ON THE QUASI-CONFORMAL CURVATURE TENSOR OF A (k, μ)-CONTACT METRIC MANIFOLD

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The object of the present paper is to study a quasi-conformally flat (k, μ)-contact metric manifold and such a manifold with vanishing extended quasi-conformal curvature tensor. Quasi-conformally semisymmetric and quasi-conformally recurrent (k, μ)-contact metric manifolds are also considered.

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1. INTRODUCTION

In [4], the authors introduced a class of contact metric manifolds for which the characteristic vector field ξ belongs to the (k, μ)-nullity distribution for some real numbers k and μ. Such manifolds are known as (k, μ)-contact metric manifolds. The class of (k, μ)-contact metric manifolds encloses both Sasakian and non-Sasakian manifolds. Before Boeckx [5], two classes of non-Sasakian (k, μ)-contact metric manifolds were known. The first class consists of the unit tangent sphere bundles of spaces of constant curvature, equipped with their natural contact metric structure, and the second class contains all the three-dimensional unimodular Lie groups, except the commutative one, admitting the structure of a left invariant (k, μ)-contact metric manifold [4], [5], [18]. A full classification of (k, μ)-contact metric manifolds was given by E. Boeckx [5]. (k, μ)-contact metric manifolds are invariant under D-homothetic transformations. Recently, in [15], the authors proved that a non-Sasakian contact metric manifold with η-parallel torsion tensor and sectional curvatures, of plane sections containing the Reeb vector field, different from 1 at some point, is a (k, μ)-contact manifold. In another recent paper [21], R. Sharma showed that if a (k, μ)-contact metric manifold admits a non-zero holomorphically planer conformal vector field, then it is either Sasakian, or, locally isometric to E^3 or $E^{n+1} \times S^n(4)$. In [8], J.T. Cho studied a conformally flat contact Riemannian (k, μ)-space and such a space with vanishing

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C-Bochner curvature tensor. In [14], the authors investigated on the conformal curvature tensor of a (k,μ)-contact metric manifold. Conformal curvature tensor of a (k,μ)-contact metric manifold was also studied in the paper [9]. In [19], D. Perrone worked on a contact Riemannian manifold satisfying $R(X,\xi)\cdot R=0$. Again, in [20], B.J. Papantoniou dealt with a contact Riemannian manifold satisfying $R(X,\xi)\cdot R=0$ and ξ belonging to (k,μ)-nullity distribution. In [1], the authors studied extended pseudo projective curvature tensor on a contact metric manifold. Recently, quasi-conformal curvature tensor on a Sasakian manifold has been studied by U.C. De, J.B. Jun and A.K. Gazi [10]. In this connection, we mention some works of H. Endo [12], [13] regarding some curvature tensors of contact and K-contact manifolds. After the Riemannian curvature tensor, Weyl conformal curvature tensor plays an important role in differential geometry as well as in theory of relativity. Yano and Sawaki [23] introduced the notion of quasi-conformal curvature tensor which is generalization of conformal curvature tensor in a Riemannian manifold. In a recent paper De and Matsuyama [11] studied quasi-conformally flat manifold satisfying certain condition on the Ricci tensor. They proved that a quasi-conformally flat manifold satisfying

$$S(X,Y)=rT(X)T(Y), \tag{1.1}$$

where S is the Ricci tensor, r is the scalar curvature and T is a nonzero 1-form defined by $T(X)=g(X,\rho)$, ρ is a unit vector field, can be expressed as a locally warped product $I\times_{e^q}M^*$, where M^* is an Einstein manifold. From this result, it easily follows that a quasi-conformally flat space-time satisfying (1.1) is a Robertson-Walker space time [17]. The aim of our present paper is to study a quasi-conformally flat (k,μ)-contact metric manifold. We also like to study a (k,μ)-contact metric manifold with vanishing extended quasi-conformal curvature tensor. We also consider quasi-conformally semisymmetric and quasi-conformally recurrent (k,μ)-contact metric manifolds. The present paper is organized as follows:

In Section 2, we discuss about a (k,μ)-contact metric manifold and give some preliminary results regarding such a manifold. Section 3 deals with a quasi-conformally flat (k,μ)-contact metric manifold. Section 4 is devoted to study a (k,μ)-contact metric manifold with vanishing extended quasi-conformal curvature tensor. Section 5 contains the study of a quasi-conformally semisymmetric (k,μ)-contact metric manifold. The last section is devoted to study a quasi-conformally recurrent (k,μ)-contact metric manifold.

2. PRELIMINARIES

Let M be a $(2n+1)$-dimensional C^∞-differentiable manifold. The manifold is said to admit an almost contact metric structure (ϕ,ξ,η,g) if it satisfies the following relations:

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(X, \xi) = \eta(X), \tag{2.1}$$

$$\phi\xi = 0, \quad \eta\phi = 0, \quad g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

where ϕ is a tensor field of type $(1, 1)$, ξ is a vector field, η is an 1-form and g is a Riemannian metric on M. A manifold equipped with an almost contact metric structure is called an almost contact metric manifold. An almost contact metric manifold is called a contact metric manifold if it satisfies

$$g(X, \phi Y) = d\eta(X, Y).$$

Given a contact metric manifold $M(\phi, \xi, \eta, g)$, we consider a $(1, 1)$ tensor field h defined by $h = \frac{1}{2} L_\xi \phi$, where L denotes Lie differentiation. h is a symmetric operator and h satisfies $h\phi = -\phi h$. If λ is an eigenvalue of h with eigenvector X, then $-\lambda$ is also an eigenvalue of h with eigenvector ϕX. Again, we have $\mathrm{tr} h = \mathrm{tr} \phi h = 0$, and $h\xi = 0$. Moreover, if ∇ denotes the Riemannian connection of g, then the following relation holds [4]:

$$\nabla_X \xi = -\phi X - \phi h X. \tag{2.4}$$

The vector field ξ is a Killing vector field with respect to g if and only if $h = 0$. A contact metric manifold $M(\phi, \xi, \eta, g)$ for which ξ is a Killing vector is said to be a K-contact manifold. A K-contact structure on M gives rise to an almost complex structure on the product $M \times \mathbb{R}$. If this almost complex structure is integrable, the contact metric manifold is said to be Sasakian. Equivalently, a contact metric manifold is said to be Sasakian if and only if

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

holds for all X, Y, where R denotes the Riemannian curvature tensor of the manifold M. The (k, μ)-nullity distribution of a contact metric manifold $M(\phi, \xi, \eta, g)$ is a distribution [4]

$$\begin{aligned} N(k, \mu) : p \to N_p(k, \mu) = \{Z \in T_p(M) : R(X, Y)Z = \\ = k(g(Y, Z)X - g(X, Z)Y) + \mu(g(Y, Z)hX - g(X, Z)hY)\}, \end{aligned} \tag{2.5}$$

for any $X, Y \in T_p M$. Hence, if the characteristic vector field ξ belongs to the (k, μ)-nullity distribution, we have

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{2.6}$$

A contact metric manifold with ξ belonging to (k, μ)-nullity distribution is called a (k, μ)-contact metric manifold. If $k = 1$, $\mu = 0$, then the manifold becomes Sasakian [4]. In particular, if $\mu = 0$, then the notion of (k, μ)-nullity distribution reduces to k-nullity distribution, introduced by S. Tanno [22]. A contact metric manifold with ξ belonging to k-nullity distribution is known as $N(k)$-contact metric manifold.

In a (k,μ)-contact metric manifold we have the following [4]:

$$h^2 = (k-1)\phi^2, \quad k \le 1, \tag{2.7}$$

$$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.8}$$

$$\eta(R(X,Y)Z) = \tag{2.9}$$
$$= k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] + \mu[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)],$$

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{2.10}$$

$$R(\xi, X)\xi = k[\eta(X)\xi - X] - \mu hX, \tag{2.11}$$

$$Q\phi - \phi Q = 2(2(n-1) + \mu)h\phi.$$

Also, for a contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ with $\xi \in N(k,\mu)$, the Ricci operator Q is given by [4]

$$\begin{aligned} QX = {} & [2(n-1) - n\mu]\, X + [2(n-1) + \mu]\, hX + \\ & + [2(1-n) + n(2k+\mu)]\, \eta(X)\xi, \quad n \ge 1, \end{aligned} \tag{2.12}$$

$$\begin{aligned} S(X,Y) = {} & [2(n-1) - n\mu]\, g(X,Y) + [2(n-1) + \mu]\, g(hX,Y) + \\ & + [2(1-n) + n(2k+\mu)]\, \eta(X)\eta(Y), \quad n \ge 1, \end{aligned} \tag{2.13}$$

$$S(X,\xi) = 2nk\eta(X), \tag{2.14}$$

where S is the Ricci tensor of the manifold. A (k,μ)-contact metric manifold is called an η-Einstein manifold if it satisfies

$$S(X,W) = a_1 g(X,W) + b_1 \eta(X)\eta(Y),$$

where a_1 and b_1 are two scalars.

For a $(2n+1)$-dimensional Riemannian manifold, the quasi-conformal curvature tensor $\tilde{C}$ is given by

$$\tilde{C}(X,Y)Z = aR(X,Y)Z + b\Big[S(Y,Z)X - S(X,Z)Y + \tag{2.15}$$
$$+ g(Y,Z)QX - g(X,Z)QY\Big] - \frac{r}{2n+1}\left[\frac{a}{2n} + 2b\right][g(Y,Z)X - g(X,Z)Y],$$

where a and b are two scalars, and r is the scalar curvature of the manifold. The notion of quasi-conformal curvature tensor was introduced by K. Yano and S. Sawaki [23]. If $a = 1$ and $b = -\frac{1}{2n-1}$, then quasi-conformal curvature tensor reduces to conformal curvature tensor.

For X, Y orthogonal to ξ, we get from (2.9) and (2.15)

$$\eta(R(X,Y)Z) = 0, \quad \eta(\tilde{C}(X,Y)Z) = 0. \tag{2.16}$$

Let us recall the following results:

LEMMA 2.1. *A contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ with $R(X,Y)\xi=0$, for all vector fields X,Y on the manifold and $n>1$, is locally isometric to the Riemannian product $E^{n+1}\times S^n(4)$* [3].

LEMMA 2.2. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a contact metric manifold with ξ belonging to the (k,μ)-nullity distribution. If $k<1$, then for any X orthogonal to ξ, the ξ-sectional curvature $K(X,\xi)$ is given by*

$$\begin{aligned} K(X,\xi) = k+\mu g(hX,X) &= k+\lambda\mu \quad \textit{if } X\in D(\lambda),\\ &= k-\lambda\mu \quad \textit{if } X\in D(-\lambda), \end{aligned}$$

where $D(\lambda)$ and $D(-\lambda)$ are two mutually orthogonal distributions defined by the eigenspace of h, and $\lambda=\sqrt{1-k}$ [4].

A (k,μ)-contact metric manifold will be called a manifold of quasi-constant curvature if the Riemannian curvature tensor $\tilde{R}$ of type $(0,4)$ satisfies the condition

$$\begin{aligned} \tilde{R}(X,Y,Z,W) &= p[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]+\\ &+q[g(X,W)T(Y)T(Z)-g(X,Z)T(Y)T(W)+\\ &+g(Y,Z)T(X)T(W)-g(Y,W)T(X)T(Z)], \end{aligned} \tag{2.17}$$

where $\tilde{R}(X,Y,Z,W)=g(R(X,Y)Z,W)$, p,q are scalars, and there exists a unit vector field ρ satisfying $g(X,\rho)=T(X)$. The notion of quasi-constant curvature for Riemannian manifolds was introduced by Chen and Yano [6].

An n-dimensional Riemannian manifold whose curvature tensor $\tilde{R}$ of type $(0,4)$ satisfies the condition

$$\tilde{R}(X,Y,Z,W)=F(Y,Z)F(X,W)-F(X,Z)F(Y,W),$$

where F is a symmetric tensor of type $(0,2)$ is called a special manifold with the associated symmetric tensor F, and is denoted by $\psi(F)_n$.

Such type of manifolds have been studied by S.S. Chern [7] in 1956.

Remark 2.1. An n-dimensional manifold of quasi-constant curvature is a $\psi(F)_n$.

Proof. Let us define $\sqrt{p}g(X,Y)+\frac{q}{\sqrt{p}}T(X)T(Y)=F(X,Y)$. Then, from (2.17), it follows that

$$\tilde{R}(X,Y,Z,W)=F(Y,Z)F(X,W)-F(X,Z)F(Y,W).$$

Hence, a manifold of quasi-constant curvature is a $\psi(F)_n$. This completes the proof of the remark. $\square$

3. QUASI-CONFORMALLY FLAT (k, μ)-CONTACT METRIC MANIFOLDS

THEOREM 3.1. *A $(2n+1)$-dimensional $(n > 1)$ quasi-conformally flat (k, μ)-contact metric manifold cannot be an η-Einstein manifold.*

Proof. For a $(2n+1)$-dimensional $(n > 1)$ quasi-conformally flat (k, μ)-contact metric manifold we have

$$\begin{aligned}(3.1)\qquad \tilde{R}(X,Y,Z,W) = \frac{b}{a}\,&[S(X,Z)g(Y,W) - S(Y,Z)g(X,W)+\\ &+S(Y,W)g(X,Z) - S(X,W)g(Y,Z)]+\\ &+\frac{r}{a(2n+1)}\left[\frac{a}{2n}+2b\right][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\,.\end{aligned}$$

By using (2.1), (2.6) and (2.14), and putting $Z = \xi$ in (3.1), we get

$$\begin{aligned}(3.2)\quad &k(g(X,W)\eta(Y) - g(Y,W)\eta(X)) + \mu(\eta(Y)g(hX,W) - \eta(X)g(hY,W)) =\\ &= \frac{b}{a}\,[2nkg(Y,W)\eta(X) - 2nkg(X,W)\eta(Y) + S(Y,W)\eta(X) - S(X,W)\eta(Y)]+\\ &+\frac{r}{a(2n+1)}\left[\frac{a}{2n}+2b\right][g(X,W)\eta(Y) - g(Y,W)\eta(X)]\,.\end{aligned}$$

Putting $Y = \xi$ in (3.2), and using (2.1) and (2.14), we obtain

$$\begin{aligned}(3.3)\qquad S(X,W) &= \frac{a}{b}\left[\frac{r}{a(2n+1)}\left(\frac{a}{2n}+2b\right) - \frac{2nkb}{a} - k\right]g(X,W)+\\ &+\frac{a}{b}\left[k + \frac{4nkb}{a} - \frac{r}{a(2n+1)}\left(\frac{a}{2n}+2b\right)\right]\eta(X)\eta(W) - \frac{a\mu}{b}g(hX,W).\end{aligned}$$

From (3.3), it follows that if $\mu = 0$, the manifold is η-Einstein. Conversely, if the manifold is η-Einstein, then we can write

$$S(X,W) = a_1 g(X,W) + b_1\eta(X)\eta(Y),$$

where a_1 and b_1 are two scalars. From the above equation and (3.3), we obtain

$$\begin{aligned}(3.4)\quad a_1 g(X,W) + b_1\eta(X)\eta(W) &= \frac{a}{b}\left[\frac{r}{a(2n+1)}\left(\frac{a}{2n}+2b\right) - \frac{2nkb}{a} - k\right]\cdot\\ \cdot g(X,W) + \frac{a}{b}\left[k + \frac{4nkb}{a} - \frac{r}{a(2n+1)}\left(\frac{a}{2n}+2b\right)\right]&\eta(X)\eta(W) - \frac{a\mu}{b}g(hX,W).\end{aligned}$$

Putting $W = \phi X$ and using (2.2), we get from (3.4)

$$\frac{a\mu}{b}g(hX,\phi X) = 0,$$

for all X. Consequently, $\mu = 0$.

Hence, we see that a $(2n+1)$-dimensional $(n > 1)$ quasi-conformally flat (k, μ)-contact metric manifold is an η-Einstein manifold, if and only if $\mu = 0$.

But from (2.13), it follows that a (k,μ)-contact metric manifold is η-Einstein if and only if $2(n-1)+\mu=0$. If we consider a $(2n+1)$-dimensional $(n>1)$ quasi-conformally flat η-Einstein (k,μ)-contact metric manifold, then $n=1$, which contradicts the fact that $n>1$. Hence, the theorem is established. □

THEOREM 3.2. *A $(2n+1)$-dimensional $(n>1)$ quasi-conformally flat (k,μ)-contact metric manifold, which is not conformally flat, is of quasi-constant curvature.*

Proof. In (3.3), let us denote

$$A=\frac{r}{b(2n+1)}\Big(\frac{a}{2n}+2b\Big)-2nk-\frac{ak}{b} \tag{3.5}$$

and

$$B=\frac{ak}{b}+4nk-\frac{r}{b(2n+1)}\Big(\frac{a}{2n}+2b\Big).$$

Then, we see that

$$A+B=2nk. \tag{3.6}$$

In (3.3), putting $X=W=e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over i, $i=1,2,3,\dots,(2n+1)$, we get

$$r=A(2n+1)+B. \tag{3.7}$$

From (3.6) and (3.7), we get

$$A=\frac{r}{2n}-k. \tag{3.8}$$

From (3.5) and (3.8), it follows that

$$\frac{r}{b(2n+1)}\Big(\frac{a}{2n}+2b\Big)-2nk-\frac{ak}{b}=\frac{r}{2n}-k.$$

The above relation gives

$$(a+2nb-b)(r-2nk(2n+1))=0.$$

Hence, either $a+2nb-b=0$, or, $r=2nk(2n+1)$.

Let us suppose that $a+2nb-b=0$. Then, we see that $b=\frac{-a}{2n-1}$. Hence, from (2.15), it follows that $\tilde{C}(X,Y)Z=aC(X,Y)Z$, where $C(X,Y)Z$ is the Weyl conformal curvature tensor. This means that in this case quasi-conformally flat manifold is equivalent to conformally flat manifold. In [4], the authors showed that a contact metric manifold M^{2n+1} with ξ belonging to (k,μ)-nullity distribution is a contact strongly pseudo convex integrable CR-manifold. In [14], the authors proved that a conformally flat contact pseudo-convex integrable CR-manifold of dimension greater than three is a space of constant curvature 1, from which it follows that a conformally flat contact metric manifold M^{2n+1} $(n>1)$ with ξ belonging to (k,μ)-nullity distribution is a space of constant curvature 1. Hence, in this case, the quasi-conformally flat manifold under consideration is also of constant curvature 1.

For the second possibility, that is, for $r = 2nk(2n+1)$, we obtain from (3.3)

$$S(X,W) = 2nkg(X,W) - \frac{a\mu}{b} g(hX,W). \tag{3.9}$$

Using (3.9) in (3.1), we have

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = k[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]-\\ -\mu[g(hX,Z)g(Y,W) - g(hY,Z)g(X,W)+\\ +g(hY,W)g(X,Z) - g(hX,W)g(Y,Z)].\end{aligned} \tag{3.10}$$

From (2.13) and (3.9), it follows that

$$g(hX,W) = l(g(X,W) - \eta(X)\eta(W)), \tag{3.11}$$

where $l = \frac{b(2nk+n\mu-2n+2)}{2nb+\mu b+\mu a-2b}$.

Using (3.11) in (3.10), we obtain

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]+\\ +q[g(X,W)\eta(Y)\eta(Z) - g(X,Z)\eta(Y)\eta(W)+\\ +g(Y,Z)\eta(X)\eta(W) - g(Y,W)\eta(X)\eta(Z)],\end{aligned} \tag{3.12}$$

where $p = k + 2l\mu$ and $q = -l\mu$. This completes the proof. □

For an $N(k)$-contact metric manifold, $\mu = 0$. Consequently, $q = 0$. Hence, we have the following:

Corollary 3.1. *A $(2n+1)$-dimensional $(n>1)$ quasi-conformally flat $N(k)$-contact metric manifold is of constant curvature.*

For $k = 1$ and $\mu = 0$, a (k,μ)-contact metric manifold becomes Sasakian. Therefore, we are in a position to state the following:

Corollary 3.2. *A $(2n+1)$-dimensional $(n>1)$ quasi-conformally flat Sasakian manifold is of constant curvature.*

The above result has been proved in the paper [10].

In view of Remark 2.1 and Theorem 3.2, we get the following:

Corollary 3.3. *A $(2n+1)$-dimensional $(n>1)$ quasi-conformally flat (k,μ)-contact metric manifold, which is not conformally flat, is a $\psi(F)_{2n+1}$.*

4. (k,μ)-CONTACT METRIC MANIFOLDS WITH VANISHING EXTENDED QUASI-CONFORMAL CURVATURE TENSOR

The object of the present section is to study a (k,μ)-contact metric manifold with vanishing extended quasi-conformal curvature tensor. In this connection it should be mentioned that the extended pseudo projective curvature tensor of a contact metric manifold has been studied in the paper [1].

Theorem 4.1. *A* $(2n+1)$*-dimensional* $(n>1)$ (k,μ)*-contact metric manifold with vanishing extended quasi-conformal curvature tensor is a Sasakian manifold.*

Proof. The extended form of the quasi-conformal curvature tensor can be written as

$$\tilde{C}_e(X,Y)Z = aR(X,Y)Z + b\Big[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - \\ -g(X,Z)QY\Big] - \frac{r}{2n+1}\left[\frac{a}{2n}+2b\right][g(Y,Z)X - g(X,Z)Y] - \\ -\eta(X)\tilde{C}(\xi,Y)Z - \eta(Y)\tilde{C}(X,\xi)Z - \eta(Z)\tilde{C}(X,Y)\xi. \tag{4.1}$$

Putting $Y=Z=\xi$, and supposing that the extended quasi-conformal curvature tensor vanishes, we get from (4.1)

$$aR(X,\xi)\xi + b[S(\xi,\xi)X - S(X,\xi)\xi + QX - \eta(X)Q\xi] - \\ -\frac{r}{2n+1}\left[\frac{a}{2n}+2b\right][X-\eta(X)\xi] - \eta(X)\tilde{C}(\xi,\xi)\xi - \tilde{C}(X,\xi)\xi - \tilde{C}(X,\xi)\xi = 0. \tag{4.2}$$

In view of (2.11), and (2.15) we have, from above

$$a[k(X-\eta(X)\xi) + \mu hX] + b[2nk(X-\eta(X)\xi) + QX - \eta(X)Q\xi] - \\ -\frac{r}{2n+1}\left[\frac{a}{2n}+2b\right][X-\eta(X)\xi] - 2\Big[a\big(k(X-\eta(X)\xi+\mu hX) + \\ +b(2nk(X-\eta(X)\xi)+QX-\eta(X)Q\xi\big) - \frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)(X-\eta(X)\xi)\Big] = 0.$$

Simplifying the above equation, we obtain

$$\left(ak + 2nbk - \frac{r}{(2n+1)}\left(\frac{a}{2n}+2b\right)\right)\cdot \\ \cdot(X-\eta(X)\xi) + a\mu hX + b(QX - \eta(X)Q\xi) = 0. \tag{4.3}$$

Using (2.12) in (4.3), we get

$$\left(ak + 2nbk - \frac{r}{(2n+1)}\left(\frac{a}{2n}+2b\right) + 2b(n-1) - nb\mu\right)\cdot \\ \cdot(X-\eta(X)\xi) + [a\mu + b(2(n-1)+\mu)]hX = 0.$$

The above equation yields

$$\left(ak + 2nbk - \frac{r}{(2n+1)}\left(\frac{a}{2n} + 2b\right) + 2b(n-1) - nb\mu\right)\phi^2 X - \\ -[a\mu - b(2(n-1)+\mu)]hX = 0. \tag{4.4}$$

In view of (2.7), (4.4) takes the form

$$h^2 X = (k-1)\frac{[a\mu + b(2(n-1)+\mu)]hX}{(ak + 2nbk - \frac{r}{(2n+1)}(\frac{a}{2n} + 2b) + 2b(n-1) - nb\mu)}.$$

Since $\operatorname{tr} h = 0$, from above we get

$$\operatorname{tr} h^2 = 0. \tag{4.5}$$

Now, $h^2 X = (k-1)\phi^2 X$. Hence, we have

$$\operatorname{tr} h^2 = 2n(1-k). \tag{4.6}$$

By virtue of (4.5) and (4.6), we get $k = 1$. Then, from (2.7), $h = 0$. This means that the manifold is Sasakian. Hence, the theorem is proved. □

From the above theorem, we also have the following:

Corollary 4.1. *A $(2n+1)$-dimensional $(n > 1)$ $N(k)$-contact metric manifold with vanishing extended quasi-conformal curvature tensor is a Sasakian manifold.*

The unit tangent sphere bundle $T_1M(c)$ of constant curvature c with standard contact metric structure is a (k,μ)-contact metric manifold with $k = c(2-c)$ and $\mu = -2c$ [4]. Then by Theorem 4.1 we obtain $c = 1$. Hence, we can state the following:

Corollary 4.2. *A unit tangent sphere bundle with vanishing extended quasi-conformal curvature tensor is of constant curvature* 1.

5. QUASI-CONFORMALLY SEMISYMMETRIC (k,μ)-CONTACT METRIC MANIFOLDS

Definition 5.1. A $(2n+1)$-dimensional $(n > 1)$ (k,μ)-contact metric manifold is said to be quasi-conformally semisymmetric if the condition $R(X,Y).\tilde{C} = 0$ holds, for any vector fields X, Y on the manifold.

Theorem 5.1. A *$(2n+1)$-dimensional $(n > 1)$ quasi-conformally semisymmetric (k,μ)-contact manifold is either locally the Riemannian product $E^{n+1} \times S^n(4)$, or, locally quasi-conformally flat.*

Proof. Let us suppose that the manifold satisfies $R(\xi,U)\cdot\tilde{C}=0$. Then, it follows that

$$R(\xi,U)\tilde{C}(X,Y)Z-\tilde{C}(R(\xi,U)X,Y)Z- \\ -\tilde{C}(X,R(\xi,U)Y)Z-\tilde{C}(X,Y)R(\xi,U)Z=0. \tag{5.1}$$

In virtue of (2.10), the above equation reduces to

$$\begin{gathered} k[g(U,\tilde{C}(X,Y)Z)\xi-\eta(\tilde{C}(X,Y)Z)U]+ \\ +\mu[g(hU,\tilde{C}(X,Y)Z)\xi-\eta(\tilde{C}(X,Y)Z)hU]- \\ -\tilde{C}(k[g(U,X)\xi-\eta(X)U]+\mu[g(hU,X)\xi-\eta(X)hU],Y)Z- \\ -\tilde{C}(X,k[g(U,Y)\xi-\eta(Y)U]+\mu[g(hU,Y)\xi-\eta(Y)hU])Z- \\ -\tilde{C}(X,Y)(k[g(U,Z)\xi-\eta(Z)U]+\mu[g(hU,Z)\xi-\eta(Z)hU])=0, \end{gathered}$$

or,

$$\begin{gathered} k[g(U,\tilde{C}(X,Y)Z)\xi-\eta(\tilde{C}(X,Y)Z)U-g(X,U)\tilde{C}(\xi,Y)Z+ \\ +\eta(X)\tilde{C}(U,Y)Z-g(U,Y)\tilde{C}(X,\xi)Z+\eta(Y)\tilde{C}(X,U)Z- \\ -g(U,Z)\tilde{C}(X,Y)\xi+\eta(Z)\tilde{C}(X,Y)U]+ \\ +\mu[g(hU,\tilde{C}(X,Y)Z)\xi-\eta(\tilde{C}(X,Y)Z)hU-g(hU,X)\tilde{C}(\xi,Y)Z+ \\ +\eta(X)\tilde{C}(hU,Y)Z-g(hU,Y)\tilde{C}(X,\xi)Z+\eta(Y)\tilde{C}(X,hU)Z- \\ -g(hU,Z)\tilde{C}(X,Y)\xi+\eta(Z)\tilde{C}(X,Y)hU]=0. \end{gathered} \tag{5.2}$$

Let us consider the following cases:

Case 1. $k=0=\mu$.

Case 2. $k=0$, $\mu\neq 0$.

Case 3. $k\neq 0$, $\mu=0$.

Case 4. $k\neq 0$, $\mu\neq 0$.

For Case 1, we observe that $R(X,Y)\xi=0$, for all X,Y. Hence, by Lemma 2.1, the manifold is locally the Riemannian product $E^{n+1}\times S^n(4)$.

For Case 2, we get from (5.2)

$$\begin{gathered} [g(hU,\tilde{C}(X,Y)Z)\xi-\eta(\tilde{C}(X,Y)Z)hU-g(hU,X)\tilde{C}(\xi,Y)Z+ \\ +\eta(X)\tilde{C}(hU,Y)Z-g(hU,Y)\tilde{C}(X,\xi)Z+\eta(Y)\tilde{C}(X,hU)Z- \\ -g(hU,Z)\tilde{C}(X,Y)\xi+\eta(Z)\tilde{C}(X,Y)hU]=0. \end{gathered} \tag{5.3}$$

In this case, in view of (2.7), $h^2 = -\phi^2$. Hence, replacing U by hU, we obtain, from (5.3)

$$\begin{aligned}(5.4)\qquad &[g(\eta(U)\xi - U, \tilde{C}(X,Y)Z)\xi - \eta(\tilde{C}(X,Y)Z)(\eta(U)\xi - U)-\\ &-g(\eta(U)\xi - U, X)\tilde{C}(\xi,Y)Z + \eta(X)\tilde{C}(\eta(U)\xi - U, Y)Z-\\ &-g(\eta(U)\xi - U, Y)\tilde{C}(X,\xi)Z + \eta(Y)\tilde{C}(X,\eta(U)\xi - U)Z-\\ &-g(\eta(U)\xi - U, Z)\tilde{C}(X,Y)\xi + \eta(Z)\tilde{C}(X,Y)(\eta(U)\xi - U)] = 0.\end{aligned}$$

Choosing X, Y, Z orthogonal to ξ and using (2.16), we have

$$g(U, \tilde{C}(X,Y)Z)\xi = 0.$$

The above equation implies

$$\tilde{C}(X,Y)Z = 0.$$

For Case 3, changing U by h^2U in (5.2), we shall get the same result as in Case 2.

In Case 4, changing U by hU, and using (2.7) in (5.2), we obtain after simplification

$$\begin{aligned}(5.5)\qquad &k[g(hU, \tilde{C}(X,Y)Z)\xi - \eta(\tilde{C}(X,Y)Z)hU - g(X,hU)\tilde{C}(\xi,Y)Z+\\ &+\eta(X)\tilde{C}(hU,Y)Z - g(hU,Y)\tilde{C}(X,\xi)Z + \eta(Y)\tilde{C}(X,hU)Z-\\ &-g(hU,Z)\tilde{C}(X,Y)\xi + \eta(Z)\tilde{C}(X,Y)hU]+\\ &+\mu(k-1)[g(U,\tilde{C}(X,Y)Z)\xi - \eta(\tilde{C}(X,Y)Z)U-\\ &-g(U,X)\tilde{C}(\xi,Y)Z + \eta(X)\tilde{C}(U,Y)Z - g(U,Y)\tilde{C}(X,\xi)Z+\\ &+\eta(Y)\tilde{C}(X,U)Z - g(U,Z)\tilde{C}(X,Y)\xi + \eta(Z)\tilde{C}(X,Y)U] = 0.\end{aligned}$$

Eliminating hU from (5.2) and the above equation, we obtain

$$\begin{aligned}(5.6)\qquad &[k^2 - \mu^2(k-1)][g(U,\tilde{C}(X,Y)Z)\xi - \eta(\tilde{C}(X,Y)Z)U-\\ &-g(U,X)\tilde{C}(\xi,Y)Z + \eta(X)\tilde{C}(U,Y)Z - g(U,Y)\tilde{C}(X,\xi)Z+\\ &+\eta(Y)\tilde{C}(X,U)Z - g(U,Z)\tilde{C}(X,Y)\xi + \eta(Z)\tilde{C}(X,Y)U] = 0.\end{aligned}$$

Hence, either

$$(5.7)\qquad k^2 - \mu^2(k-1) = 0,$$

or,

$$\begin{aligned}(5.8)\qquad &g(U,\tilde{C}(X,Y)Z)\xi - \eta(\tilde{C}(X,Y)Z)U - g(U,X)\tilde{C}(\xi,Y)Z+\\ &+\eta(X)\tilde{C}(U,Y)Z - g(U,Y)\tilde{C}(X,\xi)Z + \eta(Y)\tilde{C}(X,U)Z-\\ &-g(U,Z)\tilde{C}(X,Y)\xi + \eta(Z)\tilde{C}(X,Y)U = 0.\end{aligned}$$

We assert that $k^2-\mu^2(k-1)\neq 0$. This means that the sectional curvature of the plane section containing ξ are non vanishing. But Lemma 2.2 tells us that the sectional curvature of the plane section, containing ξ, equal to zero if and only if $k+\lambda\mu=0$ for $X\in D(\lambda)$, and $k-\lambda\mu=0$ for $X\in D(-\lambda)$, where $D(\lambda)$ is the distribution defined by the vector field hX, $\lambda=\sqrt{1-k}$, and

$$TM=[\xi]\oplus[D(\lambda)]\oplus[D(-\lambda)].$$

In this case, we have $k+\mu\sqrt{1-k}=0$ and $k-\mu\sqrt{1-k}=0$. These give $k=\mu=0$. But we have assumed $k\neq 0$, $\mu\neq 0$. Consequently, $k^2-\mu^2(k-1)\neq 0$. Hence, our assertion is true.

Therefore, for the remaining possibility, as Case 3, the manifold is quasi-conformally flat, provided X,Y,Z are orthogonal to ξ. Now, from the above discussion, we get the theorem. $\square$

By virtue of the above theorem, we obtain the following:

COROLLARY 5.1. *A $(2n+1)$-dimensional $(n>1)$ quasi-conformally semisymmetric $N(k)$-contact manifold is either locally the Riemannian product $E^{n+1}\times S^n(4)$, or, locally quasi-conformally flat.*

6. QUASI-CONFORMALLY RECURRENT (k,μ)-CONTACT METRIC MANIFOLDS

Definition 6.1. A $(2n+1)$-dimensional $(n>1)$, (k,μ)-contact metric manifold is called quasi-conformally recurrent if it satisfies

$$(\nabla_U\tilde{C})(X,Y)Z=A(U)\tilde{C}(X,Y)Z, \tag{6.1}$$

for certain non-zero 1-form A.

THEOREM 6.1. *A $(2n+1)$-dimensional $(n>1)$ quasi-conformally recurrent (k,μ)-contact metric manifold is quasi-conformally semisymmetric.*

Proof. Let us suppose that $\tilde{C}\neq 0$. We now define a function by

$$f^2=g(\tilde{C},\tilde{C}).$$

By the fact that $\nabla g=0$, it follows from above

$$2f(Uf)=2f^2(A(U)),$$

or,

$$Uf=f(A(U)), \tag{6.2}$$

because $f\neq 0$. From (6.2), we get

$$V(Uf)=\frac{1}{f}(Vf)(Uf)+(VA(U))f.$$

Hence

$$V(Uf) - U(Vf) = [VA(U) - UA(V)]f.$$

Therefore,

$$(\nabla_V \nabla_U - \nabla_U \nabla_V - \nabla_{[V,U]})f = \\ = [VA(U) - UA(V) - A([V,U])]f = 2[dA(V,U)]f. \tag{6.3}$$

Since the left hand side of (6.3) is zero, and $f \neq 0$, it follows from (6.3) that $dA(V,U) = 0$. This means that the 1-form A is closed. Now, from (6.1), we have

$$(\nabla_V \nabla_U \tilde{C})(X,Y)Z = [VA(U) + A(V)A(U)]\tilde{C}(X,Y)Z.$$

Hence

$$(\nabla_V \nabla_U \tilde{C})(X,Y)Z - (\nabla_U \nabla_V \tilde{C})(X,Y)Z - (\nabla_{[U,V]}\tilde{C})(X,Y)Z = \\ = 2dA(V,U)\tilde{C}(X,Y)Z = 0.$$

Therefore, we have from above, $R(V,U) \cdot \tilde{C} = 0$, where $R(V,U)$ is considered as a derivation of tensor algebra at each point of the manifold for the tangent vectors V, U. This completes the proof. □

In view of Theorem 5.1 and Theorem 6.1 we state

THEOREM 6.2. *A $(2n+1)$-dimensional $(n > 1)$ quasi-conformally recurrent (k,μ)-contact metric manifold is either locally the Riemannian product $E^{n+1} \times S^n(4)$, or, locally quasi-conformally flat.*

Theorem 6.2. leads us to state the following:

COROLLARY 6.2. *A $(2n+1)$-dimensional $(n > 1)$ quasi-conformally recurrent $N(k)$-contact metric manifold is either locally the Riemannian product $E^{n+1} \times S^n(4)$, or, locally quasi-conformally flat.*

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