ORDER AMONG QUASI-ARITHMETIC MEANS OF POSITIVE OPERATORS

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As a continuation of our previous research [J. Mićić, J. Pečarić and Y. Seo, *Converses of Jensen's operator inequality*, accepted to Oper. Matrices **4** (2010), *3*, 385–403], we discuss order among quasi-arithmetic means of positive operators with fields of positive linear mappings $(\phi_t)_{t\in T}$ such that $\int_T \phi_t(\mathbf{1})\,\mathrm{d}\mu(t) = k\mathbf{1}$ for some positive scalar k.

AMS 2010 Subject Classification: 47A63, 47A64.

Key words: quasi-arithmetic mean, power mean, Jensen's inequality, Mond-Pečarić method, positive linear map, positive operator.

1. INTRODUCTION

We recall some definitions. Let $\mathcal{A}$ be a C^*-algebra of operators on a Hilbert space H and $B(H)$ be the C^*-algebra of all bounded linear operators on H. A real valued function f is said to be *operator convex* on an interval I in $\mathbb{R}$ if

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1 - \lambda f(B)$$

holds for each $\lambda \in [0,1]$ and every pair of self-adjoint operators A, B in $\mathcal{A}$ with spectra in I. A real valued function f is said to be *operator monotone* on I if

$$A \leq B \quad \text{implies} \quad f(A) \leq f(B)$$

for every pair of self-adjoint operators A, B in $\mathcal{A}$ with spectra in I.

Let T be a locally compact Hausdorff space. We say that a field $(x_t)_{t\in T}$ of operators in $\mathcal{A}$ is continuous if the function $t \mapsto x_t$ is norm continuous on T. If in addition μ is a bounded Radon measure on T and the function $t \mapsto \|x_t\|$ is integrable, then we can form the Bochner integral $\int_T x_t\,\mathrm{d}\mu(t)$, which is the unique element in the multiplier algebra

$$M(\mathcal{A}) = \{a \in B(H) \mid \forall x \in \mathcal{A} : ax + xa \in \mathcal{A}\}$$

such that

$$\varphi\left(\int_T x_t\,\mathrm{d}\mu(t)\right) = \int_T \varphi(x_t)\,\mathrm{d}\mu(t) \tag{1}$$

for every linear functional φ in the norm dual $\mathcal{A}^*$.

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras on Hilbert spaces H and K. Assume furthermore that there is a field $(\phi_t)_{t\in T}$ of positive linear maps $\phi_t : \mathcal{A} \to \mathcal{B}$. We say that such a field is continuous if the function $t \mapsto \phi_t(x)$ is continuous for every $x \in \mathcal{A}$.

We denote by $\mathrm{P}_k[\mathcal{A},\mathcal{B}]$ the set of all fields $(\phi_t)_{t\in T}$ of positive linear maps $\phi_t : \mathcal{A} \to \mathcal{B}$, such that the field $t \to \phi_t(\mathbf{1})$ is integrable with $\int_T \phi_t(\mathbf{1})\,\mathrm{d}\mu(t) = k\mathbf{1}$ for some positive scalar k.

Recently, J. Mićić, J. Pečarić and Y. Seo in [6] gave a general formulation of Jensen's operator inequality and its converses shown in the next two theorems:

Theorem A ([6, Theorem 2.1]). *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras on a Hilbert spaces H and K. Let $(x_t)_{t\in T}$ be a bounded continuous field of self-adjoint elements in $\mathcal{A}$ with spectra in an interval I and $(\phi_t)_{t\in T} \in \mathrm{P}_k[\mathcal{A},\mathcal{B}]$ for some positive scalar k. If $f : I \to \mathbb{R}$ is an operator convex function defined on I, then the inequality*

$$f\left(\frac{1}{k}\int_T \phi_t(x_t)\,\mathrm{d}\mu(t)\right) \le \frac{1}{k}\int_T \phi_t(f(x_t))\,\mathrm{d}\mu(t) \tag{2}$$

holds. In the dual case (when f is operator concave) the opposite inequality holds in (2).

Theorem B ([6, Theorem 2.2]). *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras on a Hilbert spaces H and K. Let $(x_t)_{t\in T}$ be a bounded continuous field of self-adjoint elements in $\mathcal{A}$ with spectra in $[m,M]$ and $(\phi_t)_{t\in T} \in \mathrm{P}_k[\mathcal{A},\mathcal{B}]$ for some positive scalar k. Let $f : [m,M] \to \mathbb{R}$, $g : [km,kM] \to \mathbb{R}$ and $F : U\times V \to \mathbb{R}$ be functions such that $(kf)([m,M]) \subset U$, $g([km,kM]) \subset V$ and F is bounded. Let $\{conx.\}$ (resp. $\{conc.\}$) denotes the set of operator convex (resp. operator concave) functions defined on $[m,M]$. Let $f : [m,M] \to \mathbb{R}$, $g : [km,kM] \to \mathbb{R}$ and $F : U \times V \to \mathbb{R}$ be functions such that $(kf)([m,M]) \subset U$, $g([km,kM]) \subset V$ and F is bounded. If F is operator monotone in the first variable, then*

$$\begin{aligned} &\inf_{km\le z\le kM} F\left[k\cdot h_1\left(\frac{1}{k}z\right), g(z)\right]\mathbf{1} \le \\ &\le F\left[\int_T \phi_t\left(f(x_t)\right)\mathrm{d}\mu(t), g\left(\int_T \phi_t(x_t)\mathrm{d}\mu(t)\right)\right] \le \\ &\le \sup_{km\le z\le kM} F\left[k\cdot h_2\left(\frac{1}{k}z\right), g(z)\right]\mathbf{1} \end{aligned} \tag{3}$$

holds for every operator convex function h_1 on $[m,M]$ such that $h_1 \le f$ and for every operator concave function h_2 on $[m,M]$ such that $h_2 \ge f$.

The goal of this paper is to examine the order among the following generalized quasi-arithmetic operator means

$$M_\varphi(\mathbf{x},\phi) = \varphi^{-1}\left(\frac{\int_T \phi_t\left(\varphi(x_t)\right)\mathrm{d}\mu(t)}{k}\right), \tag{4}$$

under these conditions: $(x_t)_{t\in T}$ is a field of positive operators in $B(H)$ with spectra in $[m,M]$ for some scalars $0<m<M$, $(\phi_t)_{t\in T}\in \mathrm{P}_k[B(H),B(K)]$ for some positive scalar k and $\varphi\in\mathcal{C}[m,M]$ is a strictly monotone function.

We denote $M_\varphi(\mathbf{x},\phi)$ shortly with M_φ. Also, we use the notation

$$\varphi_m = \min\{\varphi(m),\varphi(M)\},\quad \varphi_M = \max\{\varphi(m),\varphi(M)\}$$

for a strictly monotone function $\varphi\in C[m,M]$.

Since $m\mathbf{1}\le x_t\le M\mathbf{1}$ for every $t\in T$ and φ is monotone, then $\varphi_m\mathbf{1}\le \varphi(x_t)\le\varphi_M\mathbf{1}$. Applying a positive linear map ϕ_t and integrating, it follows that

$$\varphi_m k\mathbf{1}\le\int_T\phi_t\left(\varphi(x_t)\right)\mathrm{d}\mu(t)\le\varphi_M k\mathbf{1},$$

since $\int_T\phi_t(\mathbf{1})\,\mathrm{d}\mu(t)=k\mathbf{1}$. Then the spectrum of $\int_T\phi_t\left(\varphi(x_t)\right)\mathrm{d}\mu(t)/\,k$ is a subset of $[\varphi_m,\varphi_M]$. Hence, the mean M_φ is well-defined with (4).

As a special case of (4), we may consider the power operator mean, see e.g. [6],

$$M_r(\mathbf{x},\phi)=\begin{cases}\left(\dfrac{\int_T\phi_t\left(x_t^r\right)\mathrm{d}\mu(t)}{k}\right)^{1/r}, & r\neq 0,\\ \exp\left(\dfrac{1}{k}\displaystyle\int_T\phi_t\left(\ln x_t\right)\mathrm{d}\mu(t)\right), & r=0.\end{cases} \tag{5}$$

2. INEQUALITIES INVOLVING THE ORDER OF QUASI-ARITHMETIC MEANS

In this section we study the monotonicity of quasi-arithmetic means.

THEOREM 2.1. *Let $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean (4). Let $\psi,\varphi\in\mathcal{C}[m,M]$ be strictly monotone functions. If one of the following conditions is satisfied:*

(i) *$\psi\circ\varphi^{-1}$ is operator convex and ψ^{-1} is operator monotone,*

(i′) *$\psi\circ\varphi^{-1}$ is operator concave and $-\psi^{-1}$ is operator monotone,*

then

$$M_\varphi\le M_\psi. \tag{6}$$

If one of the following conditions is satisfied:

(ii) *$\psi\circ\varphi^{-1}$ is operator concave and ψ^{-1} is operator monotone,*

(ii′) *$\psi\circ\varphi^{-1}$ is operator convex and $-\psi^{-1}$ is operator monotone,* *then the reverse inequality is valid in* (6).

Proof. (i): If we put $f = \psi \circ \varphi^{-1}$ and $I = [\varphi_m, \varphi_M]$ in Theorem A, we obtain

$$\psi \circ \varphi^{-1}\left(\frac{1}{k}\int_T \phi_t\left(\varphi(x_t)\right)\mathrm{d}\mu(t)\right) \leq \frac{1}{k}\int_T \phi_t\left(\psi(x_t)\right)\mathrm{d}\mu(t). \tag{7}$$

Since ψ^{-1} is operator monotone, it follows that

$$\varphi^{-1}\left(\frac{1}{k}\int_T \phi_t\left(\varphi(x_t)\right)\mathrm{d}\mu(t)\right) \leq \psi^{-1}\left(\frac{1}{k}\int_T \phi_t\left(\psi(x_t)\right)\mathrm{d}\mu(t)\right),$$

which is the desired inequality (6).

(i′): Since $\psi \circ \varphi^{-1}$ is operator concave, we obtain the reverse inequality in (7). Now, applying an operator monotone function $-\psi^{-1}$, we obtain (7) in this case too.

In cases (ii) and (ii′), the proof is essentially the same as in previous cases. □

We can give the following generalization of the previous theorem.

COROLLARY 2.2. *Let $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean* (4). *Let $\psi, \varphi \in \mathcal{C}[m, M]$ be strictly monotone functions and $F : [m, M] \times [m, M] \to \mathbb{R}$ be a bounded and operator monotone function in its first variable, such that $F(z, z) = C$ for all $z \in [m, M]$.*
If one of the following conditions is satisfied:

(i) *$\psi \circ \varphi^{-1}$ is operator convex and ψ^{-1} is operator monotone,*

(i′) *$\psi \circ \varphi^{-1}$ is operator concave and $-\psi^{-1}$ is operator monotone,*

then

$$F\left[M_\psi, M_\varphi\right] \geq C\mathbf{1}. \tag{8}$$

If one of the following conditions is satisfied:

(ii) *$\psi \circ \varphi^{-1}$ is operator concave and ψ^{-1} is operator monotone,*

(ii′) *$\psi \circ \varphi^{-1}$ is operator convex and $-\psi^{-1}$ is operator monotone,*

then the reverse inequality is valid in (8).

Proof. Suppose (i) or (i′). Then by Theorem 2.1 we have

$$M_\varphi \leq M_\psi.$$

Using assumptions about function F, it follows

$$F\left[M_\psi, M_\varphi\right] \geq F\left[M_\varphi, M_\varphi\right] \geq \inf_{m\leq z\leq M} F(z, z)\mathbf{1} = C\mathbf{1}.$$

In the remaining cases the proof is essentially the same as in previous cases. □

THEOREM 2.3. *Let $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean* (4) *and $\psi, \varphi \in \mathcal{C}[m, M]$ be strictly monotone functions.*

(i) *If φ^{-1} is operator convex and ψ^{-1} is operator concave, then*

$$M_\varphi \leq M_1 \leq M_\psi. \tag{9}$$

(ii) *If φ^{-1} is operator concave and ψ^{-1} is operator convex then the reverse inequality is valid in* (9).

Proof. We prove only the case (i): Using Theorem A for a operator convex function φ^{-1} on $[\varphi_m, \varphi_M]$, we have

$$M_\varphi = \varphi^{-1}\left(\frac{1}{k}\int_T \phi_t\left(\varphi(x_t)\right) \mathrm{d}\mu(t)\right) \leq \frac{1}{k}\int_T \phi_t\left(x_t\right) \mathrm{d}\mu(t) = M_1,$$

which gives LHS of (9). Similarly, since ψ^{-1} is operator concave on $I = [\psi_m, \psi_M]$, we have

$$M_1 = \frac{1}{k}\int_T \phi_t\left(x_t\right) \mathrm{d}\mu(t) \leq \psi^{-1}\left(\frac{1}{k}\int_T \phi_t\left(\psi(x_t)\right) \mathrm{d}\mu(t)\right) = M_\psi,$$

which gives RHS of (9). □

THEOREM 2.4. *Let $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean* (4) *and $\psi, \varphi \in \mathcal{C}[m, M]$ be strictly monotone functions.*

(i) *If $\varphi = A\psi + B$, where A, B are real numbers, then $M_\varphi = M_\psi$.*

(ii) *If $\psi \circ \varphi^{-1}$ is an operator convex function and*

$$M_\varphi = M_\psi \quad \textit{for all } (x_t)_{t\in T} \textit{ and } (\phi_t)_{t\in T},$$

then $\varphi = A\psi + B$ for some real numbers A and B.

Proof. The case (i) is obvious.

(ii) Let

$$\varphi^{-1}\left(\frac{1}{k}\int_T \phi_t\left(\varphi(x_t)\right) \mathrm{d}\mu(t)\right) = \psi^{-1}\left(\frac{1}{k}\int_T \phi_t\left(\psi(x_t)\right) \mathrm{d}\mu(t)\right)$$

for all $(x_t)_{t\in T}$ and $(\phi_t)_{t\in T}$. Setting $y_t = \varphi(x_t) \in B(H)$, $\varphi_m \mathbf{1} \leq y_t \leq \varphi_M \mathbf{1}$, we obtain

$$\psi \circ \varphi^{-1}\left(\int_T \frac{1}{k}\phi_t\left(y_t\right) \mathrm{d}\mu(t)\right) = \int_T \frac{1}{k}\phi_t\left(\psi \circ \varphi^{-1}(y_t)\right) \mathrm{d}\mu(t) \tag{10}$$

for all $(y_t)_{t\in T}$ and $(\phi_t)_{t\in T}$. As in [4, the proof of Theorem 2.1] we consider C^*-algebra $CB(T, B(H))$ of bounded continuous functions on T with values in $B(H)$ by applying the point-wise operations and the norm $\|(y_t)_{t\in T}\| = \sup_{t\in T}\|y_t\|$. Also, $f((y_t)_{t\in T}) = (f(y_t))_{t\in T}$. Since the integral is an element in the multiplier algebra $M(B(K)) = B(K)$, we can consider the mapping $\pi\colon CB(T, B(H)) \to B(K)$ defined by

$$\pi\left((y_t)_{t\in T}\right) = \frac{1}{k} \int_T \phi_t(y_t)\,\mathrm{d}\mu(t),$$

and obviously that it is a unital positive linear map. Setting $y = (y_t)_{t\in T} \in CB(T, B(H))$ and $f = \psi \circ \varphi^{-1}$ we get from (10)

$$f(\pi(y)) \quad = f\left(\pi\left((y_t)_{t\in T}\right)\right) = \pi\left((f(y_t))_{t\in T}\right) = \pi\left(f\big((y_t)_{t\in T}\big)\right) = \pi(f(y)).$$

M.D. Choi states in [2, Theorem 2.5] that a Schwarz inequality $f(\Phi(y)) \leq \Phi(f(y))$ may became an equality for all self-adjoint y in the extraordinary cases: f is affine or Φ is homomorphism. In the case (10), this means that $f = \psi \circ \varphi^{-1}$ is affine, i.e., $\psi \circ \varphi^{-1}(u) = Au + B$ for some real numbers A and B, which gives the desired connection: $\psi(v) = A\varphi(v) + B$. □

There are many results about operator monotone or operator convex functions. E.g., using [3, Section 1.2], [1, Chapter V], we can obtain the following corollary.

COROLLARY 2.5. *Let $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean* (4) *and φ, ψ be continuous strictly monotone functions from $[0, \infty)$ into itself.*
If one of the following conditions is satisfied:

(i) *$\psi \circ \varphi^{-1}$ and ψ^{-1} are operator monotone,*

(i′) *$\varphi \circ \psi^{-1}$ is operator convex, $\varphi \circ \psi^{-1}(0) = 0$ and ψ^{-1} is operator monotone,*

then

$$M_\psi \leq M_1 \leq M_\varphi.$$

Specially, if one of the following conditions is satisfied:

(ii) *ψ^{-1} is operator monotone,*

(ii′) *ψ^{-1} is operator convex, $\varphi(0) = 0$,*

then

$$M_1 \leq M_\psi.$$

Proof. (i): We use the statement: *a bounded below function $f \in C([\alpha, \infty))$ is operator monotone iff f is operator concave* and we apply Theorem 2.1(ii).

(i′): We use the statement: *if a function $f\colon [0, \infty) \to [0, \infty)$ such that $f(0) = 0$ is operator convex, then f^{-1} is operator monotone* and Theorem 2.1(ii).

(ii) or (ii′): We put that φ is an affine function in (i) or (i′), respectively. □

Example 2.6. If we put $\varphi(t) = t^r$, $\psi(t) = t^s$ or $\varphi(t) = t^s$, $\psi(t) = t^r$ in Theorem 2.1 and Theorem 2.3, then we obtain (cf. [5, Theorem 11], [6, Remark 4.4])

$$M_r(\mathbf{x}, \phi) \leq M_s(\mathbf{x}, \phi)$$

for either $r \leq s$, $r \notin (-1, 1)$, $s \notin (-1, 1)$ or $1/2 \leq r \leq 1 \leq s$ or $r \leq -1 \leq s \leq -1/2$.

3. COMPLEMENTARY INEQUALITIES

In this section we study inequalities complementary to the order of quasi-arithmetic means.

First, we will give a complementary result to (i) or (i)′ of Theorem 2.1 under the assumption that $\psi \circ \varphi^{-1}$ is only convex or concave, respectively. In the following theorem we give a general result.

THEOREM 3.1. *Let $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean* (4). *Let $\psi, \varphi \in \mathcal{C}[m, M]$ be strictly monotone functions and $F : [m, M] \times [m, M] \to \mathbb{R}$ be a bounded and operator monotone function in its first variable.*
If one of the following conditions is satisfied:
(i) *$\psi \circ \varphi^{-1}$ is convex and ψ^{-1} is operator monotone,*
(i′) *$\psi \circ \varphi^{-1}$ is concave and $-\psi^{-1}$ is operator monotone,*
then

$$F\left[M_\psi, M_\varphi\right] \leq \leq \sup_{0\leq\theta\leq 1} F\left[\psi^{-1}\left(\theta\psi(M) + (1-\theta)\psi(m)\right), \varphi^{-1}\left(\theta\varphi(M) + (1-\theta)\varphi(m)\right)\right]\mathbf{1}. \tag{11}$$

If one of the following conditions is satisfied:
(ii) *$\psi \circ \varphi^{-1}$ is concave and ψ^{-1} is operator monotone,*
(ii′) *$\psi \circ \varphi^{-1}$ is convex and $-\psi^{-1}$ is operator monotone,*
then the opposite inequality is valid in (11) *with* inf *instead of* sup.

Proof. We prove only the case (i): Since the inequality

$$f(z) \leq \frac{f(M) - f(m)}{M - m}\,(z - m) + f(m), \quad z \in [m, M],$$

holds for any convex function $f \in \mathcal{C}[m, M]$, then we have that inequality

$$f(\varphi(z)) \leq \frac{f(\varphi_M) - f(\varphi_m)}{\varphi_M - \varphi_m}\,(\varphi(z) - \varphi_m) + f(\varphi_m), \quad z \in [m, M],$$

holds for any convex function $f \in \mathcal{C}[\varphi_m, \varphi_M]$. Then for a convex function $\psi \circ \varphi^{-1} \in \mathcal{C}[\varphi_m, \varphi_M]$, we obtain

$$\psi(z) \le \frac{\psi(\varphi^{-1}(\varphi_M)) - \psi(\varphi^{-1}(\varphi_m))}{\varphi_M - \varphi_m} \left(\varphi(z) - \varphi_m\right) + \psi(\varphi^{-1}(\varphi_m)), \quad z \in [m, M].$$

Thus, using the functional calculus,

$$\psi(x_t) \le \frac{\psi(\varphi^{-1}(\varphi_M)) - \psi(\varphi^{-1}(\varphi_m))}{\varphi_M - \varphi_m} \left(\varphi(x_t) - \varphi_m\right) + \psi(\varphi^{-1}(\varphi_m)), \quad t \in T.$$

Applying the positive linear map $\frac{1}{k}\phi_t$ and integrating, we obtain

$$\int_T \frac{1}{k}\,\phi_t\left(\psi(x_t)\right) \mathrm{d}\mu(t) \le$$
$$\le \frac{\psi(M) - \psi(m)}{\varphi(M) - \varphi(m)} \left(\int_T \frac{1}{k}\,\phi_t\left(\varphi(x_t)\right) \mathrm{d}\mu(t) - \varphi_m \mathbf{1}\right) + \psi(\varphi^{-1}(\varphi_m))\mathbf{1}.$$

Then, applying the operator monotone function ψ^{-1}, it follows

$$M_\psi \le \psi^{-1}\left(\frac{\psi(M) - \psi(m)}{\varphi(M) - \varphi(m)} \left(\varphi(M_\varphi) - \varphi_m \mathbf{1}\right) + \psi(\varphi^{-1}(\varphi_m))\mathbf{1}\right).$$

Finally, operator monotonicity of $F(\cdot\,, v)$ give

$$F\left[M_\psi, M_\varphi\right] \le$$
$$\le F\left[\psi^{-1}\left(\frac{\psi(M) - \psi(m)}{\varphi(M) - \varphi(m)} \left(\varphi(M_\varphi) - \varphi_m \mathbf{1}\right) + \psi(\varphi^{-1}(\varphi_m))\mathbf{1}\right), \varphi^{-1}\left(\varphi(M_\varphi)\right)\right] \le$$
$$\le \sup_{\varphi_m \le z \le \varphi_M} F\left[\psi^{-1}\left(\frac{\psi(M) - \psi(m)}{\varphi(M) - \varphi(m)} \left(z - \varphi_m\right) + \psi(\varphi^{-1}(\varphi_m))\right), \varphi^{-1}(z)\right] \mathbf{1} =$$
$$= \sup_{0 \le \theta \le 1} F\left[\psi^{-1}\left(\theta\psi(M) + (1-\theta)\psi(m)\right), \varphi^{-1}\left(\theta\varphi(M) + (1-\theta)\varphi(m)\right)\right] \mathbf{1},$$

which is the desired inequality (11). □

Remark 3.2. We can obtain similar inequalities as in Theorem 3.1 and Corollary 3.3 when $F : [m, M] \times [m, M] \to \mathbb{R}$ is a bounded and operator monotone function in its second variable.

It is particularly interesting to observe difference and ratio type inequalities when the function F in Theorem 3.1 has the form $F(u, v) = u - v$ and $F(u, v) = v^{-1/2} u v^{-1/2}$ $(v > 0)$. In these cases we have a generalization of [7, Theorem 3.5 and Theorem 4.4].

COROLLARY 3.3. *Let $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean* (4). *Let $\psi, \varphi \in \mathcal{C}[m, M]$ be strictly monotone functions and let one of the following conditions is satisfied:*

(i) *$\psi \circ \varphi^{-1}$ be convex* (*resp. concave*) *and ψ^{-1} is operator monotone,* (i′) $\psi \circ \varphi^{-1}$ *be concave* (*resp. convex*) *and* $-\psi^{-1}$ *is operator monotone. Then*

$$M_{\psi} \leq M_{\varphi} + \max_{0\leq\theta\leq 1} \left\{\psi^{-1}\left(\theta\psi(M)+(1-\theta)\psi(m)\right) - \varphi^{-1}\left(\theta\varphi(M)+(1-\theta)\varphi(m)\right)\right\}$$

$\big($*resp.*

$$M_{\psi} \geq M_{\varphi} + \min_{0\leq\theta\leq 1} \left\{\psi^{-1}(\theta\psi(M)+(1-\theta)\psi(m)) - \varphi^{-1}(\theta\varphi(M)+(1-\theta)\varphi(m))\right\}\big).$$

If in addition $\varphi > 0$ *on* $[m, M]$, *then*

$$M_{\psi} \leq \max_{0\leq\theta\leq 1} \left\{\frac{\psi^{-1}\left(\theta\psi(M)+(1-\theta)\psi(m)\right)}{\varphi^{-1}\left(\theta\varphi(M)+(1-\theta)\varphi(m)\right)}\right\} M_{\varphi}.$$

$$\Big(\textit{resp. } M_{\psi} \geq \min_{0\leq\theta\leq 1} \left\{\frac{\psi^{-1}\left(\theta\psi(M)+(1-\theta)\psi(m)\right)}{\varphi^{-1}\left(\theta\varphi(M)+(1-\theta)\varphi(m)\right)}\right\} M_{\varphi}\Big).$$

We will give a complementary result to (i) or (i)′ of Theorem 2.1 under the assumption that $\psi \circ \varphi^{-1}$ is operator convex and ψ^{-1} is not operator monotone. In the following theorem we give a general result.

Theorem 3.4. *Let* $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$ *be as in the definition of the quasi-arithmetic mean* (4). *Let* $\psi, \varphi \in \mathcal{C}[m, M]$ *be strictly monotone functions and* $F : [m, M] \times [m, M] \to \mathbb{R}$ *be a bounded and operator monotone function in its first variable.*
If one of the following conditions is satisfied:
(i) $\psi \circ \varphi^{-1}$ *is operator convex and* ψ^{-1} *is increasing convex,*
(i′) $\psi \circ \varphi^{-1}$ *is operator concave and* ψ^{-1} *is decreasing convex,*
then

$$F[M_{\varphi}, M_{\psi}] \leq \sup_{0\leq\theta\leq 1} F[\theta M + (1-\theta)m, \psi^{-1}(\theta\psi(M) + (1-\theta)\psi(m))]\mathbf{1}. \tag{12}$$

If one of the following conditions is satisfied:
(ii) $\psi \circ \varphi^{-1}$ *is operator convex and* ψ^{-1} *is decreasing concave,*
(ii′) $\psi \circ \varphi^{-1}$ *is operator concave and* ψ^{-1} *is increasing concave,*
then the opposite inequality is valid in (12) *with* inf *instead of* sup.

Proof. Let $\psi \circ \varphi^{-1}$ be operator convex. By using Theorem A, we have

$$\psi(M_{\varphi}) = \psi \circ \varphi^{-1}\left(\frac{1}{k}\int_T \phi_t\left(\varphi(x_t)\right) \mathrm{d}\mu(t)\right) \leq \frac{1}{k}\int_T \phi_t\left(\psi(x_t)\right) \mathrm{d}\mu(t) = \psi(M_{\psi}). \tag{13}$$

(i): Since ψ^{-1} is increasing, then $\psi(m)\mathbf{1} \le \psi(M_\varphi) \le \psi(M)\mathbf{1}$, and since ψ^{-1} is also convex we have

$$\begin{aligned} M_\varphi &= \psi^{-1}(\psi(M_\varphi)) \\ &\le \frac{M-m}{\psi(M)-\psi(m)}\left(\psi(M_\varphi)-\psi(m)\right)+m \quad \text{by convexity of } \psi^{-1} \\ &\le \frac{M-m}{\psi(M)-\psi(m)}\left(\psi(M_\psi)-\psi(m)\right)+m \quad \text{by increase of } \psi \text{ and (13).} \end{aligned}$$

Now, operator monotonicity of $F(\cdot\,,v)$ give

$$\begin{aligned} F\left[M_\varphi, M_\psi\right] &\le F\left[\frac{M-m}{\psi(M)-\psi(m)}\ \left(\psi(M_\psi)-\psi(m)\right)+m,\ \psi^{-1}\left(\psi(M_\psi)\right)\right] \\ &\le \sup_{\psi(m)\le z\le\psi(M)} F\left[\frac{M-m}{\psi(M)-\psi(m)}\ \left(z-\psi(m)\right)+m,\ \psi^{-1}\left(z\right)\right]\mathbf{1} \\ &= \sup_{0\le\theta\le1} F\left[\theta M+(1-\theta)m,\ \psi^{-1}\left(\theta\psi(M)+(1-\theta)\psi(m)\right)\right]\mathbf{1}, \end{aligned}$$

which is the desired inequality (12).

(ii): Since ψ^{-1} is decreasing, then $\psi(M)\mathbf{1} \le \psi(M_\varphi) \le \psi(M)\mathbf{1}$, and since ψ^{-1} is also concave we have

$$\begin{aligned} M_\varphi &= \psi^{-1}(\psi(M_\varphi)) \\ &\ge \frac{m-M}{\psi(m)-\psi(M)}\left(\psi(M_\varphi)-\psi(m)\right)+m \quad \text{by concavity of } \psi^{-1} \\ &\ge \frac{m-M}{\psi(m)-\psi(M)}\left(\psi(M_\psi)-\psi(m)\right)+m \quad \text{by decrease of } \psi \text{ and (13).} \end{aligned}$$

Now, operator monotonicity of $F(\cdot\,,v)$ give

$$\begin{aligned} F\left[M_\varphi, M_\psi\right] &\ge F\left[\frac{M-m}{\psi(M)-\psi(m)}\left(\psi(M_\psi)-\psi(m)\right)+m, \psi^{-1}\left(\psi(M_\psi)\right)\right] \\ &\ge \inf_{\psi(M)\le z\le\psi(m)} F\left[\frac{M-m}{\psi(M)-\psi(m)}\left(z-\psi(m)\right)+m, \psi^{-1}\left(z\right)\right]\mathbf{1} \\ &= \inf_{0\le\theta\le1} F\left[\theta M+(1-\theta)m, \psi^{-1}\left(\theta\psi(M)+(1-\theta)\psi(m)\right)\right]\mathbf{1}, \end{aligned}$$

which is the desired inequality.

In cases (i′) and (ii′), the proof is essentially the same as in previous cases. □

Remark 3.5. Similar to Corollary 3.3, we have the following results by using Theorem 3.4.

Let one of the following conditions be satisfied:

(i) $\psi \circ \varphi^{-1}$ *is operator convex and* ψ^{-1} *is increasing convex,*

(i′) $\psi \circ \varphi^{-1}$ *is operator concave and* ψ^{-1} *is decreasing convex.*
Then

$$M_\varphi \le M_\psi + \max_{0\le\theta\le1} \left\{\theta M + (1-\theta)m - \psi^{-1}\left(\theta\psi(M) + (1-\theta)\psi(m)\right)\right\}\mathbf{1},$$

and if, additionally, $\psi > 0$ on $[m, M]$, then

$$M_\varphi \le \max_{0\le\theta\le1} \left\{\frac{\theta M + (1-\theta)m}{\psi^{-1}\left(\theta\psi(M) + (1-\theta)\psi(m)\right)}\right\} M_\psi.$$

Let one of the following conditions be satisfied:
(ii) $\psi \circ \varphi^{-1}$ *is operator convex and* ψ^{-1} *is decreasing concave,*
(ii′) $\psi \circ \varphi^{-1}$ *is operator concave and* ψ^{-1} *is increasing concave.*
Then

$$M_\varphi \ge M_\psi + \min_{0\le\theta\le1} \left\{\theta M + (1-\theta)m - \psi^{-1}\left(\theta\psi(M) + (1-\theta)\psi(m)\right)\right\}\mathbf{1},$$

and if, additionally, $\psi > 0$ on $[m, M]$, then

$$M_\varphi \ge \min_{0\le\theta\le1} \left\{\frac{\theta M + (1-\theta)m}{\psi^{-1}\left(\theta\psi(M) + (1-\theta)\psi(m)\right)}\right\} M_\psi.$$

There are a generalization of some results from [7, Theorem 3.1 and Theorem 3.3] and the proof given in them is different than one in Theorem 3.4.

In the following theorem we give the complementary results to those given in the above remark.

Theorem 3.6. *Let* $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$ *be as in the definition of the quasi-arithmetic mean* (4) *and* $\psi, \varphi \in \mathcal{C}[m, M]$ *be strictly monotone functions. Let one of the following conditions be satisfied:*
(i) $\psi \circ \varphi^{-1}$ *is operator convex and* ψ^{-1} *is decreasing convex,*
(i′) $\psi \circ \varphi^{-1}$ *is operator concave and* ψ^{-1} *is increasing convex.*
Then

$$M_\psi \le M_\varphi + \max_{0\le\theta\le1} \left\{\theta M + (1-\theta)m - \psi^{-1}\left(\theta\psi(M) + (1-\theta)\psi(m)\right)\right\}\mathbf{1}, \tag{14}$$

and if, additionally, $\psi > 0$ *on* $[m, M]$, *then*

$$M_\psi \le \max_{0\le\theta\le1} \left\{\frac{\theta M + (1-\theta)m}{\psi^{-1}\left(\theta\psi(M) + (1-\theta)\psi(m)\right)}\right\} M_\varphi. \tag{15}$$

Let one of the following conditions be satisfied:
(ii) $\psi \circ \varphi^{-1}$ *is operator convex and* ψ^{-1} *is increasing concave,*
(ii′) $\psi \circ \varphi^{-1}$ *is operator concave and* ψ^{-1} *is decreasing concave.*
Then

$$M_\psi \ge M_\varphi + \min_{0\le\theta\le1} \left\{\theta M + (1-\theta)m - \psi^{-1}\left(\theta\psi(M) + (1-\theta)\psi(m)\right)\right\}\mathbf{1},$$

and if, additionally, $\psi > 0$ on $[m, M]$, then

$$M_\psi \geq \min_{0\leq\theta\leq1} \left\{ \frac{\theta M + (1-\theta)m}{\psi^{-1}\left(\theta\psi(M) + (1-\theta)\psi(m)\right)} \right\} M_\varphi.$$

Proof. The proof is essentially the same as the proof of [7, Theorem 3.1].

We prove only the case (i): Mond-Pečarić [8] showed that if A is a self-adjoint operator on H such that $m\mathbf{1} \leq A \leq M\mathbf{1}$ for some scalars $m \leq M$, $f \in C[m, M]$ is convex, then every unit vector $x \in H$

$$\begin{aligned}(16)\qquad & f((Ax,x)) \leq (f(A)x,x) \leq \\ & \leq \max_{m\leq z\leq M} \left\{ \frac{f(M)-f(m)}{M-m}(z-m) + f(m) - f(z) \right\} + f((Ax,x))\end{aligned}$$

and if, additionally, $f > 0$ then

$$\begin{aligned}(17)\qquad & f((Ax,x)) \leq (f(A)x,x) \leq \\ & \leq \max_{m\leq z\leq M} \left\{ \frac{\frac{f(M)-f(m)}{M-m}(z-m) + f(m)}{f(z)} \right\} f((Ax,x)).\end{aligned}$$

Also, since $\psi \circ \varphi^{-1}$ is operator convex, then $\psi(M_\varphi) \leq \psi(M_\psi)$. Then for every unit vector $x \in H$

$$\begin{aligned}
&(M_\varphi x, x) = (\psi^{-1} \circ \psi(M_\varphi)x, x) \\
&\geq \psi^{-1}(\psi(M_\varphi)x, x) \quad \text{by convexity of } \psi^{-1} \text{ and (16)} \\
&\geq \psi^{-1}(\psi(M_\psi)x, x) \quad \text{by decrease of } \psi^{-1} \text{ and operator convexity } \psi\circ\varphi^{-1} \\
&\geq (M_\psi x, x) - \max_{\psi(M)\leq z\leq\psi(m)} \left\{ \frac{m-M}{\psi^{-1}(m)-\psi^{-1}(M)}(z-m) + \psi^{-1}(m) - \psi^{-1}(z) \right\}\mathbf{1} \\
&\quad \text{by convexity of } \psi^{-1} \text{ and (16)} \\
&= (M_\psi x, x) - \max_{0\leq\theta\leq1} \left\{ \theta M + (1-\theta)m - \psi^{-1}\left(\theta\psi(M) + (1-\theta)\psi(m)\right) \right\}\mathbf{1}
\end{aligned}$$

and hence we have the desired inequality (14).

Similarly, for every unit vector $x \in H$

$$\begin{aligned}
(M_\varphi x, x) &\geq \psi^{-1}(\psi(M_\psi)x, x) \\
&\geq 1 \Big/ \max_{\psi(M)\leq z\leq\psi(m)} \left\{ \frac{\frac{m-M}{\psi^{-1}(m)-\psi^{-1}(M)}(z-m) + \psi^{-1}(m)}{\psi^{-1}(z)} \right\} (M_\psi x, x) \\
&\qquad \text{by convexity of } \psi^{-1} \text{ and (17)} \\
&= 1 \Big/ \max_{0\leq\theta\leq1} \left\{ \frac{\theta M + (1-\theta)m}{\psi^{-1}\left(\theta\psi(M) + (1-\theta)\psi(m)\right)} \right\} (M_\psi x, x)
\end{aligned}$$

and hence we have the desired inequality (15). $\square$

We will give a complementary result to Theorem 2.3. In the following theorem we give a general result. In [7, Theorem 3.4] a different proof was given for ratio cases.

THEOREM 3.7. *Let* $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$ *be as in the definition of the quasi-arithmetic mean* (4) *and* $\psi,\varphi\in\mathcal{C}[m,M]$ *be strictly monotone functions and* $F:[m,M]\times[m,M]\to\mathbb{R}$ *be a bounded and operator monotone function in its first variable.*

(i) *If* φ^{-1} *is operator convex and* ψ^{-1} *is concave, then*

$$F[M_\varphi,M_\psi]\le\sup_{0\le\theta\le1}F\left[\theta M+(1-\theta)m,\psi^{-1}\left(\theta\psi(M)+(1-\theta)\psi(m)\right)\right]\mathbf{1}. \tag{18}$$

(ii) *If* φ^{-1} *is convex and* ψ^{-1} *is operator concave then*

$$F[M_\psi,M_\varphi]\ge\inf_{0\le\theta\le1}F\left[\theta M+(1-\theta)m,\varphi^{-1}\left(\theta\varphi(M)+(1-\theta)\varphi(m)\right)\right]\mathbf{1}. \tag{19}$$

Proof. (i): Using LHS of (9) for a operator convex function φ^{-1} and then operator monotonicity of $F(\cdot\,,v)$ we have

$$F[M_\varphi,M_\psi]\le F[M_1,M_\psi]. \tag{20}$$

If we put $\psi=I$ the identity function and replace φ by ψ in (11), then

$$F\left[M_1,M_\psi\right]\le\sup_{0\le\theta\le1}F\left[\theta M+(1-\theta)m,\psi^{-1}\left(\theta\psi(M)+(1-\theta)\psi(m)\right)\right]\ \mathbf{1}. \tag{21}$$

Combining two inequalities (20) and (21), we have the desired inequality (18).

(ii): We have (19) using a similar method as in (i). $\square$

COROLLARY 3.8. *Let* $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$ *be as in the definition of the quasi-arithmetic mean* (4) *and* $\psi,\varphi\in\mathcal{C}[m,M]$ *be strictly monotone functions. If* φ^{-1} *is convex and* ψ^{-1} *is concave, then*

$$\begin{aligned}M_\varphi\le M_\psi&+\max_{0\le\theta\le1}\left\{\theta M+(1-\theta)m-\psi^{-1}\left(\theta\psi(M)+(1-\theta)\psi(m)\right)\right\}\mathbf{1}\\&+\max_{0\le\theta\le1}\left\{\varphi^{-1}\left(\theta\varphi(M)+(1-\theta)\varphi(m)\right)-\theta M-(1-\theta)m\right\}\mathbf{1},\end{aligned} \tag{22}$$

and if, additionally, $\varphi>$ *and* $\psi>0$ *on* $[m,M]$, *then*

$$\begin{aligned}M_\varphi\le&\max_{0\le\theta\le1}\left\{\frac{\theta M+(1-\theta)m}{\psi^{-1}\left(\theta\psi(M)+(1-\theta)\psi(m)\right)}\right\}\times\\&\times\max_{0\le\theta\le1}\left\{\frac{\varphi^{-1}\left(\theta\varphi(M)+(1-\theta)\varphi(m)\right)}{\theta M+(1-\theta)m}\right\}M_\psi.\end{aligned} \tag{23}$$

Proof. If we put $F(u,v)=u-v$ and $\varphi=I$ in (18), then for any concave function ψ^{-1} we have

$$M_1-M_\psi\le\max_{0\le\theta\le1}\left\{\theta M+(1-\theta)m-\psi^{-1}\left(\theta\psi(M)+(1-\theta)\psi(m)\right)\right\}\mathbf{1}. \tag{24}$$

Similarly, if we put $\psi = I$ in (19), then for any convex function φ^{-1} we have

$$(25)\quad M_1 - M_\varphi \geq \min_{0\leq\theta\leq 1} \left\{\theta M + (1-\theta)m - \varphi^{-1}\left(\theta\varphi(M) + (1-\theta)\varphi(m)\right)\right\} \mathbf{1}.$$

Combining two inequalities (24) and (25), we have the inequality (22).

We have (23) by a similar method. □

If we directly use conversions of Jensen's operator inequality (2) given in Theorem B when the function F has the form $F(u,v) = u - v$ or $F(u,v) = v^{-1/2}uv^{-1/2}$ $(v > 0)$, then we obtain the following two corollaries.

Corollary 3.9. *Let $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean* (4) *and $\psi, \varphi \in \mathcal{C}[m, M]$ be strictly monotone functions. Let $\psi \circ \varphi^{-1}$ be convex* (*resp. concave*).

(i) *If ψ^{-1} is operator monotone and operator subadditive* (*resp. operator superadditive*) *on $\mathbb{R}^+$, then*

$$(26)\qquad M_\psi \leq M_\varphi + \psi^{-1}(\beta)\mathbf{1} \quad \left(\textit{resp. } M_\psi \geq M_\varphi + \psi^{-1}(\beta)\mathbf{1}\right),$$

(i′) *if $-\psi^{-1}$ is operator monotone and operator subadditive* (*resp. operator superadditive*) *on $\mathbb{R}^+$, then the opposite inequality is valid in* (12),

(ii) *if ψ^{-1} is operator monotone and operator superadditive* (*resp. operator subadditive*) *on $\mathbb{R}$, then*

$$(27)\qquad M_\psi \leq M_\varphi - \varphi^{-1}(-\beta)\mathbf{1} \quad \left(\textit{resp. } M_\psi \geq M_\varphi - \varphi^{-1}(-\beta)\mathbf{1}\right),$$

(ii′) *if $-\psi^{-1}$ is operator monotone and operator superadditive* (*resp. operator subadditive*) *on $\mathbb{R}$, then the opposite inequality is valid in* (27), *where*

$$(28)\quad \beta = \max_{0\leq\theta\leq 1} \left\{\theta\psi(M) + (1-\theta)\psi(m) - \psi\circ\varphi^{-1}\left(\theta\varphi(M) + (1-\theta)\varphi(m)\right)\right\}$$

$$\left(\textit{resp.}\quad \beta = \min_{0\leq\theta\leq 1} \left\{\theta\psi(M) + (1-\theta)\psi(m) - \psi\circ\varphi^{-1}\left(\theta\varphi(M) + (1-\theta)\varphi(m)\right)\right\}\right).$$

Proof. (i): We will prove only the case when $\psi\circ\varphi^{-1}$ is convex. Putting $F(u,v) = u - v$ and $f = g$ convex in Theorem B, we have (cf. also [6, Corollary 2.5], [5, Corollary 1]):

$$\frac{1}{k}\int_T \phi_t\left(f(x_t)\right) \mathrm{d}\mu(t) \leq f\left(\frac{1}{k}\int_T \phi_t(x_t)\mathrm{d}\mu(t)\right) + \\ + \max_{m\leq z\leq M}\left\{\frac{f(M) - f(m)}{M - m}(z - m) + f(m) - f(z)\right\}\mathbf{1}.$$

Since $\psi\circ\varphi^{-1}$ is convex, it follows

$$(29)\qquad \psi(M_\psi) = \int_T \frac{1}{k}\phi_t\left(\psi\circ\varphi^{-1}\left(\varphi(x_t)\right)\right)\mathrm{d}\mu(t) \leq \psi\circ\varphi^{-1}\left(\varphi(M_\varphi)\right) + \beta\mathbf{1},$$

where

$$\beta = \max_{\varphi_m \le z \le \varphi_M} \left\{ \frac{\psi(M) - \psi(m)}{\varphi(M) - \varphi(m)} (z - \varphi_m) + \psi \circ \varphi^{-1}(\varphi_m) - \psi \circ \varphi^{-1}(z) \right\}$$

which gives (28). Since ψ^{-1} is operator monotone and subadditive on $\mathbb{R}^+$, using (29) we obtain

$$M_\psi \le \psi^{-1}\left(M_\varphi + \beta \mathbf{1}\right) \le M_\varphi + \psi^{-1}(\beta)\mathbf{1}.$$

In the remaining cases the proof is essentially the same as in the previous case. □

COROLLARY 3.10. *Let $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean* (4) *and $\psi, \varphi \in \mathcal{C}[m, M]$ be strictly monotone functions. Let $\psi \circ \varphi^{-1}$ be convex and $\psi > 0$ (resp. $\psi < 0$) on $[m, M]$.*

(i) *If ψ^{-1} is operator monotone and operator submultiplicative on $\mathbb{R}^+$, then*

$$M_\psi \le \psi^{-1}(\alpha)\, M_\varphi, \tag{30}$$

(i′) *if $-\psi^{-1}$ is operator monotone and operator submultiplicative on $\mathbb{R}^+$, then the opposite inequality is valid in* (30)*,*

(ii) *if ψ^{-1} is operator monotone and operator supermultiplicative on $\mathbb{R}$, then*

$$M_\psi \le \left[\psi^{-1}(\alpha^{-1})\right]^{-1} M_\varphi, \tag{31}$$

(ii′) *if $-\psi^{-1}$ is operator monotone and operator supermultiplicative on $\mathbb{R}$, then the opposite inequality is valid in* (31)*, where*

$$\alpha = \max_{0 \le \theta \le 1} \left\{ \frac{\theta\psi(M) + (1-\theta)\psi(m)}{\psi \circ \varphi^{-1}\left(\theta\varphi(M) + (1-\theta)\varphi(m)\right)} \right\} \tag{32}$$

$$\left(\textit{resp.} \quad \alpha = \min_{0 \le \theta \le 1} \left\{ \frac{\theta\psi(M) + (1-\theta)\psi(m)}{\psi \circ \varphi^{-1}\left(\theta\varphi(M) + (1-\theta)\varphi(m)\right)} \right\}\right).$$

The *proof* is essentially the same as that of Corollary 3.9 and we omit it.

Remark 3.11. We note that we can obtain similar inequalities as in Corollary 3.10 when $\psi \circ \varphi^{-1}$ is a concave function, in the same way as we did in Corollary 3.9. E.g. if $\psi > 0$ (resp. $\psi < 0$) on $[m, M]$ is operator monotone and supermultiplicative on $\mathbb{R}^+$, then

$$M_\psi \ge \psi^{-1}(\alpha)\, M_\varphi,$$

with min instead of max in (32).

Example 3.12. If we put $\varphi(t) = t^s$ and $\psi(t) = t^r$ in inequalities involving the complementary order among quasi-arithmetic means, we can obtain the complementary order among power means. E.g. using Corollary 3.3, we obtain that

$$M_s(\mathbf{x},\phi) \le \max_{0\le\theta\le1}\left\{\frac{\sqrt[r]{(\theta M^r+(1-\theta)m^r)}}{\sqrt[s]{(\theta M^s+(1-\theta)m^s)}}\right\} M_r(\mathbf{x},\phi) = \Delta(h,r,s)M_r(\mathbf{x},\phi)$$

holds for $r \le s$, $s \ge 1$ or $r \le s \le -1$, where $\Delta(h,r,s)$ is the generalized Specht ratio defined by (see [3, (2.97)])

$$\Delta(h,r,s) = \left\{\frac{r(h^s-h^r)}{(s-r)(h^r-1)}\right\}^{\frac{1}{s}} \left\{\frac{s(h^r-h^s)}{(r-s)(h^s-1)}\right\}^{-\frac{1}{r}}, \quad h = \frac{M}{m}.$$

We obtain the same bound as in [5].

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Received 23 May 2010

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