ON A SUBCLASS OF UNIVALENT FUNCTIONS DEFINED BY A GENERALIZED DIFFERENTIAL OPERATOR

DORINA RĂDUCANU

In this paper we investigate a new subclass of univalent functions defined by a generalized differential operator. An inclusion result, structural formula, extreme points and other properties of this class of functions are obtained.

AMS 2010 Subject Classification: 30C45.

Key words: analytic functions, differential operator, convex functions, structural formula.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$.

By S and C we denote the subclasses of functions in $\mathcal{A}$ which are univalent and convex in $\mathbb{U}$, respectively.

Let $\mathcal{P}$ be the well-known Carathéodory class of normalized functions with positive real part in $\mathbb{U}$ and let $\mathcal{P}(\lambda)$, $0 \leq \lambda < 1$ be the subclass of $\mathcal{P}$ consisting of functions with real part greater than λ.

The Hadamard product or convolution of the functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}.$$

Let $f \in \mathcal{A}$. We consider the following differential operator introduced by Răducanu and Orhan [13]:

$$D^0_{\alpha\beta} f(z) = f(z),$$
$$D^1_{\alpha\beta} f(z) = D_{\alpha\beta} f(z) = \alpha\beta z^2 f''(z) + (\alpha - \beta) z f'(z) + (1 - \alpha + \beta) f(z),$$
$$D^m_{\alpha\beta} f(z) = D_{\alpha\beta}\big(D^{m-1}_{\alpha\beta} f(z)\big), \tag{2}$$

where $0 \le \beta \le \alpha$ and $m \in \mathbb{N} := \{1, 2, \ldots\}$.

If the function f is given by (1) then, from (2) we see that

$$D^m_{\alpha\beta} f(z) = z + \sum_{n=2}^{\infty} A_n(\alpha, \beta, m) a_n z^n, \tag{3}$$

where

$$A_n(\alpha, \beta, m) = [1 + (\alpha\beta n + \alpha - \beta)(n-1)]^m \,. \tag{4}$$

When $\alpha = 1$ and $\beta = 0$, we get Sălăgean differential operator [14]. When $\beta = 0$, we obtain the differential operator defined by Al-Oboudi [1].

From (3) it follows that $D^m_{\alpha\beta} f(z)$ can be written in terms of convolution as

$$D^m_{\alpha\beta} f(z) = (f * g)(z), \tag{5}$$

where

$$g(z) = z + \sum_{n=2}^{\infty} A_n(\alpha, \beta, m) z^n. \tag{6}$$

We say that a function $f \in \mathcal{A}$ is in the class $R^m(\alpha, \beta, \lambda)$ if $[D^m_{\alpha\beta} f(z)]'$ is in the class $\mathcal{P}(\lambda)$, that is, if

$$\operatorname{Re}\left[D^m_{\alpha\beta} f(z)\right]' > \lambda, \quad z \in \mathbb{U} \tag{7}$$

for $0 \le \lambda < 1$, $0 \le \beta \le \alpha$ and $m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$. For $\beta = 0$, we obtain the class of functions considered in [1].

The main object of this paper is to present a systematic investigation for the class $R^m(\alpha, \beta, \lambda)$. In particular, for this function class, we derive an inclusion result, structural formula, extreme points and other interesting properties.

2. INCLUSION RESULT

In order to obtain the inclusion result for the class $R^m(\alpha, \beta, \lambda)$, we need the following lemma due to Miller and Mocanu [12, Theorem 1f, p. 198].

Lemma 2.1. *Let $h \in \mathbb{C}$ and let $A \geq 0$. Suppose that B and D are analytic in $\mathbb{U}$, with $D(0) = 0$ and*

$$\operatorname{Re} B(z) \geq A + 4\left|\frac{D(z)}{h'(0)}\right|$$

for $z \in \mathbb{U}$. If an analytic function p, with $p(0) = h(0)$ satisfies

$$Az^2p''(z) + B(z)zp'(z) + p(z) + D(z) \prec h(z), \quad z \in \mathbb{U}$$

then $p(z) \prec h(z)$, $z \in \mathbb{U}$.

Note that the symbol "$\prec$" stands for subordination.

Theorem 2.1. *Let $0 \leq \lambda < 1$, $0 \leq \beta \leq \alpha$ and $m \in \mathbb{N}_0$. Then*

$$R^{m+1}(\alpha, \beta, \lambda) \subset R^m(\alpha, \beta, \lambda).$$

Proof. Suppose $f \in R^{m+1}(\alpha, \beta, \lambda)$. Then

$$\operatorname{Re}\left[D^{m+1}_{\alpha\beta} f(z)\right]' > \lambda$$

which is equivalent to

$$\left[D^{m+1}_{\alpha\beta} f(z)\right]' \prec h(z), \quad z \in \mathbb{U}, \tag{8}$$

where

$$h(z) := \frac{1 + (1 - 2\lambda)z}{1 - z}, \quad z \in \mathbb{U}. \tag{9}$$

From (2), we have

$$D^{m+1}_{\alpha\beta} f(z) = \alpha\beta z^2 \left[D^m_{\alpha\beta} f(z)\right]'' + (\alpha - \beta)z\left[D^m_{\alpha\beta} f(z)\right]' + (1 - \alpha + \beta)D^m_{\alpha\beta} f(z).$$

It follows that

(10)

$$\left[D^{m+1}_{\alpha\beta} f(z)\right]' = \alpha\beta z^2 \left[D^m_{\alpha\beta} f(z)\right]''' + (2\alpha\beta + \alpha - \beta)z\left[D^m_{\alpha\beta} f(z)\right]'' + \left[D^m_{\alpha\beta} f(z)\right]'.$$

Denote

$$p(z) := \left[D^m_{\alpha\beta} f(z)\right]', \quad z \in \mathbb{U}. \tag{11}$$

Making use of (10) and (11), the differential subordination (8) becomes

$$\alpha\beta z^2 p''(z) + (2\alpha\beta + \alpha - \beta)zp'(z) + p(z) \prec h(z), \quad z \in \mathbb{U}.$$

It is easy to check that the conditions of Lemma 2.1 with $h(z)$ given by (9), $p(z)$ given by (11), $A = \alpha\beta$, $B(z) \equiv 2\alpha\beta + \alpha - \beta$ and $D(z) \equiv 0$ are satisfied. Thus, we obtain $p(z) \prec h(z)$ which implies that

$$\operatorname{Re}\left[D^m_{\alpha\beta} f(z)\right]' > \lambda, \quad z \in \mathbb{U}.$$

Therefore, $f \in R^m(\alpha, \beta, \lambda)$ and the proof of our theorem is completed. $\square$

Corollary 2.1. *Let* $0 \le \lambda < 1$, $0 \le \beta \le \alpha$ *and* $m \in \mathbb{N}_0$. *Then*

$$R^m(\alpha,\beta,\lambda) \subset S.$$

Proof. Making use of Theorem 2.1, we obtain

$$R^m(\alpha,\beta,\lambda) \subset R^{m-1}(\alpha,\beta,\lambda) \subset \cdots \subset R^0(\alpha,\beta,\lambda).$$

The class $R^0(\alpha,\beta,\lambda)$ consists of functions $f \in \mathcal{A}$ for which $\mathrm{Re}[D^0_{\alpha,\beta}f(z)]' > \lambda$, that is $\mathrm{Re}\, f'(z) > \lambda$. It is known (see [9] and also [7]) that, if $\mathrm{Re}\, f'(z) > \lambda$, $0 \le \lambda < 1$, then f is univalent. Thus,

$$R^m(\alpha,\beta,\lambda) \subset R^0(\alpha,\beta,\lambda) \subset S. \quad \square$$

3. STRUCTURAL FORMULA

In this section a structural formula, extreme points and coefficient bounds for functions in $R^m(\alpha,\beta,\lambda)$ are obtained.

Theorem 3.1. *A function* $f \in \mathcal{A}$ *is in the class* $R^m(\alpha,\beta,\lambda)$ *if and only if it can be expressed as*

$$f(z) = \left[z + \sum_{n=2}^{\infty} \frac{1}{A_n(\alpha,\beta,m)} z^n\right] * \int_{|\zeta|=1} \left[z + 2(1-\lambda)\bar{\zeta} \sum_{n=2}^{\infty} \frac{(\zeta z)^n}{n}\right] \mathrm{d}\mu(\zeta), \tag{12}$$

where μ *is a positive Borel probability measure defined on the unit circle* $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$.

Proof. From (5) it follows that, $f \in R^m(\alpha,\beta,\lambda)$ if and only if

$$\frac{\left[D^m_{\alpha\beta} f(z)\right]' - \lambda}{1-\lambda} \in \mathcal{P}.$$

Using Herglotz integral representation of functions in Carathéodory class $\mathcal{P}$ (see [8] and also [10]), there exists a positive Borel probability measure μ such that

$$\frac{\left[D^m_{\alpha\beta} f(z)\right]' - \lambda}{1-\lambda} = \int_{|\zeta|=1} \frac{1+\zeta z}{1-\zeta z}\, \mathrm{d}\mu(\zeta), \quad z \in \mathbb{U}$$

which is equivalent to

$$\left[D^m_{\alpha\beta} f(z)\right]' = \int_{|\zeta|=1} \frac{1+(1-2\lambda)\zeta z}{1-\zeta z}\, \mathrm{d}\mu(\zeta).$$

Integrating this last equality, we obtain

$$D^m_{\alpha\beta}f(z) = \int_0^z \left[\int_{|\zeta|=1} \frac{1+(1-2\lambda)\zeta u}{1-\zeta u} \mathrm{d}\mu(\zeta)\right] \mathrm{d}u =$$
$$= \int_{|\zeta|=1} \left[\int_0^z \frac{1+(1-2\lambda)\zeta u}{1-\zeta u} \mathrm{d}u\right] \mathrm{d}\mu(\zeta)$$

that is

$$D^m_{\alpha\beta}f(z) = \int_{|\zeta|=1} \left[z + 2(1-\lambda)\bar{\zeta} \sum_{n=2}^{\infty} \frac{(\zeta z)^n}{n}\right] \mathrm{d}\mu(\zeta). \tag{13}$$

From (5), (6) and (13) it follows that

$$f(z) = \left[z + \sum_{n=2}^{\infty} \frac{1}{A_n(\alpha,\beta,m)} z^n\right] * \int_{|\zeta|=1} \left[z + 2(1-\lambda)\bar{\zeta} \sum_{n=2}^{\infty} \frac{(\zeta z)^n}{n}\right] \mathrm{d}\mu(\zeta).$$

Since this deductive process can be converse, we have proved our theorem. □

Corollary 3.1. *The extreme points of the class $R^m(\alpha,\beta,\lambda)$ are*

$$f_\zeta(z) = z + 2(1-\lambda)\bar{\zeta} \sum_{n=2}^{\infty} \frac{(\zeta z)^n}{nA_n(\alpha,\beta,m)}, \quad z \in \mathbb{U},\ |\zeta| = 1. \tag{14}$$

Proof. Consider the functions

$$g_\zeta(z) = z + 2(1-\lambda)\bar{\zeta} \sum_{n=2}^{\infty} \frac{(\zeta z)^n}{n}$$

and

$$g_\mu(z) = \int_{|\zeta|=1} g_\zeta(z) \mathrm{d}\mu(\zeta).$$

Since the map $\mu \to g_\mu$ is one-to-one, making use of (5), (6) and (13), the assertion follows from (12) (see [5]).

From Corollary 3.1 we can obtain coefficient bounds for the functions in the class $R^m(\alpha,\beta,\lambda)$.

Corollary 3.2. *If $f \in R^m(\alpha,\beta,\lambda)$ is given by (1) then*

$$|a_n| \le \frac{2(1-\lambda)}{nA_n(\alpha,\beta,m)}, \quad n \ge 2.$$

The result is sharp.

Proof. The coefficient bounds are maximized at an extreme point. Therefore, the result follows from (14).

Corollary 3.3. *If* $f \in R^m(\alpha, \beta, \lambda)$ *then, for* $|z| = r < 1$

$$r - 2(1-\lambda)r^2 \sum_{n=2}^{\infty} \frac{1}{nA_n(\alpha,\beta,m)} \le |f(z)| \le r + 2(1-\lambda)r^2 \sum_{n=2}^{\infty} \frac{1}{nA_n(\alpha,\beta,m)}$$

and

$$1 - 2(1-\lambda)r \sum_{n=2}^{\infty} \frac{1}{A_n(\alpha,\beta,m)} \le |f'(z)| \le 1 + 2(1-\lambda)r \sum_{n=2}^{\infty} \frac{1}{A_n(\alpha,\beta,m)}.$$

4. CONVOLUTION PROPERTY

In order to prove a convolution property for the class $R^m(\alpha, \beta, \lambda)$, we need the following result.

Lemma 4.1 [15]. *If* $p(z)$ *is analytic in* $\mathbb{U}$, $p(0) = 1$ *and* $\operatorname{Re} p(z) > \frac{1}{2}$ *then, for any function* F *analytic in* $\mathbb{U}$*, the function* $F * p$ *takes values in the convex hull of* $F(\mathbb{U})$.

Theorem 4.1. *The class* $R^m(\alpha, \beta, \lambda)$ *is closed under the convolution with a convex function. That is, if* $f \in R^m(\alpha, \beta, \lambda)$ *and* $g \in C$ *then* $f * g \in R^m(\alpha, \beta, \lambda)$.

Proof. Let $g \in C$. Then (see [12])

$$\operatorname{Re} \frac{g(z)}{z} > \frac{1}{2}.$$

Suppose $f \in R^m(\alpha, \beta, \lambda)$. Making use of the convolution properties, we have

$$\operatorname{Re} \left[D^m_{\alpha\beta}(f * g)(z)\right]' = \operatorname{Re} \left[\left(D^m_{\alpha\beta} f(z)\right)' * \frac{g(z)}{z}\right].$$

By applying Lemma 4.1, the result follows.

Corollary 4.1. *The class* $R^m(\alpha, \beta, \lambda)$ *is invariant under Bernardi integral operator* [4]:

$$F_c(f)(z) = \frac{1+c}{z^c} \int_0^z t^{c-1} f(t) \mathrm{d}t, \quad \operatorname{Re} c > 0.$$

Proof. Assume $f \in R^m(\alpha, \beta, \lambda)$. It is easy to check that $F_c(f)(z) = (f * g)(z)$, where

$$g(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n = \frac{1+c}{z^c} \int_0^z \frac{t^c}{1-t} \mathrm{d}t, \quad z \in \mathbb{U},\ \operatorname{Re} c > 0.$$

Since the function $\phi(z) = \frac{z}{1-z}, z \in \mathbb{U}$ is convex, it follows (see [11]) that the function g is also convex. From Theorem 4.1 we obtain $F_c(f) \in R^m(\alpha, \beta, \lambda)$. Therefore, $F_c\left[R^m(\alpha, \beta, \lambda)\right] \subset R^m(\alpha, \beta, \lambda)$.

Acknowledgement. The author would like to thank the referee for his/hers helpful remarks and suggestions.

REFERENCES

[1] F.M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator.* Internat. J. Math. Math. Sci. **27** (2004), 1429–1436.
[2] O.P. Ahuja, *Univalent functions whose derivatives have a positive real part.* Rend. Mat. **7** (1982) *1*, 173–187.
[3] R.W. Barnard and C. Kellogg, *Applications of convolution operators to problems in univalent function theory.* Michigan Math. J. **27(1)** (1980), 81–94.
[4] S.D. Bernardi, *Convex and starlike functions.* Trans. Amer. Math. Soc. **135** (1969), 429–446.
[5] L. Brickman, T.H. MacGregor and D.R. Wilken, *Convex hull of some classical families of univalent functions.* Trans. Amer. Math. Soc. **156** (1971), 91–107.
[6] C.-Y. Gao and S.-Q. Zhou, *Certain subclass of starlike functions.* Appl. Math. Comput. **187** (2007), 176–182.
[7] A.W. Goodman, *Univalent Functions.* Mariner Publishing Company Inc., 1984.
[8] D.J. Hallenbeck and T.H. MacGregor *Linear problems and Convexity Techniques in Geometric Function Theory.* Pitman, 1984.
[9] W. Kaplan, *Close-to-convex schlicht functions.* Michigan Math. J. **1** (1953), 169–185.
[10] W. Keopf, *A uniqueness theorem for functions of positive real part.* J. Math. Sci. **28** (1994), 78–90.
[11] Z. Lewandowski and S.S. Miller, E. Zlotkiewicz, *Generating functions for some classes of univalent functions.* Proc. Amer. Math. Soc. **56** (1976), 111–117.
[12] S.S. Miller and P.T. Mocanu, *Differential Subordinations. Theory and Applications.* Marcel-Dekker, New York, 2000.
[13] D. Răducanu and H. Orhan, *Subclasses of analytic functions defined by a generalized differential operator.* Int. J. Math. Anal. **4** (2010), *1-2*, 1–15.
[14] G.S. Sălăgean, *Subclasses of univalent functions.* Complex Analysis 5th Romanian-Finnish Seminar, Part. I (Bucharest, 1981), Lect. Notes Math. **1013**, Springer-Verlag, 1983, pp. 362–372.
[15] R. Singh and S. Singh, *Convolution properties of a class of starlike functions.* Proc. Amer. Math. Soc. **106** (1989), *1*, 145–152.

Received 15 September 2009

"Transilvania" University of Braşov
Faculty of Mathematics and Computer Science
Str. Iuliu Maniu 50, 50091 Braşov, Romania
dorinaraducanu@yahoo.com