < \varepsilon, \quad \forall m, n \geq 0, \ \forall i \in S,$$

because $P_{m,m+n+n_1}$ is a stochastic matrix, $\forall m, n \geq 0$. Therefore, $\lim_{n\to\infty} (P_{m,n})^+_{i2} = 0$ uniformly with respect to $m \geq 0$, $\forall i \in S$. Hence, the chain is uniformly strongly ergodic on (R, T) and has limit $(e', 0)$. $\square$

Remark 1.24. Another proof of Theorem 1.23 is as follows. Theorem 2.7 in [13] says, in other words, that the chain is uniformly strongly ergodic on $\left(R, (\{i\})_{i\in T}\right)$ and has limit $(e', 0, \ldots, 0)$. But this implies, obviously, that the chain is uniformly strongly ergodic on (R, T) and has limit $(e', 0)$ because $|S| < \infty$.

Theorem 1.23 and its proof lead to the next result.

THEOREM 1.25. *We have*

(i) *$P(X_n \in R) > 1 - \varepsilon$ and $P(X_n \in T) < \varepsilon$, $\forall n \geq n_0 + n_1$;*

(ii) *$\lim_{n\to\infty} P(X_n \in R) = 1$ and $\lim_{n\to\infty} P(X_n \in T) = 0$ and do not depend on the initial distribution p_0 of chain.*

Proof. (i) Let $n \geq n_0 + n_1$. By Theorem 1.23 and its proof,

$$P(X_n \in T) = p_0 \left((P_{0,n})^+\right)^{\{2\}} = \sum_{i \in S} (p_0)_i (P_{0,n})^+_{i2} < \sum_{i \in S} (p_0)_i \varepsilon = \varepsilon.$$

(ii)

$$\lim_{n\to\infty} P(X_n \in T) = \lim_{n\to\infty} p_0 \left((P_{0,n})^+\right)^{\{2\}} = p_0 \lim_{n\to\infty} \left((P_{0,n})^+\right)^{\{2\}} = p_0 \cdot 0 = 0. \square$$

Set

$$a^+ = \begin{cases} a & \text{if } a > 0, \\ 0 & \text{if } a \leq 0, \end{cases}$$

where $a \in \mathbf{R}$.

Let $H : S \to \mathbf{R}$ be a nonconstant function. We want to find $\min_{y \in S} H(y)$. A stochastic optimization technique for solving this problem approximately when S is very large is the simulated annealing (see, e.g., [2–3], [5–6], [13], and [16–19]). Consider a sequence $(\beta_n)_{n\geq 1}$ of positive real numbers with $\beta_n \to \infty$ as $n \to \infty$ ($(\beta_n)_{n\geq 1}$ is called the *cooling schedule*), an irreducible stochastic matrix $G = (G_{ij})_{i,j\in S}$ (G is called the *generation matrix*) and a Markov chain $(X_n)_{n\geq 0}$ with state space S and transition matrices $(P_n)_{n\geq 1}$, where

$$(P_n)_{ij} = \begin{cases} G_{ij} e^{-\beta_n (H(j)-H(i))^+} & \text{if } i \neq j \\ 1 - \sum\limits_{k \neq i} (P_n)_{ik} & \text{if } i = j, \end{cases}$$

$\forall i, j \in S$. $(X_n)_{n\geq 0}$ (or, by convention, $(P_n)_{n\geq 1}$) is called the (*classical*) *simulated annealing chain* (the (*classical*) *simulated annealing* for short).

We have

$$\lim_{n\to\infty} (P_n)_{ij} = \begin{cases} 0 & \text{if } i \neq j,\ H(j) > H(i), \\ G_{ij} & \text{if } i \neq j,\ H(j) \leq H(i), \\ 1 - \sum\limits_{k \neq i, H(k) \leq H(i)} G_{ik} & \text{if } i = j, \end{cases}$$

$\forall i, j \in S$.

Set

$$P = \lim_{n\to\infty} P_n,$$

where $(P_n)_{n\geq 1}$ is the simulated annealing,

$$T_1 = \{i \mid i \in S \text{ and } \exists p \geq 2, \exists \{i_1, i_2, \ldots, i_p\} \subseteq S \text{ such that } i_1 = i,$$
$$G_{i_1 i_2}, G_{i_2 i_3}, \ldots, G_{i_{p-1} i_p} > 0, \text{ and } H(i_1) \geq H(i_2) \geq \cdots \geq H(i_{p-1}) > H(i_p)\}$$

(the condition $\{i_1, i_2, \ldots, i_p\} \subseteq S$ implies $i_1, i_2, \ldots, i_p \in S$ and $i_k \neq i_l$, $\forall k, l \in \{1, 2, \ldots, p\}$, $k \neq l$), and

$T_2 = \{i \mid i \in S - T_1$ and $\exists j \in S,\ j \neq i,$ for which $\exists p \geq 2,\ \exists \{i_1, i_2, \ldots, i_p\} \subseteq S$

such that $i_1 = i, i_p = j, G_{i_1i_2}, G_{i_2i_3}, \ldots, G_{i_{p-1}i_p} > 0,$ and $H(i_1) = H(i_2) = \cdots = H(i_p)$ and $\forall q \geq 2,\ \forall \{j_1, j_2, \ldots, j_q\} \subseteq S$ such that $j_1 = j, j_q = i$, and $H(j_1) = H(j_2) = \cdots = H(j_q),\ \exists u \in \{1, 2, \ldots, q-1\}$ with $G_{j_uj_{u+1}} = 0\big\}$.

Let R and T be the sets of recurrent and transient states of P, respectively.

Theorem 1.26 (*see also* [13]). *Consider the simulated annealing* (*chain*) *above. Then we have*

$$T = T_1 \cup T_2.$$

Proof. "$\subseteq$" Let $i \in T$ ($T \neq \emptyset$ because H is a nonconstant function and G is an irreducible matrix). We show that either $i \in T_1$ or $i \in T_2$ ($T_1 \cap T_2 = \emptyset$). As $i \in T$, $\exists j \in S,\ j \neq i$, for which

(c1) $\exists p \geq 2,\ \exists \{i_1, i_2, \ldots, i_p\} \subseteq S$ such that $i_1 = i,\ i_p = j$, and $P_{i_1i_2}, P_{i_2i_3}, \ldots, P_{i_{p-1}i_p} > 0$

and

(c2) $\forall q \geq 2,\ \forall \{j_1, j_2, \ldots, j_q\} \subseteq S$ such that $j_1 = j,\ j_q = i,\ \exists u \in \{1, 2, \ldots, q-1\}$ for which $P_{j_uj_{u+1}} = 0$.

The definition of P and (c1) imply $H(i) \geq H(j)$.

Case 1. $H(i) > H(j)$. The definition of P, (c1), and $H(i) > H(j)$ imply $G_{i_1i_2}, G_{i_2i_3}, \ldots, G_{i_{p-1}i_p} > 0$ and $H(i_1) \geq H(i_2) \geq \cdots \geq H(i_{u-1}) > H(i_u) \geq \cdots \geq H(i_p)$. Therefore, $i \in T_1$.

Case 2. $H(i) = H(j)$. The definition of P, (c1), and $H(i) = H(j)$ imply $G_{i_1i_2}, G_{i_2i_3}, \ldots, G_{i_{p-1}i_p} > 0$ and $H(i_1) = H(i_2) = \cdots = H(i_p)$ while the definition of P and (c2) when $H(j_1) = H(j_2) = \cdots = H(j_q)$ imply that $\exists u \in \{1, 2, \ldots, q-1\}$ for which $G_{j_uj_{u+1}} = 0$. Therefore, $i \in T_2$.

"$\supseteq$" Let $i \in T_1 \cup T_2$. We show that $i \in T$.

Case 1. $i \in T_1$. Then $\exists p \geq 2,\ \exists \{i_1, i_2, \ldots, i_p\} \subseteq S$ such that $i_1 = i,\ G_{i_1i_2}, G_{i_2i_3}, \ldots, G_{i_{p-1}i_p} > 0$, and $H(i_1) \geq H(i_2) \geq \cdots \geq H(i_{p-1}) > H(i_p)$. Further, it follows that $P_{i_1i_2}, P_{i_2i_3}, \ldots, P_{i_{p-1}i_p} > 0$. Let $q \geq 2$ and $\{j_1, j_2, \ldots, j_q\} \subseteq S$ such that $j_1 = i_p,\ j_q = i$, and $G_{j_1j_2}, G_{j_2j_3}, \ldots, G_{j_{q-1}j_q} > 0$. Since $H(i_{p-1}) > H(i_p),\ \exists u \in \{1, 2, \ldots, q-1\}$ such that $H(j_{u+1}) > H(j_u)$. Consequently, $P_{j_uj_{u+1}} = 0$. Therefore, $i \in T$.

Case 2. $i \in T_2$. Then $\exists j \in S,\ j \neq i$, for which $\exists p \geq 2,\ \exists \{i_1, i_2, \ldots, i_p\} \subseteq S$ such that $i_1 = i,\ i_p = j,\ G_{i_1i_2}, G_{i_2i_3}, \ldots, G_{i_{p-1}i_p} > 0$, and $H(i_1) = H(i_2) = \cdots = H(i_p)$ and $\forall q \geq 2,\ \forall \{j_1, j_2, \ldots, j_q\} \subseteq S$ such that $j_1 = j,\ j_q = i$, and $H(j_1) = H(j_2) = \cdots = H(j_q),\ \exists u \in \{1, 2, \ldots, q-1\}$ with $G_{j_uj_{u+1}} = 0$. Further, it follows that $P_{i_1i_2}, P_{i_2i_3}, \ldots, P_{i_{p-1}i_p} > 0$. Let $q \geq 2$ and $\{j_1, j_2, \ldots, j_q\} \subseteq S$ such that $j_1 = j$ and $j_q = i$. If $H(j_1) = H(j_2) = \cdots = H(j_q)$, then, by $i \in T_2$, $\exists u \in \{1, 2, \ldots, q-1\}$ such that $G_{j_uj_{u+1}} = 0$. Consequently, $P_{j_uj_{u+1}} = 0$.

If $\exists v \in \{2,3,\ldots,q-1\}$ such that $H(j_v) \neq H(j_1) = H(j_q)$, then $\exists w \in \{1,2,\ldots,q-1\}$ such that $H(j_w) < H(j_{w+1})$. Consequently, $P_{j_w j_{w+1}} = 0$. So that $i \in T$ because there always exists $t \in \{1,2,\ldots,q-1\}$ such that $P_{j_t j_{t+1}} = 0$. $\square$

The irreducible matrix G determines a directed graph G'. Further, the graph G' determines an undirected graph G'' if we define

$$[u,v] \text{ is an edge of } G'' \text{ if and only if } (u,v) \text{ or } (v,u) \text{ is an edge of } G'.$$

Now, the undirected graph G'' determines a neighbourhood system $\mathcal{N} = \{N(i) \mid i \in S\}$ on S, where

$$N(i) := \left\{ j \mid j \in S,\ j \neq i, \text{ and } [i,j] \text{ is an edge of } G'' \right\}.$$

This neighbourhood system $\mathcal{N}$ is symmetric, i.e., $j \in N(i)$ if and only if $i \in N(j)$. Thus, the irreducible matrix G determines a symmetric neighbourhood system $\mathcal{N}$ on S.

Definition 1.27. Let $i \in S$. We say that i is a *local minimum of* H (with respect to a neighbourhood system $\mathcal{N}$) if $\forall j \in N(i)$ we have $H(i) \leq H(j)$.

Set

$$S^* = S^*(H) = \left\{ i \mid i \in S \text{ and } H(i) = \min_{y \in S} H(y) \right\},$$

i.e., S^* is the set of global minima of H (it only depends on H) and

$$S^{**} = S^{**}(H,G) = \{ i \mid i \in S \text{ and } i \text{ is a local minimum of } H \},$$

i.e., S^{**} is the set of local minima of H (it only depends on H and G).

Remark 1.28. (a) We have $T_1 \cap S^* = \emptyset$.

(b) If

$$G_{ij} > 0 \text{ if and only if } G_{ji} > 0, \quad \forall i,j \in S,$$

then $R \subseteq S^{**}$.

(c) It is possible as $T_2 \cap S^* \neq \emptyset$ (therefore, it is possible as $T \cap S^* \neq \emptyset$; note that $T \cap S^* \neq \emptyset$ if and only if $S^* \nsubseteq R$), or $R \nsubseteq S^*$, or $T_1 \cap S^{**} \neq \emptyset$, or $T_2 \cap S^{**} \neq \emptyset$ (therefore, it is possible as $T \cap S^{**} \neq \emptyset$; note that $T \cap S^{**} \neq \emptyset$ if and only if $S^{**} \nsubseteq R$), or $R \nsubseteq S^{**}$.

Let $\varepsilon > 0$. By Theorem 1.25(i) applied to the simulated annealing we have

$$P(X_n \in R) > 1 - \varepsilon \quad \text{and} \quad P(X_n \in T) < \varepsilon, \quad \forall n \geq n_0 + n_1 := \bar{n},$$

i.e., the chain stick in a subset of the recurrent states (R is a union of recurrent classes) with a very large probability if ε is very small and $n \geq \bar{n}$. Clearly, this result is interesting from a practical point of view if the threshold $\bar{n}$ is small (?) in comparison with $|S|$. (Warning! $\bar{n}$ depends on ε, H, G, and $(\beta_n)_{n\geq 1}$.)

Remark 1.29. Since the analysis of simulated annealing is very intricate (even when S is small), one way to understand it is to find as many thresholds as possible which characterizes it. A threshold is given in [16] (see Remark 2.27(b) there) and another one is $\bar{n}$ above (at present we do not know hardly anything about the latter threshold).

THEOREM 1.30. *Consider the simulated annealing above* (H *is a nonconstant function,* G *is an* (*aperiodic or cyclic*) *irreducible stochastic matrix, and* $\beta_n \to \infty$ *and* $P_n \to P$ *as* $n \to \infty$). *Then each recurrent class of* P *is aperiodic.*

Proof. Let K be a recurrent class of P. Since H is a nonconstant function, we have $K \neq S$. Now, $\exists i \in K$, $\exists j \notin K$ such that $G_{ij} > 0$ because $K \neq S$ and G is irreducible. We show that $H(i) < H(j)$. Suppose that $H(i) \geq H(j)$.

Case 1. $H(i) = H(j)$. By definition of P, $P_{ij} = G_{ij}$. It follows that $P_{ij} > 0$, and we reached a contradiction.

Case 2. $H(i) > H(j)$. This implies $i \in T_1$, and we reached a contradiction.

Now, since $H(i) < H(j)$, G is a stochastic matrix, and $G_{ij} > 0$, we have $P_{ii} > 0$ (see the definition of P). Consequently, K is an aperiodic class. □

Theorem 1.30 says that the (classical) simulated annealing (chain) belongs to the collection of (finite) Markov chains $(P_n)_{n\geq 1}$ for which there exists a (stochastic) matrix P such that $P_n \to P$ as $n \to \infty$ and P has each recurrent class aperiodic. Hence, it raises the next problem related, in particular, to the convergence of simulated annealing (see also [5] for the convergence of simulated annealing).

Problem 1.31. Consider a Markov chain $(P_n)_{n\geq 1}$ with $P_n \to P$ as $n \to \infty$. Let R and T ($T \neq \emptyset$ or $T = \emptyset$) be the sets of recurrent and transient states of P, respectively. Suppose that each recurrent class is aperiodic. Is there $\Delta \in \mathrm{Par}(S)$ such that $(P_n)_{n\geq 1}$ is strongly Δ-ergodic?

Related to the above open problem we note.

Remark 1.32. (a) If R itself is a recurrent class, then is well-known that the answer to Problem 1.31 is in the affirmative with $\Delta = (S)$, i.e., the chain is strongly ergodic (see [4] and, in a more general setting [1, Proposition 7.13, p. 226]). Moreover, the chain is even uniformly strongly ergodic (see, e.g., [8]).

(b) We do not expect to obtain a stronger result than that from Problem 1.31, i.e., uniform strong Δ-ergodicity (see also (a)) instead of strong Δ-ergodicity. Indeed, let, e.g.,

$$P_n = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{n} & 1-\frac{1}{n} \end{pmatrix}, \quad \forall n \geq 1.$$

Then

$$P_n \to \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} := P \quad \text{as } n \to \infty,$$

so that P satisfies the conditions of Problem 1.31. Note that the chain $(P_n)_{n\geq 1}$ is strongly $(\{1,2\},\{3,4\})$-ergodic, but $\nexists \Delta \in \operatorname{Par}(\{1,2,3,4\})$ such that it is uniformly strongly Δ-ergodic. To prove this, we see that the chain $(P_n)_{n\geq 1}$ is uniformly weakly $(\{1,2\},\{3\},\{4\})$-ergodic (the states 3 and 4 are not in the same uniformly weakly ergodic class because

$$Q_n := \begin{pmatrix} 1 & 0 \\ \frac{1}{n} & 1-\frac{1}{n} \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} := Q \quad \text{as } n \to \infty,$$

and Q is not a mixing matrix (see [8, Theorem 2.7 and Example 2.9]; $P \in S_r$ is a *mixing matrix* if $\exists n \geq 1$ such that $\bar{\alpha}(P^n) < 1$)). However, the chain $(P_n)_{n\geq 1}$ has a limit behaviour which is partly uniform because the classes $\{1,2\}$ and $\{3\}$ are even uniformly strongly ergodic. Coming back to simulated annealing we can also expect such a limit behaviour in some cases; obviously, we need at least an example. This is an open problem.

(c) If the answer to Problem 1.31 is in the affirmative, then the answer to Problem 3.7 in [13] is also in the affirmative, so that we would have a strong ergodicity criterion for the case $P_n \to P$ as $n \to \infty$ with P having each recurrent class aperiodic.

2. LIMIT Δ-ERGODIC THEORY

In this section we set forth a simplified version of the limit Δ-ergodic theory given in [14]. But not only that; here we consider four limit equivalence relations (two of them are new and for the other ones see also [10], [12], and [14]). Also, we give some results (see [10–12] and [14] for others).

We shall agree that when writing

$$\lim_{u\to\infty} \lim_{v\to\infty} a_{uv},$$

where $a_{uv} \in \mathbf{R}$, $\forall u, v \in \mathbf{N}$ with $u \geq u_1, v \geq v_1(u)$, we assume that $\exists u_0 \geq u_1$ such that

$$\lim_{v \to \infty} a_{uv} \text{ exists}, \quad \forall u \geq u_0.$$

As in Section 1, we consider that $\Sigma \in \operatorname{Par}(S)$ and is, moreover, an ordered set. (Equivalently, we can use a σ-algebra, say $\mathcal{F}$, on S instead of $\Sigma \in \operatorname{Par}(S)$.) The three definitions below are simplified versions of Definitions 2.1–3 in [14], respectively.

Definition 2.1. Let $i, j \in S$. We say that i and j are in the same *limit weakly ergodic class on* Σ if $\forall K \in \Sigma$ we have

$$\lim_{m \to \infty} \lim_{n \to \infty} \sum_{k \in K} \left[(P_{m,n})_{ik} - (P_{m,n})_{jk} \right] = 0.$$

Write $i \overset{l,\Sigma}{\sim} j$ when i and j are in the same limit weakly ergodic class on Σ. Then $\overset{l,\Sigma}{\sim}$ is an equivalence relation and determines a partition $\bar{\Delta} = \bar{\Delta}(\Sigma) = (L_1, L_2, \ldots, L_u)$ of S. The sets $L_1, L_2, \ldots, L_u$ are called *limit weakly ergodic classes on* Σ.

Definition 2.2. Let $\bar{\Delta} = (L_1, L_2, \ldots, L_u)$ be the partition of limit weakly ergodic classes on Σ. We say that the chain is *limit weakly* $\bar{\Delta}$*-ergodic on* Σ. In particular, a limit weakly (S)-ergodic chain on Σ is called *limit weakly ergodic on* Σ for short.

Definition 2.3. Let $(L_1, L_2, \ldots, L_u)$ be the partition of limit weakly ergodic classes on Σ of a Markov chain with state space S and $\bar{\Delta} \in \operatorname{Par}(S)$. We say that the chain is *limit weakly* $[\bar{\Delta}]$*-ergodic on* Σ if $\bar{\Delta} \preceq (L_1, L_2, \ldots, L_u)$.

In the above definitions we have used $\bar{\Delta}$ only for differing from Section 1, where we have used Δ. This section is called 'Limit Δ-ergodic theory', but not 'Limit $\bar{\Delta}$-ergodic theory' because the former is simply a generic name.

The next definition is a generalization of one given in [12] (see below of Remark 1.29 there).

Definition 2.4. Let $i, j \in S$. We say that i and j are in the same *limit weakly ergodic class on* Σ *in a generalized sense* if $\forall K \in \Sigma$ we have

$$\limsup_{m \to \infty} \limsup_{n \to \infty} \left| \sum_{k \in K} \left[(P_{m,n})_{ik} - (P_{m,n})_{jk} \right] \right| = 0.$$

Write $i \overset{l,\Sigma,g}{\sim} j$ when i and j are in the same limit weakly ergodic class on Σ in a generalized sense. Then $\overset{l,\Sigma,g}{\sim}$ is an equivalence relation and determines a partition $\bar{\Delta} = \bar{\Delta}(\Sigma) = (M_1, M_2, \ldots, M_v)$ of S. The sets $M_1, M_2, \ldots, M_v$ are called *limit weakly ergodic classes on* Σ *in a generalized sense.*

Definition 2.5. Let $\bar{\Delta} = (M_1, M_2, \ldots, M_v)$ be the partition of limit weakly ergodic classes on Σ in a generalized sense. We say that the chain is *limit weakly* $\bar{\Delta}$*-ergodic on* Σ *in a generalized sense.* In particular, a limit weakly (S)-ergodic chain on Σ in a generalized sense is called *limit weakly ergodic on* Σ *in a generalized sense* for short.

Definition 2.6. Let $(M_1, M_2, \ldots, M_v)$ be the partition of limit weakly ergodic classes on Σ in a generalized sense of a Markov chain with state space S and $\bar{\Delta} \in \mathrm{Par}(S)$. We say that the chain is *limit weakly* $[\bar{\Delta}]$*-ergodic on* Σ *in a generalized sense* if $\bar{\Delta} \preceq (M_1, M_2, \ldots, M_v)$.

The two limit equivalence relations below are new.

Definition 2.7. Let $i, j \in S$. We say that i and j are in the same *limit uniformly weakly ergodic class on* Σ if $\forall K \in \Sigma$ we have

$$\lim_{l\to\infty} \lim_{n\to\infty} \sum_{k\in K} [(P_{n,n+l})_{ik} - (P_{n,n+l})_{jk}] = 0.$$

Write $i \overset{l,u,\Sigma}{\sim} j$ when i and j are in the same limit uniformly weakly ergodic class on Σ. Then $\overset{l,u,\Sigma}{\sim}$ is an equivalence relation and determines a partition $\bar{\Delta} = \bar{\Delta}(\Sigma) = (V_1, V_2, \ldots, V_s)$ of S. The sets $V_1, V_2, \ldots, V_s$ are called *limit uniformly weakly ergodic classes on* Σ.

Definition 2.8. Let $\bar{\Delta} = (V_1, V_2, \ldots, V_s)$ be the partition of limit uniformly weakly ergodic classes on Σ. We say that the chain is *limit uniformly weakly* $\bar{\Delta}$*-ergodic on* Σ. In particular, a limit uniformly weakly (S)-ergodic chain on Σ is called *limit uniformly weakly ergodic on* Σ for short.

Definition 2.9. Let $(V_1, V_2, \ldots, V_s)$ be the partition of limit uniformly weakly ergodic classes on Σ of a Markov chain with state space S and $\bar{\Delta} \in \mathrm{Par}(S)$. We say that the chain is *limit uniformly weakly* $[\bar{\Delta}]$*-ergodic on* Σ if $\bar{\Delta} \preceq (V_1, V_2, \ldots, V_s)$.

Definition 2.10. Let $i, j \in S$. We say that i and j are in the same *limit uniformly weakly ergodic class on* Σ *in a generalized sense* if $\forall K \in \Sigma$ we have

$$\limsup_{l\to\infty} \limsup_{n\to\infty} \left| \sum_{k\in K} \left[(P_{n,n+l})_{ik} - (P_{n,n+l})_{jk} \right] \right| = 0.$$

Write $i \overset{l,u,\Sigma,g}{\sim} j$ when i and j are in the same limit uniformly weakly ergodic class on Σ in a generalized sense. Then $\overset{l,u,\Sigma,g}{\sim}$ is an equivalence relation and determines a partition $\bar{\Delta} = \bar{\Delta}(\Sigma) = (W_1, W_2, \ldots, W_t)$ of S. The sets $W_1, W_2, \ldots, W_t$ are called *limit uniformly weakly ergodic classes on* Σ *in a generalized sense.*

Definition 2.11. Let $\bar{\Delta} = (W_1, W_2, \ldots, W_t)$ be the partition of limit uniformly weakly ergodic classes on Σ in a generalized sense. We say that the chain is *limit uniformly weakly* $\bar{\Delta}$*-ergodic on* Σ *in a generalized sense.* In particular, a limit uniformly weakly (S)-ergodic chain on Σ is called *limit uniformly weakly ergodic on* Σ *in a generalized sense* for short.

Definition 2.12. Let $(W_1, W_2, \ldots, W_t)$ be the partition of limit uniformly weakly ergodic classes on Σ in a generalized sense of a Markov chain with state space S and $\bar{\Delta} \in \mathrm{Par}(S)$. We say that the chain is *limit uniformly weakly* $\left[\bar{\Delta}\right]$*-ergodic on* Σ *in a generalized sense* if $\bar{\Delta} \preceq (W_1, W_2, \ldots, W_t)$.

$\overset{l,\Sigma}{\sim}$, $\overset{l,\Sigma,g}{\sim}$, $\overset{l,u,\Sigma}{\sim}$, and $\overset{l,u,\Sigma,g}{\sim}$ are called the *limit equivalence relations on* Σ; $\overset{l,\Sigma}{\sim}$ and $\overset{l,\Sigma,g}{\sim}$ are called the *simple limit equivalence relations on* Σ while $\overset{l,u,\Sigma}{\sim}$ and $\overset{l,u,\Sigma,g}{\sim}$ are called the *uniform limit equivalence relations on* Σ.

If $\Sigma = (\{i\})_{i\in S}$, then in the above definitions we can omit '*on* Σ' and can write $\overset{l}{\sim}$, $\overset{l,g}{\sim}$, $\overset{l,u}{\sim}$, and $\overset{l,u,g}{\sim}$ instead of $\overset{l,(\{i\})_{i\in S}}{\sim}$, $\overset{l,(\{i\})_{i\in S},g}{\sim}$, $\overset{l,u,(\{i\})_{i\in S}}{\sim}$, and $\overset{l,u,(\{i\})_{i\in S},g}{\sim}$, respectively.

The definition below is a simplified version of Definition 2.5 in [14].

Definition 2.13. Let L be a limit weakly ergodic class on Σ. Let $\emptyset \neq \Sigma_0 \subseteq \Sigma$. We say that L is a *limit strongly ergodic class on* Σ_0 *with respect to* Σ if $\forall i \in L$, $\forall K \in \Sigma_0$ the limit

$$\lim_{m\to\infty} \lim_{n\to\infty} \sum_{j\in K} (P_{m,n})_{ij} := \sigma_K = \sigma_K(L)$$

exists and does not depend on i.

Definition 2.14. Let L be a limit weakly ergodic class on Σ in a generalized sense. Let $\emptyset \neq \Sigma_0 \subseteq \Sigma$. We say that L is a *limit strongly ergodic class on* Σ_0 *with respect to* Σ *in a generalized sense* if $\forall i \in L$, $\forall K \in \Sigma_0, \exists \pi_K = \pi_K(L) \in [0,1]$ depending on K (and L) but not on i such that

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \left| \sum_{j\in K} (P_{m,n})_{ij} - \pi_K \right| = 0.$$

Definition 2.15. Let L be a limit uniformly weakly ergodic class on Σ. Let $\emptyset \neq \Sigma_0 \subseteq \Sigma$. We say that L is a *limit uniformly strongly ergodic class on* Σ_0 *with respect to* Σ if $\forall i \in L$, $\forall K \in \Sigma_0$ the limit

$$\lim_{l\to\infty} \lim_{n\to\infty} \sum_{j\in K} (P_{n,n+l})_{ij} := \tau_K = \tau_K(L)$$

exists and does not depend on i.

Definition 2.16. Let L be a limit uniformly weakly ergodic class on Σ in a generalized sense. Let $\emptyset \neq \Sigma_0 \subseteq \Sigma$. We say that L is a *limit uniformly strongly ergodic class on* Σ_0 *with respect to* Σ *in a generalized sense* if $\forall i \in L$, $\forall K \in \Sigma_0, \exists \pi_K = \pi_K(L) \in [0,1]$ depending on K (and L) but not on i such that

$$\limsup_{l\to\infty}\limsup_{n\to\infty}\left|\sum_{j\in K}(P_{n,n+l})_{ij} - \pi_K\right| = 0.$$

For simplification, in Definitions 2.13–16 we can replace '*on* Σ_0 *with respect to* Σ' with '*on* Σ' when $\Sigma_0 = \Sigma$ and can omit '*on* Σ' when $\Sigma_0 = \Sigma = (\{i\})_{i\in S}$.

Definition 2.17 (see also [14, Definition 2.6]). Let $(P_n)_{n\geq 1}$ be a limit weakly (respectively, uniformly weakly) $\bar{\Delta}$-ergodic Markov chain on Σ. We say that the chain is *limit strongly* (respectively, *uniformly strongly*) $\bar{\Delta}$*-ergodic on* Σ if any $L \in \bar{\Delta}$ is a limit strongly (respectively, uniformly strongly) ergodic class on Σ. In particular, a limit strongly (respectively, uniformly strongly) (S)-ergodic chain on Σ is called *limit strongly* (respectively, *uniformly strongly*) *ergodic on* Σ for short.

Definition 2.18 (see also [14, Definition 2.7]). Let $(P_n)_{n\geq 1}$ be a limit weakly (respectively, uniformly weakly) $[\bar{\Delta}]$-ergodic Markov chain on Σ. We say that the chain is *limit strongly* (respectively, *uniformly strongly*) $[\bar{\Delta}]$*-ergodic on* Σ if any $L \in \bar{\Delta}$ is included in a limit strongly (respectively, uniformly strongly) ergodic class on Σ.

Definition 2.19. Let $(P_n)_{n\geq 1}$ be a limit weakly (respectively, uniformly weakly) $\bar{\Delta}$-ergodic Markov chain on Σ in a generalized sense. We say that the chain is *limit strongly* (respectively, *uniformly strongly*) $\bar{\Delta}$*-ergodic on* Σ *in a generalized sense* if any $L \in \bar{\Delta}$ is a limit strongly (respectively, uniformly strongly) ergodic class on Σ in a generalized sense. In particular, a limit strongly (respectively, uniformly strongly) (S)-ergodic chain on Σ in a generalized sense is called *limit strongly* (respectively, *uniformly strongly*) *ergodic on* Σ *in a generalized sense* for short.

Definition 2.20. Let $(P_n)_{n\geq 1}$ be a limit weakly (respectively, uniformly weakly) $[\bar{\Delta}]$-ergodic Markov chain on Σ in a generalized sense. We say that the chain is *limit strongly* (respectively, *uniformly strongly*) $[\bar{\Delta}]$*-ergodic on* Σ *in a generalized sense* if any $L \in \bar{\Delta}$ is included in a limit strongly (respectively, uniformly strongly) ergodic class on Σ in a generalized sense.

Definition 2.21 (see also [14, Definition 2.19]). Let $(P_n)_{n\geq 1}$ be a limit strongly $[\bar{\Delta}]$- or $\bar{\Delta}$-ergodic Markov chain on Σ. We say that the chain has (*iterated*) *limit* Π if

$$\lim_{m\to\infty}\lim_{n\to\infty}(P_{m,n})^+ = \Pi.$$

(There always exists a Π with the above property and is unique because of Definitions 2.13 and 2.17–18. Moreover, Π is a $\left[\bar{\Delta}\right]$- or $\bar{\Delta}$-stable matrix.)

Definition 2.22. Let $(P_n)_{n\geq 1}$ be a limit strongly $\left[\bar{\Delta}\right]$- or $\bar{\Delta}$-ergodic Markov chain on Σ in a generalized sense. We say that the chain has (*iterated*) *limit* Π if

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\left|\left\|(P_{m,n})^+ - \Pi\right\|\right|_\infty = 0.$$

(There always exists a Π with the above property and is unique because of Definitions 2.14 and 2.19–20. Moreover, Π is a $\left[\bar{\Delta}\right]$- or $\bar{\Delta}$-stable matrix.)

Definition 2.23. Let $(P_n)_{n\geq 1}$ be a limit uniformly strongly $\left[\bar{\Delta}\right]$- or $\bar{\Delta}$-ergodic Markov chain on Σ. We say that the chain has (*iterated*) *limit* Π if

$$\lim_{l\to\infty}\lim_{n\to\infty}(P_{n,n+l})^+ = \Pi.$$

Definition 2.24. Let $(P_n)_{n\geq 1}$ be a limit uniformly strongly $\left[\bar{\Delta}\right]$- or $\bar{\Delta}$-ergodic Markov chain on Σ in a generalized sense. We say that the chain has (*iterated*) *limit* Π if

$$\limsup_{l\to\infty}\limsup_{n\to\infty}\left|\left\|(P_{n,n+l})^+ - \Pi\right\|\right|_\infty = 0.$$

An example of a limit uniformly strongly $\bar{\Delta}$-ergodic Markov chain on Σ in a generalized sense is $(P_n)_{n\geq 1}$ with $\Sigma = (R, T)$ from Theorem 1.23. It has the iterated limit Π, where $\Pi = \lim_{l\to\infty}\left(P^l\right)^+$ $(P = \lim_{n\to\infty} P_n)$. (The proof is straightforward; for another proof see Theorem 2.31(iv).)

In Definitions 2.17–24 we can omit '*on* Σ' if $\Sigma = (\{i\})_{i\in S}$.

The next result makes some basic connections between Δ-ergodic theory and limit Δ-ergodic theory.

Theorem 2.25. *Let* $(P_n)_{n\geq 1}$ be *a Markov chain.*

(i) *The next statements are equivalent* (see *also* [10, *Theorem* 2.24]).

(i1) *The chain is weakly ergodic on* Σ.

(i2) *The chain is limit weakly ergodic on* Σ.

(i3) *The chain is limit weakly ergodic on* Σ *in a generalized sense.*

(ii) *The chain is strongly ergodic on* Σ *and has limit* Π *if and only if it is limit strongly ergodic on* Σ *and has limit* Π.

(iii) *The chain is uniformly weakly ergodic if and only if it is limit uniformly weakly ergodic in a generalized sense* (see *also* [10, *Theorem* 2.29]).

(iv) *The chain is uniformly strongly ergodic and has limit* Π *if and only if it is limit uniformly strongly ergodic in a generalized sense and has limit* Π.

Proof. (i) (i1)⇒(i2) Obvious.

(i2)⇒(i3) Obvious.

(i3)⇒(i1) Obviously, the chain is limit weakly ergodic on Σ in a generalized sense if and only if

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\bar{\alpha}\left((P_{m,n})^{+}\right)=0.$$

Let $\varepsilon>0$ and $a_m=\limsup_{n\to\infty}\bar{\alpha}\left((P_{m,n})^{+}\right)$, $\forall m\geq 0$. Since $\lim_{m\to\infty}a_m=0$, $\exists m_\varepsilon\geq 0$ such that $a_m<\varepsilon$, $\forall m\geq m_\varepsilon$. It follows that $\exists n_{\varepsilon,m}>m$ such that $\bar{\alpha}\left((P_{m,n})^{+}\right)<\varepsilon$, $\forall m\geq m_\varepsilon$, $\forall n\geq n_{\varepsilon,m}$.

Let $l\geq 0$.

Case 1. $l<m_\varepsilon$ (when $m_\varepsilon>0$). Using the inequality

$$\bar{\alpha}\left(PQ\right)\leq\bar{\alpha}\left(P\right)\bar{\alpha}\left(Q\right),\quad\forall P\in S_{m,n},\ \forall Q\in S_{n,p}$$

(see, e.g., [1, pp. 58–59] or [9]), we have

$$\bar{\alpha}\left((P_{l,n})^{+}\right)\leq\bar{\alpha}\left(P_{l,m_\varepsilon}\right)\bar{\alpha}\left((P_{m_\varepsilon,n})^{+}\right)\leq\bar{\alpha}\left((P_{m_\varepsilon,n})^{+}\right)<\varepsilon,\quad\forall n\geq n_{\varepsilon,m_\varepsilon}.$$

Case 2. $l\geq m_\varepsilon$. In this case, we have

$$\bar{\alpha}\left((P_{l,n})^{+}\right)<\varepsilon,\quad\forall n\geq n_{\varepsilon,l}.$$

Consequently, from Cases 1 and 2 we have $\lim_{n\to\infty}\bar{\alpha}\left((P_{l,n})^{+}\right)=0$, $\forall l\geq 0$. Obviously, the chain is weakly ergodic on Σ if and only if $\lim_{n\to\infty}\bar{\alpha}\left((P_{l,n})^{+}\right)=0$, $\forall l\geq 0$. Therefore, (i1) holds.

(ii) See [14, Theorem 2.20].

(iii) The chain is uniformly weakly ergodic if and only if

$$\limsup_{l\to\infty}\limsup_{n\to\infty}\bar{\alpha}(P_{n,n+l})=0$$

(see [7, Theorem 3.3]). Obviously, the chain is limit uniformly weakly ergodic in a generalized sense if and only if

$$\limsup_{l\to\infty}\limsup_{n\to\infty}\bar{\alpha}(P_{n,n+l})=0.$$

Consequently, (iii) holds.

(iv) See [12, Theorem 2.6]. □

Remark 2.26. To define the uniform limit equivalence relations a starting point was [7, Theorem 3.3] (then see also the proof of Theorem 2.25(iii) above).

The limit equivalence relations together with the notions derived from these are good tools for the study of perturbed Markov chains. More precisely, the simple limit equivalence relations are useful when the perturbation is of the first type while the uniform limit equivalence relations are useful when the perturbation is of the second type (see mainly the looping method in [12]). Here we only state a few basic results (see also [12] and [14] for others).

Definition 2.27 ([12]). Let $(P_n)_{n\geq 1}$ and $(P'_n)_{n\geq 1}$ be two Markov chains. We say that $(P'_n)_{n\geq 1}$ is a *perturbation of the first type of* $(P_n)_{n\geq 1}$ if

$$\sum_{n\geq 1} \left|\left|\left|P_n - P'_n\right|\right|\right|_\infty < \infty.$$

Definition 2.28 ([12]). Let $(P_n)_{n\geq 1}$ and $(P'_n)_{n\geq 1}$ be two Markov chains. We say that $(P'_n)_{n\geq 1}$ is a *perturbation of the second type of* $(P_n)_{n\geq 1}$ if

$$\left|\left|\left|P_n - P'_n\right|\right|\right|_\infty \to 0 \quad \text{as } n \to \infty$$

(this is equivalent to $P_n - P'_n \to 0$ as $n \to \infty$).

THEOREM 2.29. *Let* $(P_n)_{n\geq 1}$ *and* $(P'_n)_{n\geq 1}$ *be two Markov chains. Then*

$$\left|\left|\left|(P_{m,n})^+ - (P'_{m,n})^+\right|\right|\right|_\infty \leq \left|\left|\left|P_{m,n} - P'_{m,n}\right|\right|\right|_\infty \leq \sum_{u=1}^{n-m} \left|\left|\left|P_{m+u} - P'_{m+u}\right|\right|\right|_\infty .$$

Proof. See [14, Proposition 2.21]. $\square$

THEOREM 2.30. *Let* $(P_n)_{n\geq 1}$ *be a Markov chain and* $(P'_n)_{n\geq 1}$ *a perturbation of the first type of it.*

(i) $(P_n)_{n\geq 1}$ *is limit weakly* $\bar{\Delta}$*-ergodic on* Σ *in a generalized sense if and only if* $(P'_n)_{n\geq 1}$ *is limit weakly* $\bar{\Delta}$*-ergodic on* Σ *in a generalized sense.*

(ii) $(P_n)_{n\geq 1}$ *is limit strongly* $\bar{\Delta}$*-ergodic on* Σ *in a generalized sense and has limit* Π *if and only if* $(P_n)_{n\geq 1}$ *is limit strongly* $\bar{\Delta}$*-ergodic on* Σ *in a generalized sense and has limit* Π.

Proof. By symmetry, it is sufficient to only prove an implication for both statements.

(i) "$\Rightarrow$" Let $i, j \in S$. Suppose that $i \overset{l,\Sigma,g}{\sim} j$ for $(P_n)_{n\geq 1}$ and prove that $i \overset{l,\Sigma,g}{\sim} j$ for $(P'_n)_{n\geq 1}$. (The conclusion (i) is equivalent to $i \overset{l,\Sigma,g}{\sim} j$ for $(P_n)_{n\geq 1}$ if and only if $i \overset{l,\Sigma,g}{\sim} j$ for $(P'_n)_{n\geq 1}$.) From

$$\left|\sum_{k\in K}\left[(P'_{m,n})_{ik} - (P'_{m,n})_{jk}\right]\right| \leq \left|\sum_{k\in K}\left[(P'_{m,n})_{ik} - (P_{m,n})_{ik}\right]\right| +$$

$$+\left|\sum_{k\in K}\left[(P_{m,n})_{ik} - (P_{m,n})_{jk}\right]\right| + \left|\sum_{k\in K}\left[(P_{m,n})_{jk} - \left(P'_{m,n}\right)_{jk}\right]\right| \leq$$

$$\leq \left| \sum_{k \in K} [(P_{m,n})_{ik} - (P_{m,n})_{jk}] \right| + 2 \left|\left|\left| (P_{m,n})^+ - (P'_{m,n})^+ \right|\right|\right|_\infty ,$$

$\forall m, n,\ 0 \leq m < n,\ \forall K \in \Sigma$, using the hypothesis and Theorem 2.29, we have

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \left| \sum_{k \in K} \left[(P'_{m,n})_{ik} - (P'_{m,n})_{jk} \right] \right| = 0, \quad \forall K \in \Sigma,$$

i.e., $i \overset{l,\Sigma,g}{\sim} j$ for $(P'_n)_{n\geq 1}$.

(ii) "⇒" From

$$\left|\left|\left| (P'_{m,n})^+ - \Pi \right|\right|\right|_\infty \leq \left|\left|\left| (P'_{m,n})^+ - (P_{m,n})^+ \right|\right|\right|_\infty + \left|\left|\left| (P_{m,n})^+ - \Pi \right|\right|\right|_\infty ,$$

$\forall m, n,\ 0 \leq m < n$, using the hypothesis and Theorem 2.29, we have

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \left|\left|\left| (P'_{m,n})^+ - \Pi \right|\right|\right|_\infty = 0.$$

By (i), $(P'_n)_{n\geq 1}$ is limit weakly $\bar{\Delta}$-ergodic on Σ in a generalized sense. Further, $\limsup_{m\to\infty} \limsup_{n\to\infty} \left|\left|\left| (P'_{m,n})^+ - \Pi \right|\right|\right|_\infty = 0$ and the fact that $(P'_n)_{n\geq 1}$ is limit weakly $\bar{\Delta}$-ergodic on Σ in a generalized sense imply that $(P'_n)_{n\geq 1}$ is limit strongly $\bar{\Delta}$-ergodic on Σ in a generalized sense and has limit Π. □

THEOREM 2.31. *Let $(P_n)_{n\geq 1}$ be a Markov chain and $(P'_n)_{n\geq 1}$ a perturbation of the second type of it.*

(i) *$(P_n)_{n\geq 1}$ is limit uniformly weakly $\bar{\Delta}$-ergodic on Σ if and only if $(P'_n)_{n\geq 1}$ is limit uniformly weakly $\bar{\Delta}$-ergodic on Σ.*

(ii) *$(P_n)_{n\geq 1}$ is limit uniformly strongly $\bar{\Delta}$-ergodic on Σ and has limit Π if and only if $(P'_n)_{n\geq 1}$ is limit uniformly strongly $\bar{\Delta}$-ergodic on Σ and has limit Π.*

(iii) *$(P_n)_{n\geq 1}$ is limit uniformly weakly $\bar{\Delta}$-ergodic on Σ in a generalized sense if and only if $(P'_n)_{n\geq 1}$ is limit uniformly weakly $\bar{\Delta}$-ergodic on Σ in a generalized sense.*

(iv) *$(P_n)_{n\geq 1}$ is limit uniformly strongly $\bar{\Delta}$-ergodic on Σ in a generalized sense and has limit Π if and only if $(P'_n)_{n\geq 1}$ is limit uniformly strongly $\bar{\Delta}$-ergodic on Σ in a generalized sense and has limit Π.*

Proof. By symmetry, it is sufficient to only prove an implication for all statements.

(i) "⇒" Let $i, j \in S$. Suppose that $i \overset{l,u,\Sigma}{\sim} j$ for $(P_n)_{n\geq 1}$ and prove that $i \overset{l,u,\Sigma}{\sim} j$ for $(P'_n)_{n\geq 1}$. By $i \overset{l,u,\Sigma}{\sim} j$,

$$\lim_{l\to\infty} \lim_{n\to\infty} \sum_{k \in K} \left[(P_{n,n+l})_{ik} - (P_{n,n+l})_{jk} \right] = 0, \quad \forall K \in \Sigma.$$

Further, setting $P'_n = P_n + Q_n$, $\forall n \geq 1$, we have $\lim_{n\to\infty} (P'_{n,n+l} - P_{n,n+l}) = 0$, $\forall l \geq 1$, because $\lim_{n\to\infty} Q_n = 0$.

Finally, from

$$\sum_{k\in K} \left[(P'_{n,n+l})_{ik} - \left(P'_{n,n+l}\right)_{jk} \right] = \sum_{k\in K} \left[\left(P'_{n,n+l}\right)_{ik} - (P_{n,n+l})_{ik} \right] +$$

$$+ \sum_{k\in K} \left[(P_{n,n+l})_{ik} - (P_{n,n+l})_{jk} \right] + \sum_{k\in K} \left[(P_{n,n+l})_{jk} - \left(P'_{n,n+l}\right)_{jk} \right],$$

$\forall l \geq 1,\ \forall n \geq 0,\ \forall K \in \Sigma$, we have

$$\lim_{l\to\infty} \lim_{n\to\infty} \sum_{k\in K} \left[\left(P'_{n,n+l}\right)_{ik} - \left(P'_{n,n+l}\right)_{jk} \right] = 0, \quad \forall K \in \Sigma,$$

i.e., $i \overset{l,u,\Sigma}{\sim} j$ for $(P'_n)_{n\geq 1}$.

(ii) "$\Rightarrow$" By hypothesis,

$$\lim_{l\to\infty} \lim_{n\to\infty} \left[(P_{n,n+l})^+ - \Pi \right] = 0.$$

Obviously, $\lim_{n\to\infty} \left(P'_{n,n+l} - P_{n,n+l}\right) = 0$, $\forall l \geq 1$ (see the proof of (i)), implies that

$$\lim_{n\to\infty} \left[\left(P'_{n,n+l}\right)^+ - (P_{n,n+l})^+ \right] = 0, \quad \forall l \geq 1.$$

Now, from

$$\left(P'_{n,n+l}\right)^+ - \Pi = \left[\left(P'_{n,n+l}\right)^+ - (P_{n,n+l})^+ \right] + \left[(P_{n,n+l})^+ - \Pi \right], \quad \forall l \geq 1,\ \forall n \geq 0,$$

we have

$$\lim_{l\to\infty} \lim_{n\to\infty} \left[\left(P'_{n,n+l}\right)^+ - \Pi \right] = 0.$$

By (i), $(P'_n)_{n\geq 1}$ is limit uniformly weakly $\bar{\Delta}$-ergodic on Σ. Further, $\lim_{l\to\infty} \lim_{n\to\infty} \left[(P'_{n,n+l})^+ - \Pi \right] = 0$ and the fact that $(P'_n)_{n\geq 1}$ is limit uniformly weakly $\bar{\Delta}$-ergodic on Σ imply that $(P'_n)_{n\geq 1}$ is limit uniformly strongly $\bar{\Delta}$-ergodic on Σ and has limit Π.

(iii) "$\Rightarrow$" Let $i, j \in S$. Suppose that $i \overset{l,u,\Sigma,g}{\sim} j$ for $(P_n)_{n\geq 1}$ and prove that $i \overset{l,u,\Sigma,g}{\sim} j$ for $(P'_n)_{n\geq 1}$. From

$$\left| \sum_{k\in K} \left[\left(P'_{n,n+l}\right)_{ik} - \left(P'_{n,n+l}\right)_{jk} \right] \right| \leq \left| \sum_{k\in K} \left[\left(P'_{n,n+l}\right)_{ik} - (P_{n,n+l})_{ik} \right] \right| +$$

$$+ \left| \sum_{k\in K} \left[(P_{n,n+l})_{ik} - (P_{n,n+l})_{jk} \right] \right| + \left| \sum_{k\in K} \left[(P_{n,n+l})_{jk} - \left(P'_{n,n+l}\right)_{jk} \right] \right|,$$

$\forall l \geq 1,\ \forall n \geq 0,\ \forall K \in \Sigma$, we have

$$\limsup_{l\to\infty} \limsup_{n\to\infty} \left| \sum_{k\in K} \left[\left(P'_{n,n+l}\right)_{ik} - \left(P'_{n,n+l}\right)_{jk} \right] \right| = 0, \quad \forall K \in \Sigma,$$

i.e., $i \overset{l,u,\Sigma,g}{\sim} j$ for $(P'_n)_{n\geq 1}$.

(iv) "$\Rightarrow$" By hypothesis,

$$\limsup_{l\to\infty} \limsup_{n\to\infty} \left|\left|\left| (P_{n,n+l})^+ - \Pi \right|\right|\right|_\infty = 0.$$

Now, from

$$\left|\left|\left| (P'_{n,n+l})^+ - \Pi \right|\right|\right|_\infty \leq \left|\left|\left| (P'_{n,n+l})^+ - (P_{n,n+l})^+ \right|\right|\right|_\infty + \\ + \left|\left|\left| (P_{n,n+l})^+ - \Pi \right|\right|\right|_\infty, \quad \forall l \geq 1,\ \forall n \geq 0,$$

we have

$$\limsup_{l\to\infty} \limsup_{n\to\infty} \left|\left|\left| (P'_{n,n+l})^+ - \Pi \right|\right|\right|_\infty = 0.$$

By (iii), $(P'_n)_{n\geq 1}$ is limit uniformly weakly $\bar{\Delta}$-ergodic on Σ in a generalized sense. Further, $\limsup_{l\to\infty} \limsup_{n\to\infty} |||(P'_{n,n+l})^+ - \Pi|||_\infty = 0$ and the fact that $(P'_n)_{n\geq 1}$ is limit uniformly weakly $\bar{\Delta}$-ergodic on Σ in a generalized sense imply that $(P'_n)_{n\geq 1}$ is limit uniformly strongly $\bar{\Delta}$-ergodic on Σ in a generalized sense and has limit Π. $\square$

Theorems 2.30–31 say that under certain conditions the chains $(P_n)_{n\geq 1}$ and $(P'_n)_{n\geq 1}$ have an identical iterated limit behaviour (concerning the matrix products $P_{m,n}$ and $P'_{m,n}$, respectively (see Section 1)). This is an important fact (see the looping method in [12], Theorem 2.25 above, a.s.o.).

We conclude this article specifying that much remains to be done both in general Δ-ergodic theory and in simulated annealing theory (obviously, the former is the natural framework for the latter).

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