NON-DEGENERATE HYPERSURFACES OF A SEMI-RIEMANNIAN MANIFOLD WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

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We consider a non-degenerate hypersurface of a semi-Riemannian manifold with a semi-symmetric non-metric connection. We obtain a relation between the Ricci and the scalar curvatures of a semi-Riemannian manifold and of its non-degenerate hypersurface with respect to a semi-symmetric non-metric connection. We also show that the Ricci tensor of a non-degenerate hypersurface of a semi-Riemannian space form admitting a semi-symmetric non-metric connection is symmetric, but not parallel. Finally, we get the conditions under which a non-degenerate hypersurface with a semi-symmetric non-metric connection is projectively flat.

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Key words: semi-symmetric non-metric connection, Levi-Civita connection, Ricci tensor, projectively flat.

1. INTRODUCTION

In 1924, Friedmann and Schouten [5] introduced the idea of semi-symmetric linear connection on a differentiable manifold. In 1992, Ageshe and Chafle [1] defined a linear connection on a Riemannian manifold and studied some properties of the curvature tensors of a Riemannian manifold with respect to the semi-symmetric non-metric connection. In 1994, they also considered submanifolds of a Riemannian manifold and obtained the equations of Gauss, Codazzi and Ricci associated with a semi-symmetric non-metric connection and gave some properties of the submanifolds of a space form admitting a semi-symmetric non-metric connection [2]. In 1995, De and Kamilya [4] studied the properties of hypersurfaces of a Riemannian manifold with a semi-symmetric non-metric connection. In 2000, Sengupta, De and Binh [9] defined a semi-symmetric non-metric connection which is a generalization of the notion of the semi-symmetric non-metric connection introduced by Agashe and Chafle [1]. They also studied some properties of the curvature tensor and Weyl projective curvature tensor with respect to the semi-symmetric non-metric connection. Finally, they obtained certain conditions under which two semi-symmetric non-metric connections are equal. In 2004, Prasad and Verma [8] obtained a necessary and sufficient condition for the equality of the Weyl projective curvature tensor of the semi-symmetric non-metric connection with the Weyl projective curvature of the Riemannian connection. Moreover, they showed that if the curvature tensor with respect to the semi-symmetric non-metric connection vanishes, then the Riemannian manifold is projectively flat.

In the present paper, we studied a non-degenerate hypersurface of a semi-Riemannian manifold admitting a semi-symmetric non-metric connection. We gave equations of Gauss and Weingarten for a non-degenerate hypersurface of a semi-Riemannian manifold with a semi-symmetric non-metric connection. We also derived the equations of Gauss curvature and Codazzi Mainardi with respect to a semi-symmetric non-metric connection on a semi-Riemannian manifold and the induced ones of a non-degenerate hypersurface. Then we showed that the Ricci tensor of a non-degenerate hypersurface of a semi-Riemannian space form with a semi-symmetric non-metric connection is symmetric but not parallel. Eventually, we observed that a totally umbilical non-degenerate hypersurface of a projectively flat semi-Riemannian manifold with a semi-symmetric non-metric connection is projectively flat.

2. PRELIMINARIES

Let $\widetilde{M}$ be an $(n+1)$-dimensional differentiable manifold of class C^∞ and M an n-dimensional differentiable manifold immersed in $\widetilde{M}$ by a differentiable immersion

$$i : M \to \widetilde{M}.$$

$i(M)$, identical to M, is said to be a hypersurface of $\widetilde{M}$. The differential di of the immersion i will be denoted by B so that a vector field X in M corresponds to a vector field BX in $\widetilde{M}$. We suppose that the manifold $\widetilde{M}$ is a semi-Riemannian manifold with the semi-Riemannian metric $\widetilde{g}$ of index ν, $0 \leq \nu \leq n+1$. Thus the index of $\widetilde{M}$ is the ν, which will be denoted by $ind\widetilde{M} = \nu$. If the induced metric tensor $g = \widetilde{g}_{|M}$ defined by

$$g(X,Y) = \widetilde{g}(BX,BY), \quad \forall\, X,Y \in \chi(M)$$

is non-degenerate, then the hypersurface M is called a *non-degenerate hypersurface*. Also, M is a semi-Riemannian manifold with the induced semi-Riemannian metric g (see [7]). If the semi-Riemannian manifolds $\widetilde{M}$ and M are both orientable, we can choose a unit vector field N defined along M such that

$$\widetilde{g}(BX,N) = 0, \quad \widetilde{g}(N,N) = \varepsilon = \begin{cases} +1, & \text{for spacelike } N \\ -1, & \text{for timelike } N \end{cases}$$

for $\forall X \in \chi(M)$, where N is called the unit normal vector field to M, and $\operatorname{ind} M = \operatorname{ind}\widetilde{M}$ if $\varepsilon = 1$, $\operatorname{ind} M = \operatorname{ind}\widetilde{M} - 1$ if $\varepsilon = -1$.

3. SEMI-SYMMETRIC NON-METRIC CONNECTION

Let $\widetilde{M}$ denotes an $(n+1)$-dimensional semi-Riemannian manifold with semi-Riemannian metric $\widetilde{g}$ of index ν, $0 \leq \nu \leq n+1$. A linear connection $\widetilde{\nabla}$ on $\widetilde{M}$ is called a semi-symmetric non-metric connection if

$$(\widetilde{\nabla}_{\widetilde{X}}\widetilde{g})(\widetilde{Y},\widetilde{Z}) = -\widetilde{\pi}(\widetilde{Y})\widetilde{g}(\widetilde{X},\widetilde{Z}) - \widetilde{\pi}(\widetilde{Z})\widetilde{g}(\widetilde{X},\widetilde{Y})$$

and the torsion tensor $\widetilde{T}$ of $\widetilde{\nabla}$ satisfies

$$\widetilde{T}(\widetilde{X},\widetilde{Y}) = \widetilde{\pi}(\widetilde{Y})\widetilde{X} - \widetilde{\pi}(\widetilde{X})\widetilde{Y}$$

for any $\widetilde{X},\widetilde{Y},\widetilde{Z} \in \chi(\widetilde{M})$, where $\widetilde{\pi}$ is a 1-form associated with the vector field $\widetilde{Q}$ on $\widetilde{M}$ defined by

$$\widetilde{g}(\widetilde{Q},\widetilde{X}) = \widetilde{\pi}(\widetilde{X})$$

(see [1]).

Throughout the paper, we will denote by $\widetilde{M}$ the semi-Riemannian manifold admitting a semi-symmetric non-metric connection given by

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = \overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Y} + \widetilde{\pi}(\widetilde{Y})\widetilde{X} \tag{1}$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ of $\widetilde{M}$, where $\overset{\circ}{\widetilde{\nabla}}$ denotes the Levi-Civita connection with respect to the semi-Riemannian metric $\widetilde{g}$. When M is a non-degenerate hypersurface, we have the following orthogonal direct sum:

$$\chi(\widetilde{M}) = \chi(M) \oplus \chi(M)^{\perp}. \tag{2}$$

According to (2), the vector field $\widetilde{Q}$ on $\widetilde{M}$ can be decomposed as:

$$\widetilde{Q} = BQ + \mu N,$$

where Q and μ are a vector field and a function in M, respectively.

We denote by $\overset{\circ}{\nabla}$ the connection on the non-degenerate hypersurface M induced from the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ on $\widetilde{M}$ with respect to the unit spacelike or timelike normal vector field N. We have the equality

$$\overset{\circ}{\widetilde{\nabla}}_{BX}BY = B(\overset{\circ}{\nabla}_X Y) + \overset{\circ}{h}(X,Y)N \tag{3}$$

for arbitrary vector fields X and Y of M, where $\overset{\circ}{h}$ is the second fundamental form of the non-degenerate hypersurface M. Let us define the connection ∇ on M which is induced by the semi-symmetric non-metric connection $\widetilde{\nabla}$ on $\widetilde{M}$ with respect to the unit spacelike or timelike normal vector field N. We obtain the equation

$$\widetilde{\nabla}_{BX}BY = B(\nabla_X Y) + h(X,Y)N \tag{4}$$

for arbitrary vector fields X and Y of M, where h is the second fundamental form of the non-degenerate hypersurface M. We call (4) the *equation of Gauss* with respect to the induced connection ∇.

According to (1), we have

$$\widetilde{\nabla}_{BX}BY = \overset{\circ}{\widetilde{\nabla}}_{BX}BY + \widetilde{\pi}(BY)BX. \tag{5}$$

Hence, by applying (3) and (4) into (5), we get the relation

$$B(\nabla_X Y) + h(X,Y)N = B(\overset{\circ}{\nabla}_X Y) + \overset{\circ}{h}(X,Y)N + \widetilde{\pi}(BY)BX.$$

which implies

$$\nabla_X Y = \overset{\circ}{\nabla}_X Y + \pi(Y)X, \tag{6}$$

where $\pi(X) = \widetilde{\pi}(BX)$ and

$$h(X,Y) = \overset{\circ}{h}(X,Y).$$

By virtue of (6), we conclude that

$$(\nabla_X g)(Y,Z) = -\pi(Y)g(X,Z) - \pi(Z)g(X,Y), \tag{7}$$

and

$$T(X,Y) = \pi(Y)X - \pi(X)Y. \tag{8}$$

for any $X, Y, Z \in \chi(M)$. Consequently, using (7) and (8), we can state the following theorem:

Theorem 3.1. *The connection induced on a non-degenerate hypersurface of a semi-Riemannian manifold with a semi-symmetric non-metric connection with respect to the unit spacelike or timelike normal vector field is also a semi-symmetric non-metric connection.*

The equation of Weingarten with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ is

$$\overset{\circ}{\widetilde{\nabla}}_{BX}N = -B(\overset{\circ}{A}_N X) \tag{9}$$

for any vector field X in M, where $\overset{\circ}{A}_N$ is a tensor field of type $(1,1)$ of M which is defined by

$$\overset{\circ}{h}(X,Y) = \varepsilon g(\overset{\circ}{A}_N X, Y)$$

(see [7]). By using (1), we have

$$\widetilde{\nabla}_{BX}N = \overset{\circ}{\widetilde{\nabla}}_{BX}N + \varepsilon\mu BX$$

because of

$$\widetilde{\pi}(N) = \widetilde{g}(\widetilde{Q}, N) = \widetilde{g}(BQ + \mu N, N) = \mu\widetilde{g}(N,N) = \varepsilon\mu.$$

Thus, substituting (9) into above equation, we get

$$\widetilde{\nabla}_{BX}N = -B((\overset{\circ}{A}_N - \varepsilon\mu I)X), \quad \varepsilon = \mp 1, \tag{10}$$

where I is the unit tensor. Applying the tensor field A_N of type $(1,1)$ of M defined

$$A_N = \overset{\circ}{A}_N - \varepsilon\mu I, \tag{11}$$

into (10), the *equation of Weingarten* with respect to the semi-symmetric non-metric connection can be obtained as

$$\widetilde{\nabla}_{BX}N = -B(A_N X) \tag{12}$$

for $X \in \chi(M)$. Indeed, using (11) we get the relation

$$h(X,Y) = \varepsilon g(A_N X, Y) + \mu g(X,Y). \tag{13}$$

From (11), we have the following corollary:

Corollary 3.2. *Let M be a non-degenerate hypersurface of a semi-Riemannian manifold $\widetilde{M}$. Then,*

i) *If M has a spacelike normal vector field, the shape operator A_N with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ is*

$$A_N = \overset{\circ}{A}_N - \mu I.$$

ii) *If M has a timelike normal vector field, the shape operator A_N with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ is*

$$A_N = \overset{\circ}{A}_N + \mu I.$$

Now, we suppose that $E_1, E_2, \ldots, E_\nu, E_{\nu+1}, \ldots, E_n$ are principal vector fields corresponding to unit spacelike or timelike normal vector field N with respect to $\overset{\circ}{\widetilde{\nabla}}$. By using (11), we have

$$A_N(E_i) = \overset{\circ}{A}_N(E_i) - \varepsilon\mu E_i = \overset{\circ}{k}_i E_i - \varepsilon\mu E_i = (\overset{\circ}{k}_i - \varepsilon\mu)E_i, \quad 1 \le i \le n,$$

where $\overset{\circ}{k}_i$, $1 \le i \le n$, are the principal curvatures corresponding to the unit spacelike or timelike normal vector field N with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$. If we take

$$k_i = \overset{\circ}{k}_i - \varepsilon\mu, \quad 1 \le i \le n,$$

then we obtain

$$A_N(E_i) = k_i E_i, \quad 1 \le i \le n,$$

where k_i, $1 \le i \le n$, are the principal curvatures corresponding to the normal vector field N (spacelike or timelike) with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$. Therefore, we can give the following corollary:

Corollary 3.3. *Let M be a non-degenerate hypersurface of the semi-Riemannian manifold $\widetilde{M}$. Then,*

i) *If M has a spacelike normal vector field, the principal curvatures corresponding to unit spacelike normal N with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ are $k_i = \overset{\circ}{k}_i - \mu$, $1 \leq i \leq n$.*

ii) *If M has a timelike normal vector field, the principal curvatures corresponding to unit timelike normal N with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ are $k_i = \overset{\circ}{k}_i + \mu$, $1 \leq i \leq n$.*

4. EQUATIONS OF GAUSS CURVATURE AND CODAZZI-MAINARDI

We denote the curvature tensor of $\widetilde{M}$ with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ by

$$\overset{\circ}{\widetilde{R}}(\widetilde{X},\widetilde{Y})\widetilde{Z} = \overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}}\overset{\circ}{\widetilde{\nabla}}_{\widetilde{Y}}\widetilde{Z} - \overset{\circ}{\widetilde{\nabla}}_{\widetilde{Y}}\overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Z} - \overset{\circ}{\widetilde{\nabla}}_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}$$

and that of M with respect to the Levi-Civita connection $\overset{\circ}{\nabla}$ by

$$\overset{\circ}{R}(X,Y)Z = \overset{\circ}{\nabla}_X\overset{\circ}{\nabla}_Y Z - \overset{\circ}{\nabla}_Y\overset{\circ}{\nabla}_X Z - \overset{\circ}{\nabla}_{[X,Y]}Z.$$

Then *the equation of Gauss curvature* is given by

$$\overset{\circ}{R}(X,Y,Z,U) = \overset{\circ}{\widetilde{R}}(BX,BY,BZ,BU) + \varepsilon\{\overset{\circ}{h}(X,U)\overset{\circ}{h}(Y,Z) - \overset{\circ}{h}(Y,U)\overset{\circ}{h}(X,Z)\},$$

where

$$\overset{\circ}{\widetilde{R}}(BX,BY,BZ,BU) = \widetilde{g}(\overset{\circ}{\widetilde{R}}(BX,BY)BZ,BU),$$
$$\overset{\circ}{R}(X,Y,Z,U) = g(\overset{\circ}{R}(X,Y)Z,U),$$

and *the equation of Codazzi-Mainardi* is given by

$$\overset{\circ}{\widetilde{R}}(BX,BY,BZ,N) = \varepsilon\{(\overset{\circ}{\nabla}_X\overset{\circ}{h})(Y,Z) - (\overset{\circ}{\nabla}_Y\overset{\circ}{h})(X,Z)\}$$

(see [7]).

Next, we find the equation of Gauss curvature and Codazzi-Mainardi with respect to the semi-symmetric non-metric connection. The curvature tensor of the semi-symmetric non-metric connection $\widetilde{\nabla}$ of $\widetilde{M}$ is, by definition,

$$\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Z} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{Z} - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla}_{\widetilde{X}}\widetilde{Z} - \widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}.$$

By taking $\widetilde{X} = BX$, $\widetilde{Y} = BY$, $\widetilde{Z} = BZ$, and using (4), (12), we obtain the curvature tensor of the semi-symmetric non-metric connection $\widetilde{\nabla}$ as

$$\begin{aligned}(14)\qquad &\widetilde{R}(BX, BY)BZ = B(R(X,Y)Z + h(X,Z)A_N Y - h(Y,Z)A_N X) + \\ &+\{(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) + h(\pi(Y)X - \pi(X)Y, Z)\}N,\end{aligned}$$

where

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is the curvature tensor of the semi-symmetric non-metric connection ∇.

Setting

$$\widetilde{R}(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{U}) = \widetilde{g}(\widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z}, \widetilde{U}), \quad R(X,Y,Z,U) = g(R(X,Y)Z, U),$$

and using equations (13) and (14), we obtain *the equation of Gauss curvature and Codazzi-Mainardi* with respect to the semi-symmetric non-metric connection given by

$$\begin{aligned}(15)\qquad &\widetilde{R}(BX, BY, BZ, BU) = R(X,Y,Z,U) + \varepsilon\{h(X,Z)h(Y,U) - \\ &-h(Y,Z)h(X,U) + \mu h(Y,Z)g(X,U) - \mu h(X,Z)g(Y,U)\},\end{aligned}$$

and

$$\widetilde{R}(BX, BY, BZ, N) = \varepsilon\{(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) + h(\pi(Y)X - \pi(X)Y, Z)\}.$$

5. THE RICCI AND SCALAR CURVATURES

We denote the Riemannian curvature tensor of a non-degenerate hypersurface M with respect to the semi-symmetric non-metric connection ∇ by R and that of M by $\overset{\circ}{R}$ with respect to the Levi-Civita connection $\overset{\circ}{\nabla}$. Then, by direct computation, we get

$$(16)\qquad R(X,Y)Z = \overset{\circ}{R}(X,Y)Z + \alpha(X,Z)Y - \alpha(Y,Z)X,$$

where

$$\alpha(X,Y) = (\overset{\circ}{\nabla}_X \pi)Y - \pi(X)\pi(Y) = (\nabla_X \pi)Y.$$

THEOREM 5.1. *The Ricci tensor of a non-degenerate hypersurface M with respect to the semi-symmetric non-metric connection is symmetric if and only if π is closed.*

Proof. The Ricci tensor of a non-degenerate hypersurface M with respect to semi-symmetric non-metric connection is given by

$$Ric(X,Y) = \sum_{i=1}^{n} \varepsilon_i g(R(E_i, X)Y, E_i).$$

Using (16) in above equation of Ricci tensor, we have

$$Ric(X,Y) = \overset{\circ}{Ric}(X,Y) - (n-1)\alpha(X,Y),$$

where $\overset{\circ}{Ric}$ denotes the Ricci tensor of M with respect to the Levi-Civita connection. Since $\overset{\circ}{Ric}$ is symmetric, we obtain

$$Ric(X,Y) - Ric(Y,X) = (n-1)\{\alpha(Y,X) - \alpha(X,Y)\} = 2(n-1)d\pi(Y,X)$$

which completes the proof. □

THEOREM 5.2. *Let M be a non-degenerate hypersurface of a semi-Riemannian manifold $\widetilde{M}$. If $\widetilde{Ric}$ and Ric are the Ricci tensor of $\widetilde{M}$ and M with respect to the semi-symmetric non-metric connection, respectively, then for $\forall X, Y \in \chi(M)$*

$$\widetilde{Ric}(BX,BY) = Ric(X,Y) - \varepsilon f h(X,Y) + \\ + h(A_N X, Y) + \varepsilon n\mu h(X,Y) + \varepsilon\widetilde{g}(\widetilde{R}(N,BX)BY,N), \tag{17}$$

where if N is spacelike, $\varepsilon = +1$ or if N is timelike, $\varepsilon = -1$ and $f = \sum\limits_{i=1}^{n} \varepsilon_i h(E_i, E_i)$.

Proof. Suppose that $\{BE_1, \ldots, BE_\nu, BE_{\nu+1}, \ldots, BE_n, N\}$ is an orthonormal basis of $\chi(\widetilde{M})$. Then the Ricci curvature of $\widetilde{M}$ with respect to the semi-symmetric non-metric connection is

$$\widetilde{Ric}(BX,BY) = \sum_{i=1}^{n} \varepsilon_i \widetilde{g}(\widetilde{R}(BE_i,BX)BY,BE_i) + \varepsilon\widetilde{g}(\widetilde{R}(N,BX)BY,N) \tag{18}$$

for $\forall X, Y \in \chi(M)$. By taking account of (18), (15), (13) and considering the symmetry of shape operator we get (17). □

THEOREM 5.3. *Let M be non-degenerate hypersurface of a semi-Riemannian manifold $\widetilde{M}$. If $\widetilde{\rho}$ and ρ are the scalar curvatures of $\widetilde{M}$ and M with respect to the semi-symmetric non-metric connection, respectively, then*

$$\widetilde{\rho} = \rho + f^* + \varepsilon(n\mu - f)f + 2\varepsilon\widetilde{Ric}(N,N), \tag{19}$$

where $f^* = \sum\limits_{i=1}^{n} \varepsilon_i h(A_N E_i, E_i)$.

Proof. Assume that $\{BE_1, \ldots, BE_\nu, BE_{\nu+1}, \ldots, BE_n, N\}$ is an orthonormal basis of $\chi(\widetilde{M})$. Then the scalar curvature of $\widetilde{M}$ with respect to the semi-symmetric non-metric connection is

$$\widetilde{\rho} = \sum_{i=1}^{n} \varepsilon_i \widetilde{Ric}(E_i,E_i) + \varepsilon\widetilde{Ric}(N,N). \tag{20}$$

By virtue of (17), (20), we obtain (19). □

We now assume that the 1-form π is closed. In this case, we can define the sectional curvature for a section with respect to the semi-symmetric non metric connection (see [1]).

Suppose that the semi-symmetric non-metric connection $\widetilde{\nabla}$ is of constant sectional curvature. Then $\widetilde{R}(X,Y)Z$ should be of the form

$$\widetilde{R}(X,Y)Z = c\{\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\}, \tag{21}$$

where c is a certain scalar. Thus, $\widetilde{M}$ is a semi-Riemannian manifold of constant curvature c with respect to the semi-symmetric non-metric connection, which we denote by $\widetilde{M}(c)$.

THEOREM 5.4. *Let M be a non-degenerate hypersurface of a semi-Riemannian space form $\widetilde{M}(c)$ with a semi-symmetric non-metric connection. Then we get*

$$Ric(X,Y) = c(n-1)g(X,Y) + \varepsilon f h(X,Y) - h(A_N X, Y) - \varepsilon n \mu h(X,Y) \tag{22}$$

for $\forall X, Y \in \chi(M)$, where $\varepsilon_i = g(E_i, E_i)$, $\varepsilon_i = 1$, if E_i is spacelike or $\varepsilon_i = -1$, if E_i is timelike, and $f = \sum_{i=1}^{n} \varepsilon_i h(E_i, E_i)$.

Proof. Taking into account of (17) and (21), we have (22). □

From (22), the following corollaries can be stated:

COROLLARY 5.5. *Let M be a non-degenerate hypersurface of a semi-Riemannian space form $\widetilde{M}(c)$ with a semi-symmetric non-metric connection. Then Ricci tensor of M is symmetric.*

COROLLARY 5.6. *Let M be a non-degenerate hypersurface of a semi-Riemannian space form $\widetilde{M}(c)$ with a semi-symmetric non-metric connection. Then Ricci tensor of M is not parallel.*

6. THE WEYL PROJECTIVE CURVATURE TENSOR OF A NON-DEGENERATE HYPERSURFACE WITH RESPECT TO A SEMI-SYMMETRIC NON-METRIC CONNECTION

We denote the Weyl projective curvature tensor of the $(n+1)$-dimensional semi-Riemannian manifold $\widetilde{M}$ with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ by

$$\overset{\circ}{\widetilde{P}}(\widetilde{X},\widetilde{Y})\widetilde{Z} = \overset{\circ}{\widetilde{R}}(\widetilde{X},\widetilde{Y})\widetilde{Z} - \frac{1}{n}\{\overset{\circ}{\widetilde{Ric}}(\widetilde{Y},\widetilde{Z})\widetilde{X} - \overset{\circ}{\widetilde{Ric}}(\widetilde{X},\widetilde{Z})\widetilde{Y}\}, \tag{23}$$

$\forall \widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \chi(\widetilde{M})$, where $\widetilde{\overset{\circ}{Ric}}$ is Ricci tensor of $\widetilde{M}$ with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ (see [10]).

Analogous to (23), we define the Weyl projective curvature tensor of $\widetilde{M}$ with respect to the semi-symmetric non-metric connection as

$$\widetilde{P}(\widetilde{X},\widetilde{Y})\widetilde{Z} = \widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Z} - \frac{1}{n}\{\widetilde{Ric}(\widetilde{Y},\widetilde{Z})\widetilde{X} - \widetilde{Ric}(\widetilde{X},\widetilde{Z})\widetilde{Y}\}, \tag{24}$$

$\forall \widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \chi(\widetilde{M})$, where $\widetilde{Ric}$ is the Ricci tensor $\widetilde{M}$ with respect to the connection $\widetilde{\nabla}$. Thus, from (24), the Weyl projective curvature tensors with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ and induced connection ∇, respectively, are given by

$$\widetilde{P}(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) = \widetilde{R}(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) - \frac{1}{n}\{\widetilde{Ric}(\widetilde{Y},\widetilde{Z})\widetilde{g}(\widetilde{X},\widetilde{U}) - \widetilde{Ric}(\widetilde{X},\widetilde{Z})\widetilde{g}(\widetilde{Y},\widetilde{U})\} \tag{25}$$

and

$$P(X,Y,Z,U) = R(X,Y,Z,U) - \frac{1}{n-1}\{Ric(Y,Z)g(X,U) - Ric(X,Z)g(Y,U)\}, \tag{26}$$

where for all X, Y, $Z \in \chi(M)$,

$$\widetilde{P}(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) = \widetilde{g}(\widetilde{P}(\widetilde{X},\widetilde{Y})\widetilde{Z},\widetilde{U}), \quad P(X,Y,Z,U) = g(P(X,Y)Z,U)$$

and Ric is the Ricci tensor of M with respect to induced connection ∇.

By using (25), we obtain

$$\widetilde{P}(N,BY,BZ,N) = \widetilde{R}(N,BY,BZ,N) - \frac{\varepsilon}{n}\widetilde{Ric}(BY,BZ). \tag{27}$$

Applying (17) in (27), we have

$$\begin{aligned} Ric(Y,Z) = {} & \frac{n-1}{n}\widetilde{Ric}(Y,Z) - \varepsilon\widetilde{P}(N,BY,BZ,N) + \\ & + f\varepsilon h(Y,Z) - h(A_N Y,Z) - \mu n\varepsilon h(Y,Z). \end{aligned} \tag{28}$$

Then, taking account of (26), (25), (28) and (15), we get

$$\begin{aligned} P(X,Y,Z,U) = {} & \widetilde{P}(BX,BY,BZ,BU) - \varepsilon\{h(X,Z)h(Y,U) - h(Y,Z)h(X,U) \\ & + \mu h(Y,Z)g(X,U) - \mu h(X,Z)g(Y,U)\} \\ & + \frac{\varepsilon}{n-1}\{\widetilde{P}(N,BY,BZ,N)g(X,U) - \widetilde{P}(N,BX,BZ,N)g(Y,U)\} \\ & + \frac{1}{n-1}\{\varepsilon f h(X,Z) - \varepsilon\mu n h(X,Z) - h(A_N X,Z)\}g(Y,U) \\ & - \frac{1}{n-1}\{\varepsilon f h(Y,Z) - \varepsilon\mu n h(Y,Z) - h(A_N Y,Z)\}g(X,U). \end{aligned} \tag{29}$$

From (29), we have the following theorem:

THEOREM 6.1. *A totally umbilical non-degenerate hypersurface in a projectively flat semi-Riemannian manifold with a semi-symmetric non-metric connection is projectively flat.*

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