GRADIENT FLOWS WITH JUMPS
ASSOCIATED WITH NONLINEAR
HAMILTON-JACOBI EQUATIONS WITH JUMPS

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We analyze gradient flows with jumps generated by a finite set of complete vector fields in involution using some Radon measures \( u \in U \) as admissible perturbations. Both the evolution of a bounded gradient flow \( \{ x^n(t, \lambda) \in B(x^*, 3\gamma) : t \in [0, T], \lambda \in B(x^*, 2\gamma) \} \) and the unique solution \( \lambda = \psi^n(t, x) \in B(x^*, 2\gamma) \subseteq \mathbb{R}^n \) of integral equation \( x^n(t, \lambda) = x \in B(x^*, \gamma), t \in [0, T], \) are described using the corresponding gradient representation associated with flow and Hamilton-Jacobi equations.

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1. INTRODUCTION

For a given finite set of complete vector fields \( \{g_1, \ldots, g_m\} \subseteq C^\infty(\mathbb{R}^n; \mathbb{R}^n) \) consider the corresponding local flows \( \{G_i(t_1)[x], \ldots, G_m(t_m)[x] : |t_i| \leq a_i, x \in B(x^*, 3\gamma) \leq \mathbb{R}^n, 1 \leq i \leq m \} \) generated by \( \{g_1, \ldots, g_m\} \) correspondingly and satisfying

(1) \[ |G_i(t_i)[x] - x| \leq \frac{\gamma}{2m}, \quad x \in B(x^*, 3\gamma), \quad |t_i| \leq a_i, \quad 1 \leq i \leq m \]

for some fixed constants \( a_i > 0 \) and \( \gamma > 0 \).

Denote by \( U_a \) the set of admissible perturbations consisting of all piecewise right-continuous mappings (of \( t \geq 0 \)) \( u(t, x) : [0, \infty) \times \mathbb{R}^n \rightarrow \bigcup_{i=1}^{m} [-a_i, a_i] \) fulfilling

(2) \[ u(0, \lambda) = 0, \quad u(t, \cdot) \in C_b^1(\mathbb{R}^n; \mathbb{R}^n) \quad \text{and} \]
\[ |\partial_\lambda u_i(t, \lambda)| \leq K_1, \quad t \geq 0, \quad \lambda \in \mathbb{R}^n, \quad 1 \leq i \leq m, \]

for some fixed constant \( K_1 > 0 \).

For each admissible perturbation \( u \in \mathcal{U}_u \), we associate a piecewise right-continuous trajectory (for \( t \geq 0 \))

\[
x^u(t, \lambda) = G(u(t, \lambda))[\lambda], \quad t \geq 0, \ \lambda \in B(x^*, 2\gamma),
\]

where the smooth mapping \( G(p)[x] : \bigcup B(x^*, 2\gamma) \to B(x^*, 3\gamma) \) is defined by

\[
G(p)[x] = G_1(t_1) \circ \cdots \circ G_m(t_m)[x], \quad p = (t_1, \ldots, t_m) \in \bigcup, \ x \in B(x^*, 2\gamma)
\]

verifying \( G(p)[x] \in B(x^*, 3\gamma) \) (see (1)).

We are going to introduce some nonlinear ODE with jumps fulfilled by the bounded flow \( \{x^u(t, \lambda) : t \in [0, T], \ \lambda \in B(x^*, 2\gamma)\} \) defined in (3), when \( u \in \mathcal{U}_u \) has a bounded variation property. In addition, the unique solution \( \{\lambda = \psi(t, x) \in B(x^*, 2\gamma) : t \in [0, T], \ x \in B(x^*, \gamma)\} \) of the integral equation

\[
x^u(t, \lambda) = x \in B(x^*, \gamma), \quad t \in [0, T]
\]

fulfils a quasilinear Hamilton-Jacobi (H-J) equation on each continuity interval \( t \in [t_k, t_{k+1}] \subseteq [0, T] \). These result are contained in the last section of this paper (see Theorems 3.1, 3.3 and 3.4). In the case that we assume \( \{g_1, \ldots, g_m\} \subset C^\infty(\mathbb{R}^n; \mathbb{R}^n) \) are commuting using Lie bracket then the result are more or less contained in [1].

Here, in this paper, the vector fields \( \{g_1, \ldots, g_m\} \subset C^\infty(\mathbb{R}^n; \mathbb{R}^n) \) are supposed to be in involution over reals which lead us to make use of algebraic representation for gradient systems in a finite dimensional Lie algebra (see [1]) without involving a global nonsingularity or local times. The analysis performed here reveals the meaningful connection between dynamical systems and partial differential equations.

2. FORMULATION OF PROBLEMS AND SOME AUXILIARY RESULTS

Consider a finite set of complete vector fields \( g_i \in C^\infty(\mathbb{R}^n; \mathbb{R}^n), \ 1 \leq i \leq m \), and let \( \{G_i(t_i)[x] : |t_i| \leq a_i, \ x \in B(x^*, 3\gamma) \subseteq \mathbb{R}^n\} \) be the local flow generated by \( g_i \) satisfying

\[
|G_i(t_i)[x] - x| \leq \frac{\gamma}{2m}, \quad x \in B(x^*, 3\gamma), \ |t_i| \leq a_i, \ 1 \leq i \leq m
\]

for some fixed constants \( a_i > 0 \) and \( \gamma > 0 \).

Denote by \( \mathcal{U}_u \) the set of admissible perturbations consisting of all piecewise right-continuous mappings (of \( t \geq 0 \)) \( u(t, \lambda) : [0, \infty) \times \mathbb{R}^n \to \bigcup = \prod_{i=1}^m [-a_i, a_i] \) fulfilling

\[
u(0, \lambda) = 0, \quad u(t, \cdot) \in C^1_b(\mathbb{R}^n; \mathbb{R}^n) \quad \text{and} \quad |\partial_\lambda u_i(t, \lambda)| \leq K_1, \quad t \geq 0, \ \lambda \in \mathbb{R}^n, \ 1 \leq i \leq m,
\]
for some fixed constant $K_1 > 0$. For each admissible perturbation $u \in U_a$, we associate a piecewise right-continuous trajectory (for $t \geq 0$)

$$x^u(t, \lambda) = G(u(t, \lambda))|\lambda|, \quad t \geq 0, \ \lambda \in B(x^*, 2\gamma).$$

Here the gradient smooth mapping $G(\cdot)$ is defined as follows

$$G(p)[x] = G_1(t_1) \circ \cdots \circ G_m(t_m)[x], \quad p = (t_1, \ldots, t_m) \in \bigcup, \ x \in B(x^*, 2\gamma),$$

and satisfies (see (6)) $G(\cdot)[\lambda] \in B(x^*, 3\gamma)$ for any $p \in \bigcup$ and $\lambda \in B(x^*, 2\gamma)$.

The flow with jumps represented as in (8) stands for a gradient flow with jumps and it relies on the smooth mapping defined in (9) which is the unique solution of an associated integrable gradient system

$$\partial_t y = g_1(y), \quad \partial_t y = g_2(t; y), \ldots, \partial_t y = g_m(t_1, \ldots, t_{m-1}; y),$$

$$y(0) = \lambda \in B(x^*, 2\gamma), \quad p = (t_1, \ldots, t_m) \in \bigcup, \ y \in \mathbb{R}^n.$$

We are looking for sufficient condition on \{g_1, \ldots, g_m\} (see [2]) such that the vector fields with parameters given in (10) can be represented as follows

$$\{g_1, g_2(t_1), \ldots, g_m(t_1, \ldots, t_{m-1})\}(y) = \{g_1, \ldots, g_m\}(y)A(p), \ y \in B(x^*, 3\gamma),$$

$$p = (t_1, \ldots, t_m) \in \bigcup, \text{ where the } (m \times m) \text{ matrix } A(p) \text{ satisfies}$$

$$A(0) = I_m, \quad A(p) = [b_1b_2(t_1) \ldots b_m(t_1, \ldots, t_{m-1})],$$

$$b_j \in C^\infty(\bigcup \mathbb{R}^n), \quad b_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$ 

This algebraic representation help us to define each integral

$$\int_0^t b_j'(u_1(s, \lambda), \ldots, u_{j-1}(s, \lambda))d_s u_j(s, \lambda) = \alpha^u_{ij}(t, \lambda),$$

$1 \leq j \leq m, \ 1 \leq i \leq m, \ t \in [0, T]$, as a bounded variation function with respect to $t \in [0, T]$, provided we assume that

$$\text{each } u_i(t, \lambda), \ t \in [0, T], \ 1 \leq i \leq m, \text{ has a bounded variation property.}$$

In addition, the algebraic representation (11) (see [2]) can be obtained assuming $g_i \in C^\infty(\mathbb{R}^n; \mathbb{R}^n), \ 1 \leq i \leq m, \text{ and}$

$$\{g_1(x), \ldots, g_m(x) : x \in B(x^*, 3\gamma) \subseteq \mathbb{R}^n\} \text{ are in involution over reals, i.e., each Lie bracket can be written as}$$

$$[g_i, g_j](x) = \sum_{k=1}^m \gamma_{ij}^k g_k(x) \text{ for } x \in B(x^*, 3\gamma), \text{ using } \gamma_{ij} \in \mathbb{R}. $$

Let $[t_k, t_{k+1}] \subset [0, T], \ 0 \leq k \leq N - 1$, be the continuity intervals of $u \in U_a$. 

Problem $P_1$. Under the hypothesis (14) and (15), describe the evolution of the gradient flow with jumps in (8) as follows

$$\begin{cases}
d_t x^u(t,\lambda) = \sum_{i=1}^m g_i(x^u(t,\lambda)) d_t \beta^u_i(t,\lambda), & t \in [t_k, t_{k+1}), 0 \leq k \leq N-1, \\
x^u(0,\lambda) = \lambda \in B(x^*,2\gamma), \text{ where } \beta^u_i(t,\lambda) = \sum_{k=1}^m \alpha^u_{ij}(t,\lambda), 1 \leq i \leq m.
\end{cases}$$

Here the matrix $\{\alpha^u_{ij}(t,\lambda) : i,j \in \{1,\ldots,m\}, t \in [0,T]\}$ of bounded variation and piecewise right-continuous function of $t \in [0,T]$ are defined in (13).

Problem $P_2$. Under the hypothesis (14), (15) and for $K_1 > 0$ sufficiently small (see (7)), prove that the integral equations (with respect to $\lambda \in B(x^*,2\gamma)$ see (8))

$$x^u(t,\lambda) = \psi^u(t,x) \in B(x^*,2\gamma); t \in [0,T]$$

are reversibly with respect to $\lambda \in B(x^*,2\gamma)$.

The unique bounded variation and piecewise right-continuous solution $\{\lambda = \psi^u(t,x) \in B(x^*,2\gamma) : t \in [0,T]\}$ is first order continuously differentiable of $x \in \text{int} B(x^*,\gamma)$.

Remark 2.1. One may wonder about the Hamilton-Jacobi equation with jumps satisfied by the unique solution $\{\lambda = \psi^u(t,x) \in B(x^*,2\gamma) : t \in [0,T]\}$, $x \in B(x^*,\gamma)$ found in ($P_2$). This will be presented at the end of the following section. The next Lemma lead us to the solution of the problem ($P_1$).

Lemma 2.2. Assume that the hypothesis (14) and (15) are fulfilled and consider the gradient flow with jumps $\{x^u(t,\lambda) : t \in [0,T], \lambda \in B(x^*,2\gamma)\}$ defined in (8), where $u \in \mathcal{U}_a$ and $T > 0$ are fixed arbitrarily. Then there exists an $(m \times m)$ matrix composed by bounded variation and piecewise right-continuous functions $\{\alpha^u_{ij}(t,\lambda) : \alpha^u_{ij}(0,\lambda) = 0, 1 \leq i,j \leq m, t \in [0,T], \lambda \in B(x^*,2\gamma)\}$ (see (13)) such that

$$\begin{cases}
d_t x^u(t,\lambda) = \sum_{i=1}^m g_i(x^u(t,\lambda)) d_t \beta^u_i(t,\lambda), & t \in [t_k, t_{k+1}), 0 \leq k \leq N-1, \\
x^u(0,\lambda) = \lambda, \text{ where } \beta^u_i(t,\lambda) = \sum_{j=1}^m \alpha^u_{ij}(t,\lambda) 1 \leq i \leq m,
\end{cases}$$

and $[t_k, t_{k+1}] \subseteq [0,T], 0 \leq k \leq N - 1$, are the continuity intervals of $u \in \mathcal{U}_a$.

Proof. By definition, $x^u(t,\lambda) \in B(x^*,3\gamma)$, $t \geq 0$, $\lambda \in B(x^*,2\gamma)$ (see (8)) where $x^u(t,\lambda) = G(u(t,\lambda))|\lambda$ defined in (9) fulfills the integrable gradient system given in (10) (see [2]). In addition, using the hypothesis (15) (see [2]) we may and do represent the vector fields of (10) as in (11). As far as $x^u(t,\lambda) = y_{\lambda}(u(t,\lambda)), t \in [0,T], \lambda \in B(x^*,2\gamma)$, where $\{y_{\lambda}(p) : p \in \mathcal{U}\}$ is the unique solution of (10), we get the conclusion (18) provided the algebraic representation (11) and (12) is used. The proof is complete. $\square$
Remark 2.3. For solving integral equation $x^u(t, \lambda) = x \in B(x^*, \gamma)$ (for some fixed $u \in U_a$) using integral representation (8), we notice that these are equivalent with the following integral equations

$$\lambda = H(u(t, \lambda))[x], \quad t \in [0, T], \; x \in B(x^*, \gamma)$$

with respect to $\lambda \in B(x^*, 2\gamma)$. Here $H(p)[x] = [G(p)]^{-1}(x)$ satisfies

$$H(p)[x] \overset{\text{def}}{=} G_m(-t_m) \circ \cdots \circ G_1(-t_1)[x] \in B(x^*, 2\gamma), \; \forall p = (t_1, \ldots, t_m) \in \mathbb{U}$$

and $x \in B(x^*, \gamma)$.

In addition, using the hypothesis (15) and writing the corresponding integrable gradient system for $y(p; \lambda) = G(p)[\lambda]$ (see (10) and (11)) we get each $\partial_i(H(p)[x])$ as follows

$$\partial_i H(p)[x] = -\partial_x(H(p)[x])g_1(x), \; \partial_x H(p)[x] = -\partial_x(H(p)[x])Y_2(t_1; x),$$

,..., $\partial_{tm} H(p)[x] = -\partial_x(H(p)[x])Y_m(t_1, \ldots, t_{m-1}; x)$.

Here a direct computation applied to the identity $H(p)[G(p)(\lambda)] = \lambda$ and write $0 = \partial_i H(p)[x] + \partial_x(H(p)[x])Y_i(t_1, \ldots, t_{i-1}; x)$ for each $i \in \{1, \ldots, m\}$, where $Y_i(x) = g_i(x)$ and (see (10) and (11))

$$\{g_1(x), Y_2(t_1; x), \ldots, Y_m(t_1, \ldots, t_{m-1}; x)\} = \{g_1(x), \ldots, g_m(x)\} A(p), \; p \in \mathbb{U}.$$  

Denote $z(p, x) = H(p)[x]$.

Lemma 2.4. Assume that the hypothesis (15) is satisfied and define $H(p)[x] = [G(p)]^{-1}(x) = G_m(-t_m) \circ \cdots \circ G_1(-t_1)[x]$, $x \in B(x^*, \gamma)$, $p = (t_1, \ldots, t_m) \in \mathbb{U}$, where $y = G(p)[\lambda]$, $p \in \mathbb{U}$, $\lambda \in B(x^*, 2\gamma)$, verifies (9) and is the unique solution of the integrable gradient system (10) and (11). Then there exists an $(m \times m)$ analytic matrix $A(p)$ verifying (22) such that the following system of (H-J) equation is fulfilled

$$\begin{cases}
\partial_x z(p; x) + \partial_z(z(p; x))\{g_1(x), \ldots, g_m(x)\} A(p) = 0, \; p \in \mathbb{U}, \; x \in B(x^*, \gamma) \\
z(0; x) = x
\end{cases}$$

Proof. A direct computation applied to the identity $H(p)[G(p)(\lambda)] = \lambda$ lead us to the following system of (H-J) equations (see $z(p; x) = H(p)[x]$)

$$\partial_i z(p; x) + \partial_x(z(p; x))Y_i(t_1, \ldots, t_{i-1}; x) = 0, \; 1 \leq i \leq m, \; \forall p \in \mathbb{U}, \; x \in B(x^*, \gamma).$$

Here the vector fields with parameters $\{Y_1, \ldots, Y_m\}$ are defined in (10) and fulfills the algebraic representation given in (11). Using (11), we rewrite (24) as follows

$$\begin{cases}
\partial_p z(p; x) + \partial_x(z(p; x))\{g_1(x), \ldots, g_m(x)\} A(p) = 0, \; p \in \mathbb{U}, \; x \in B(x^*, \gamma) \\
z(0; x) = x
\end{cases}$$
and the proof is complete. \(\square\)

**Lemma 2.5.** Under the conditions assumed in Lemma 2.4, define
\[ V^u(t, x; \lambda) = z(u(t, \lambda); x), \quad t \geq 0, \lambda \in \mathbb{R}^n, x \in B(x^*, \lambda), \]
where \(u \in \mathcal{U}_a\) is fixed and \(u(p; x), p \in \bigcup_{i}, x \in B(x^*, \lambda)\), satisfies (H-J) equations (23). Then the \((n \times n)\) matrix \(M^u(t, x; \lambda) \equiv \partial_{\lambda} V^u(t, x; \lambda)\), verifies the following inequality
\[ |M^u(t, x; \lambda)| \leq C_1 C_2 K_1, \quad t \geq 0, \lambda \in \mathbb{R}^n, x \in B(x^*, \gamma), \]
where \(K_1 > 0\) is fixed in (7) (see definition of \(\mathcal{U}_a\)) and
\[ C_1 \overset{\text{def}}{=} \max\{|\partial_x(z(p; x))g_i(x)| : p \in \bigcup_{i}, x \in B(x^*, \gamma), 1 \leq i \leq m\}, \]
\[ C_2 \overset{\text{def}}{=} \max\{|A(p)| : p \in \bigcup_{i}\} (A(p) \text{ is given in (22) and used in (23)}). \]

**Proof.** By hypothesis, the mapping \(z(p; x) = H(p)[x]\) defined in Lemma 2.4 fulfils (H-J) equation (23) and for an arbitrary \(u \in \mathcal{U}_a\), we get
\[ M^u(t, x; \lambda) = \partial_p z(u(t, \lambda); x) \partial_{\lambda} u(t, \lambda), \quad t \geq 0, \lambda \in \mathbb{R}^n, x \in B(x^*, \gamma). \]

Here \(u(t, \lambda) \in \bigcup_{i} \subseteq \mathbb{R}^m\) and \(|\partial_{\lambda} u_i(t, \lambda)| \leq K_1, 1 \leq i \leq m, \text{ for any} t \geq 0, \lambda \in \mathbb{R}^n (\text{see definition of } \mathcal{U}_a \text{ in (7)}). \) On the other hand, using (23) of Lemma 2.4, the following inequality is valid
\[ |\partial_p z(u(t, \lambda); x)| \leq C_1 C_2, \quad t \geq 0, \lambda \in \mathbb{R}^n, x \in B(x^*, \gamma), \]
where the constants \(C_1, C_2\) are given in (28), (29). A direct computation applied to (30) leads us to
\[ |M^u(t, x; \lambda)| \leq C_1 C_2 K_1, \quad t \geq 0, \lambda \in \mathbb{R}^n, x \in B(x^*, \gamma) \]
and the proof is complete. \(\square\)

**Lemma 2.6.** Assume that \(u \in \mathcal{U}_a\) and \(\{g_1, \ldots, g_m\} \subseteq \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R}^n)\) satisfies (14) and (15). Consider \(z(p; x) = H(p)[x]\) which verifies (H-J) equations (23) of Lemma 2.4 and define
\[ V^u(t, x; \lambda) = z(u(t, \lambda); x), \quad t \in [0, T], \lambda \in \mathbb{R}^n, x \in B(x^*, \gamma) \subseteq \mathbb{R}^n. \]
Let \(\{\alpha_{ij}^u(t, \lambda) : t \in [0, T], \lambda \in \mathbb{R}^n, 1 \leq i, j \leq m\}\) be the \((m \times m)\) matrix given in (13) and define new bounded variation piecewise right-continuous function
\[ \beta_i^u(t, \lambda) = \sum_{j=1}^{m} \alpha_{ij}^u(t, \lambda), \quad t \in [0, T], 1 \leq i \leq m. \] Then \(\{V^u(t, x; \lambda) : t \in [0, T]\}\) is
a bounded variation piecewise right-continuous mapping satisfying the following (H-J) equations with jumps

\[
\begin{align*}
V^u(0, x; \lambda) &= x, \\
V^u(t, x; \lambda) &= x + \int_{t_k}^{t} g_i(x) dt \beta_i^u(t, \lambda), \\
\end{align*}
\]

where \([t_k, t_{k+1}] \subseteq [0, T], 0 \leq k \leq N-1,\) are the continuity intervals of \(u \in \mathcal{U}_a.\)

**Proof.** By hypothesis, the conclusion (23) of Lemma 2.4 is valid. By a direct computation, we get \(V^u(0, x; \lambda) = x\) and

\[
V^u(t, x; \lambda) = \sum_{j=1}^{m} \partial_i z(u(t, \lambda); x) d_i u_i(t, \lambda), \quad t \in [t_k, t_{k+1}).
\]

Using (23), rewrite (35) as follows

\[
d_t V^u(t, x; \lambda) = -\partial_x V^u(t, x; \lambda) \{g_1(x), \ldots, g_m(x)\} A(u(t, \lambda)) \begin{pmatrix} d_t u_1(t, \lambda) \\ \vdots \\ d_t u_m(t, \lambda) \end{pmatrix},
\]

where \(A(p) = [b_1, b_2(t_1), \ldots, b_m(t_1, \ldots, t_{m-1})], b_j \in C^\infty([\mathbb{R}], \mathbb{R}^m), t \in [t_k, t_{k+1}).\)

Using (13), write

\[
\alpha_i^u(t, \lambda) = \int_{0}^{t} b_i^j(u_1(s-\lambda), \ldots, u_{j-1}(s-\lambda)) d_s u_j(s, \lambda), \quad 1 \leq i, j \leq m,
\]

for \(t \in [0, T], \lambda \in \mathbb{R}^n.\) Rewrite (36) (using (37)) and we get conclusion (34). The proof is complete. \(\square\)

**Remark 2.7.** The (H-J) equations with jumps satisfied by the unique solution of Problem \((P_2)\) are strongly connected with Lemmas 2.4 and 2.6. On the other hand, the existence of a solution for Problem \((P_2)\) relies on Lemma 2.5 and it will be analyzed in the next Lemma assuming that \(K_1 > 0\) satisfies

\[
C_1 C_2 K_1 = \rho \in [0, \frac{1}{2}].
\]

**Lemma 2.8.** Assume that \(u \in \mathcal{U}_a\) and \(\{g_1, \ldots, g_m\} \subseteq C^\infty(\mathbb{R}^n, \mathbb{R}^n)\) fulfil (14), (15) and (38). Then there exists a unique bounded variation piecewise right continuous (of \(t \in [0, T]\)) mapping \(\{\lambda = \psi^u(t, x) \in B(x^*, 2\gamma) : t \in [0, T], x \in B(x^*, \gamma)\}\) which is first order continuously differentiable of \(x \in \text{int} B(x^*, \gamma),\) satisfying integral equations

\[
\begin{align*}
x^u(t-, \psi^u(t-, x)) &= x \in B(x^*, \gamma), \\
\psi^u(t-, x) &= V^u(t-, x; \psi(t-, x)), \psi^u(t, x) = V^u(t, x; \psi^u(t-, x)), \quad t \in [0, T].
\end{align*}
\]
Proof. By hypothesis, the conclusion of Lemma 2.5 is valid for \( V^u(t, x; \lambda) = z(u(t, \lambda); x), \ t \geq 0, \lambda \in \mathbb{R}^n, x \in B(x^*, \gamma). \) Notice that \( x^u(t, \lambda) = x \) can be rewritten as

\[
\lambda = V^u(t, x; \lambda), \quad t \in [0, T], \ x \in B(x^*, \gamma),
\]

where the \((n \times n)\) matrix \( \partial_\lambda V^u(t, x; \lambda) = M^u(t, x; \lambda) \) fulfils the conclusion (27), i.e.,

\[
|M^u(t, x; \lambda)| \leq C_1 C_2 K_1, \quad t \geq 0, \ \lambda \in \mathbb{R}^n, x \in B(x^*, \gamma).
\]

Assuming that \( K_1 > 0 \) is sufficiently small such that

\[
\rho = C_1 C_2 K_1 \in [0, \frac{1}{2}] \ \text{(see (38))},
\]

then the contraction mapping theorem can be applied for solving integral equations (39). Construct the convergent sequence \( \{\lambda_k(t, x) : t \in [0, T], x \in B(x^*, \gamma)\} \) such that

\[
\left\{ \begin{array}{l}
\lambda_k(t, x) \in B(x^*, 2\gamma), \quad \lambda_0(t, x) = x, \\
\lambda_{k+1}(t, x) = V^u(t, x; \lambda_k(t, x)), \quad k \geq 0.
\end{array} \right.
\]

(42)

Notice that the following estimates are valid

\[
|\lambda_{k+1}(t, x) - \lambda_k(t, x)| \leq \rho^k |\lambda_1(t, x) - \lambda_0(t, x)|, \quad k \geq 0,
\]

\[
|\lambda_1(t, x) - \lambda_0(t, x)| \leq \max\{|G(p)|x| - x| : x \in B(x^*, \gamma)\} \leq \frac{\gamma}{2},
\]

(see (9)) which lead us to

\[
|\lambda_k(t, x) - x| \leq \frac{1}{1 - \rho} \left( \frac{\gamma}{2} \right) \leq \gamma, \quad t \in [0, T], x \in B(x^*, \gamma), k \geq 0.
\]

(44)

Combining (43) and (44), we get \( \{\lambda_k(t, x)\}_{k \geq 0} \) fulfils (42) and passing \( k \to \infty \) in (42), we obtain

\[
\psi^u(t, x) = \lim_{k \to \infty} \lambda_k(t, x) \in B(x^*, 2\gamma), \quad t \in [0, T], x \in B(x^*, \gamma)
\]

satisfying integral equations

\[
\left\{ \begin{array}{l}
\psi^u(t, x) = V^u(t, x; \psi(t, x)), \\
\psi^u(t, x) = V^u(t, x; \psi(t, x)), \quad t \in [0, T], x \in B(x^*, \gamma).
\end{array} \right.
\]

(46)

The proof is complete. \( \square \)
3. MAIN THEOREMS

With the same notations as in Section 1, we reconsider the problems \(P_1\) and \(P_2\) in more general setting. Consider the local flow \(\{G_0(t)[x] : |t| \leq T, x \in B(x^*, 3\gamma) \subseteq \mathbb{R}^n\}\) generated by a complete vector field \(g_0 \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)\). Assume that

\[
I_1 = \left\{ g_1, \ldots, g_m \right\} \subseteq C^\infty(\mathbb{R}^n; \mathbb{R}^n) \text{ satisfies (15) and } |g_0, g_i| = 0, 1 \leq i \leq m, \text{ for any } x \in B(x^*, 3\gamma).
\]

\[
I_2 = \left\{ u \in U_a \text{ fulfils (14)} \right\} \text{ where } T > 0 \text{ and } \bigcup_1^m [-a_i, a_i] \subseteq \mathbb{R}^n,
\]

are fixed such that \(|G_0(t[x]) - x| \leq \frac{\gamma}{2(m+1)}, |G_i(t_i)[x] - x| \leq \frac{\gamma}{2(m+1)}\).

Let \([t_k, t_{k+1}] \subseteq [0, T], 0 \leq k \leq N - 1\), be the continuity intervals of \(u \in U_a\).

**Problem (R1).** Under the hypothesis \((I_1)\) and \((I_2)\), describe the evolution of the gradient flow with jumps

\[
y^{u}(t, \lambda) \overset{\text{def}}{=} G_0(t) \circ (u(t, \lambda)) \vert \lambda : t \in [0, T], \lambda \in B(x^*, 2\gamma) \subseteq \mathbb{R}^n
\]

(see \(G(p) = G_1(t_1) \circ \cdots \circ G_m(t_m)\)) as a solution of the following system with jumps

\[
d_t y^{u}(t, \lambda) = g_0(y^{u}(t, \lambda))dt + \sum_{i=1}^{m} g_i(y^{u}(t, \lambda))dt_b^u(t, \lambda), \quad y^{u}(t, \lambda) \in B(x^*, 3\gamma),
\]

\[
y^{u}(0, \lambda) = \lambda \in B(x^*, 2\gamma), \quad t \in [t_k, t_{k+1}], \quad b^u_i(t, \lambda) = \sum_{j=1}^{m} a_{ij}^u(t, \lambda), 1 \leq k \leq N - 1,
\]

where the matrix \(\{a_{ij}^u(t, \lambda) : 1 \leq i, j \leq m\}\) is defined in (13).

**Theorem 3.1.** Assume that \(\{g_0, g_1, \ldots, g_m\} \subseteq C^\infty(\mathbb{R}^n; \mathbb{R}^n)\) and \(u \in U_a\) fulfil the hypothesis \((I_1)\) and \((I_2)\). Then the gradient flow with jumps \(\{y^{u}(t, \lambda)\}\) defined in (47) verifies \(y^{u}(t, \lambda) \in B(x^*, 3\gamma)\) and is a solution of the system (48).

**Proof.** By hypothesis, the gradient flow with jumps \(\{y^{u}(t, \lambda)\}\) can be rewritten \(y^{u}(t, \lambda) = G(u(t, \lambda)) \circ G_0(t)[\lambda]\) and using \((I_2)\), we get \(y^{u}(t, \lambda) \in B(x^*, 3\gamma), t \in [0, T], \lambda \in B(x^*, 2\gamma)\). On the other hand, a direct computation applied to \(\{y^{u}(t, \lambda)\}\) lead us to the following equations

\[
d_t y^{u}(t, \lambda) = g_0(y^{u}(t, \lambda))dt + \sum_{i=1}^{m} \partial_{t_i} G(u(t, \lambda))[G_0(t)(\lambda)]d_{t_i}u_i(t, \lambda), \quad t \in [t_k, t_{k+1}],
\]

where \(y(p; \mu) \overset{\text{def}}{=} G(p)[\mu]\) satisfies an integrable gradient system (see (10)),

\[
\partial_{t_1} y = g_1(y), \quad \partial_{t_2} y = Y_2(t; y), \ldots, \partial_{t_m} = Y_m(t_1, \ldots, t_{m-1}; y),
\]
fulfilling the algebraic representation given in (11) and (12). As a consequence, we may and do rewrite the second term in the right-hand side of (49) as a follows
\[
\sum_{i=1}^{m} \partial_t G(u(t, \lambda)) [G_0(t)(\lambda)] d_t u_i(t, \lambda) = \sum_{i=1}^{m} g_i(y^u(t, \lambda)) d_t \beta_i^u(t, \lambda), \quad t \in [t_k, t_{k+1}),
\]
\[
\lambda \in B(x^*, 2\gamma), \quad 0 \leq k \leq N - 1 \quad \text{where} \quad \beta_i^u(t, \lambda) \stackrel{\text{def}}{=} \sum_{j=1}^{m} \alpha_{ij}(t, \lambda) \text{ and the matrix}
\{
\alpha_{ij}(t, \lambda) : 1 \leq i, j \leq m
\}
\text{is defined in (13). The proof is complete. □}

Remark 3.2. The evolution of the gradient flow defined in (47) satisfies the same system with jumps given in (48) if the commutative condition
\[
[g_0, g_i](x) = 0, \quad 1 \leq i \leq m, \quad x \in B(x^*, 3\gamma) \subseteq \mathbb{R}^n, \quad \text{assumed in (I_1)}
\]
is replaced by
\[
[g_0, g_i](x) = \sum_{k=1}^{m} \gamma^k_i g_k(x), \quad x \in B(x^*, 3\gamma), \quad 1 \leq i \leq m,
\]
where \(\gamma^k_i \in \mathbb{R}\) are same constants.

The only change which appears is reflected in the algebraic representation corresponding to the gradient integrable system associated with smooth mapping
\[
y(t, p)[\lambda] = G_0(t) \circ G(p)[\lambda], \quad |t| \leq T, \quad p = (t_1, \ldots, t_m) \in \bigcup, \quad \lambda \in B(x^*, 2\gamma).
\]
We get
\[
\partial_t y = g_0(y), \quad \partial_{t_1} y = Y_1(t; y), \quad \partial_{t_2} y = Y_2(t, t_1; y), \ldots,
\]
\[
\partial_{t_m} y = Y_m(t, t_1, \ldots, t_{m-1}; y).
\]
\[
\{g_0, Y_1(t), Y_2(t, t_1), \ldots, Y_m(t, t_1, \ldots, t_{m-1})\}(y) = \{g_0, g_1, \ldots, g_m\}(y)V(t, \mu).
\]

This time, the \((m+1) \times (m+1)\) analytic matrix \(V(t, p)\) has the following structure
\[
\left\{V(t, p) = [V_0, V_1(t), V_2(t, t_1), \ldots, V_m(t, t_1, \ldots, t_{m-1})] \quad V_j \in \mathbb{R}^{m+1},
\right.
\]
\[
\left. V_0 = \begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}
\right\} \quad \text{and} \quad V_i(t, t_1, \ldots, t_{i-1}) = \begin{pmatrix}
0 \\
b_i(t, t_1, \ldots, t_{i-1})
\end{pmatrix}, \quad b_i \in \mathbb{R}^m,
\]
1 \leq i \leq m. A standard computation applied to \( y^u(t, \lambda) = y(t, u(t, \lambda))[\lambda] = G_0(t) \circ G(u(t, \lambda))[\lambda], t \in [0, T], \) leads to

\[
dty^u(t, \lambda) = \partial_t y(t, u(t, \lambda))[\lambda] dt + \sum_{i=1}^{m} (\partial_t y(t, u(t, \lambda))[\lambda]) dt u_i(t, \lambda) = g_0(y^u(t, \lambda)) dt + \sum_{i=1}^{m} g_i(y^u(t, \lambda)) dt \beta^u_i(t, \lambda), \quad t \in [t_k, t_{k+1}),
\]

0 \leq k \leq N - 1, where \( \beta^u_i(t, \lambda) = \sum_{j=1}^{m} \alpha^u_{ij}(t, \lambda) \) and

\[
\alpha^u_{ij}(t, \lambda) = \defint{0}{t} b_i^j(s, u_1(s-\lambda), \ldots, u_{i-1}(s-\lambda)) ds u_j(s, \lambda), \quad 1 \leq i, j \leq m.
\]

Here \( \{b_1(t), b_2(t, t_1), \ldots, b_m(t, t_1, \ldots, t_{m-1})\} \subseteq \mathbb{R}^m \) are given in (56).

Problem (R2). Under the hypothesis (I1) and (I2) and for \( K_1 > 0 \) sufficiently small (see (38) of Lemma 2.8) prove that the integral equation with respect to \( \lambda \in B(x^*, 2\gamma) \) (see (47))

\[
y^u(t, \lambda) = \defint{0}{t} b_i^j(s, u_1(s-\lambda), \ldots, u_{i-1}(s-\lambda)) ds u_j(s, \lambda), \quad 1 \leq i, j \leq m.
\]

has a unique bounded variation and piecewise right continuous solution \( \{\lambda = \psi^u(t, x) \in B(x^*, 2\gamma) : t \in [0, T]\} \) of (58) which is first order continuously differentiable of \( x \in \text{int} B(x^*, \gamma). \)

In addition, the following equations are valid

\[
\begin{align*}
V^u(t, x; \lambda) &\defeq H(u(t, \lambda))[\lambda] = H(p)[\lambda] = [G(p)^{-1}] = [G(p)]^{-1}, t \in [0, T], p \in \mathbb{R}, \\
\psi^u(t, x) &\defeq V^u(t, x; \psi(t, x)), \quad \psi^u(t, x) = V^u(t, x, \psi(t, x)), \\
y^u(t, \psi^u(t, x)) &\defeq V^u(t, x, \psi(t, x)), \quad \text{for any } t \in [0, T].
\end{align*}
\]

Define the following two constants

\[
\begin{align*}
C_1 &\defeq \max \left\{ |\partial_y(z(p, y)) g_i(y)| : p \in \mathbb{R}, y \in B(x^*, 2\gamma) \right\}, \\
C_2 &\defeq \max \left\{ |A(p)| : p \in \mathbb{R} \right\},
\end{align*}
\]

where \( z(p, y) = H(p)[y] \) and the analytic \((m \times m)\) matrix \( A(p) \) is given in (2).

Assume that \( K_1 > 0 \) used in the definition of admissible set \( \mathcal{U}_a \) (see (7)) satisfies

\[
C_1 C_2 K_1 = \rho \in [0, \frac{1}{2}].
\]

Theorem 3.3. Assume that \( \{g_0, g_1, \ldots, g_m\} \subseteq C^\infty(\mathbb{R}^n, \mathbb{R}^n) \) and \( u \in \mathcal{U}_a \) fulfil the hypothesis (I1), (I2) and (61). Then there exists a unique bounded variation and piecewise right-continuous solution \( \{\lambda = \psi^u(t, x) \in B(x^*) : t \in [0, T]\} \) of (58) which is first order continuously differentiable of \( x \in \)
int $B(x^*, \gamma)$ such that the integral equations (59) are satisfied. In addition $V(t, x) \overset{\text{def}}{=} V^u(t, x; \lambda)$ satisfy the following system of (H-J) equations with jumps

$$d_t V(t, x) + \partial_x V(t, x) \left[ g_0(x) dt + \sum_{i=1}^{m} g_i(x) d_i \beta_i^u(t, \lambda) \right] = 0, \; x \in \text{int} B(x^*, \gamma),$$

$t \in [t_k, t_{k+1}], \; 0 \leq k \leq N - 1$, where $\beta_i^u(t, \lambda), \; 1 \leq i \leq m$, are defined in (48) (see Problem (R1)).

**Proof.** Define $\hat{V}^u(t, y; \lambda) = H(u(t, \lambda))[y]$, where $H(p) = |G(p)|^{-1}, \; t \in [0, T], \; p \in \mathbb{R}$ and $y \in B(x^*, 2\gamma)$. The integral equation (58) can be replaced by the following

$$\lambda = \hat{V}^u(t, y_0(t, x); \lambda) \overset{\text{def}}{=} V^u(t, x; \lambda),$$

where $y_0(t, x) = G_0(-t)(x) \in B(x^*, \gamma_1)$, for any $t \in [0, T], \; x \in B(x^*, \gamma)$ and $\gamma_1 = \gamma(1 + \frac{1}{2(m+1)})$ (see (I2)). It allows us to get the unique solution $\lambda = \psi^u(t, x)$ as a composition

$$\psi^u(t, x) = \hat{\psi}^u(t, y_0(t, x)), \; t \in [0, T], \; x \in B(x^*, \gamma),$$

where $\lambda = \hat{\psi}^u(t, y), \; t \in [0, T], \; y \in B(x^*, \gamma_1)$, is the unique solution of the following integral equations

$$\lambda = \hat{V}^u(t, y; \lambda), \; \lambda \in B(x^*, 2\gamma), \; y \in B(x^*, \gamma_1), \; t \in [0, T].$$

By hypothesis, the mapping $\{\hat{V}^u(t, y; \lambda), \; t \in [0, T], \; y \in B(x^*, \gamma_1)\}$ fulfills the hypothesis (14), (15) and (38) of Lemmas 2.6 and 2.8 for any $\lambda \in \mathbb{R}^n$. We get the corresponding (H-J) equations (see (34) of Lemma 2.6).

$$d_t \hat{V}^u(t, y; \lambda) + \partial_y \hat{V}^u(t, y; \lambda) \left[ \sum_{i=1}^{m} g_i(y) d_i \beta_i^u(t, \lambda) \right] = 0,$$

$$\hat{V}^u(0, y; \lambda) = y, \; t \in [t_k, t_{k+1}], \; y \in \text{int} B(x^*, \gamma_1), \; \lambda \in \mathbb{R}^n, \; 0 \leq k \leq N - 1.$$

In addition, there exists a unique bounded variation piecewise right-continuous (of $t \in [0, T]$) mapping $\{\lambda = \hat{\psi}^u(t, y) \in B(x^*, 2\gamma) : t \in [0, T], \; y \in B(x^*, \gamma_1)\}$ (see Lemma 2.8)

$$\hat{\psi}^u(t, y) = \hat{V}^u(t, y; \hat{\psi}^u(t, y)), \; t \in [0, T], \; y \in B(x^*, \gamma_1),$$

notice that $\lambda = \hat{\psi}^u(t, y)$ is first order continuously differentiable of $y \in \text{int} B(x^*, \gamma_1)$ and, using (66) and (67), we get the corresponding equations satisfied by $\lambda = \psi^u(t, x) \overset{\text{def}}{=} \hat{\psi}^u(t, G_0(-t)(x)), \; t \in [0, T]$, as follows

$$\psi^u(t, y) = V^u(t, x; \psi^u(t, y)), \psi^u(t, x) = V^u(t, x; \hat{\psi}^u(t, y)), \; t \in [0, T],$$
for any matrix used in the right-hand side of (74) relies on (27) in Lemma 2.5 and 

\[ x \forall \]

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\[ \text{The equations (68) stands for integral equation (59) and the first conclusion} \]

\[ \text{particular, the (H-J) equations (62) can be rewritten} \]

\[ (74) \]

\[ d_t V^u(t, x; \lambda) = d_t V^u(t, G_0(-t)(x); \lambda) - \partial_y \tilde{V}^u(t, G_0(-t)(x); \lambda)g_0(G_0(-t)(x)) dt, \]

\[ \text{for any } t \in [t_k, t_{k+1}], 0 \leq k \leq N - 1. \] In addition, using \((1)\), we get

\[ (70) \]

\[ \begin{cases} 
\partial_y \tilde{V}^u(t, G_0(-t)(x); \lambda)g_0(G_0(-t)(x)) = \partial_x V^u(t, x; \lambda)[\partial_x (G_0(-t)(x))]^{-1}. \\
\partial_t V^u(t, x; \lambda)g_i(x) = \partial_t V^u(t, x; x)g_i(x), \; t \in [0, T], \; 0 \leq i \leq m. 
\end{cases} \]

Rewrite (69) (using (70) and (66)) we get (H-J) equations (62). The proof is complete. \[ \square \]

**Theorem 3.4.** Under the hypothesis of Theorem 3.3 and assume that \( u \in \mathcal{U}_n \) is continuously differentiable on each continuity interval \([t_k, t_{k+1}) \subseteq [0, T], 0 \leq k \leq N - 1. \) Then the unique solution \( \{ \lambda = \psi^u(t, x) \in B(x^*, 2\gamma) : [t_k, t_{k+1}], x \in \text{int } B(x^*, \gamma) \} \) of integral equations (58) (see (59) also) satisfies the following (H-J) equations

\[ (71) \]

\[ \partial_t \psi^u(t, x) + \partial_x \psi^u(t, x) \left[ g_0(x) + \sum_{i=1}^m g_i(x) \partial_i \beta^u_i(t, \psi^u(t, x)) \right] = 0, \]

\[ t \in [t_k, t_{k+1}), \; x \in \text{int } B(x^*, \gamma), \; 0 \leq k \leq N - 1. \] Here \( \beta^u_i(t, \lambda), 1 \leq i \leq m, \) are defined in Problem (R1) (see (48)).

**Proof.** By hypothesis, the conclusion of Theorem 3.3 are valid and, in particular, the (H-J) equations (62) can be rewritten

\[ (72) \]

\[ \partial_t V(t, x) + \partial_x V(t, x) \left[ g_0(x) + \sum_{i=1}^m g_i(x) \partial_i \beta^u_i(t, \psi^u(t, x)) \right] = 0, \; t \in [t_k, t_{k+1}), \]

\[ x \in \text{int } B(x^*, \gamma), \; 0 \leq k \leq N - 1, \] where \( V(t, x) \equiv V^u(t, x; \lambda), \; \lambda \in B(x^*, 2\gamma). \)

On the other hand, using integral equations (59), we get

\[ (73) \]

\[ \psi^u(t, x) = V^u(t, x; \psi^u(t, x)), \; t \in [t_k, t_{k+1}), \; x \in B(x^*, \gamma), \; 0 \leq k \leq N - 1. \]

By a direct derivation, from (73) we obtain

\[ (74) \]

\[ \begin{cases} 
\partial_t \psi^u(t, x) = [I_n - \partial_t V^u(t, x; \psi^u(t, x))]^{-1} \partial_t V^u(t, x; \psi^u(t, x)), \\
\partial_x \psi^u(t, x) = [I_n - \partial_x V^u(t, x; \psi^u(t, x))]^{-1} \partial_x V^u(t, x; \psi^u(t, x)), 
\end{cases} \]

for any \( t \in [t_k, t_{k+1}), x \in \text{int } B(x^*, \gamma), 0 \leq k \leq N - 1. \) Here the nonsingular matrix used in the right-hand side of (74) relies on (27) in Lemma 2.5 and
(61) of Theorem 3.3. The equations (72) are valid for any $\lambda \in B(x^*, 2\gamma)$ and, in particular for $\lambda = \psi^u(t, x) \in B(x^*, 2\gamma)$, we get (75)

$$
\partial_t V(t, x; \psi^u(t, x)) + \partial_x V^u(t, x, \psi^u(t, x)) \left[ g_0(x) + \sum_{i=1}^{m} g_i(x) \partial_t \beta_i^u(t, \psi^u(t, x)) \right] = 0,
$$

for any $t \in [t_k, t_{k+1})$, $x \in \text{int} B(x^*, \gamma)$, $0 \leq k \leq N - 1$.

Using (75) and multiplying the second equation in (74) by $[g_0(x) + \sum_{i=1}^{m} g_i(x) \partial_t \beta_i^u(t, \psi^u(t, x))]$, we obtain the conclusion (71). The proof is complete. □

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