ON THE COVARIANCE STRUCTURE OF UAR(2) PROCESSES

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Stationary AR(2) processes with uniform marginal distribution have been introduced by Ristic and Popovic in 2002, as an extension of Chernick's model UAR(1). In this paper we investigate the autocorrelation structure of UAR(2) time series, both theoretical and by simulation techniques. The speed of convergence to zero of the autocorrelation function is discussed, depending the position of the parameters with respect to the border of the stationarity domain.

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1. INTRODUCTION

Non-gaussian autoregressive processes present an increasing theoretical and practical interest, due to their new and sometimes complex stochastic properties, as well as to their wide applications. Among the stationary AR time series with specified non-gaussian marginal distribution we mention the EAR processes discussed in [1] and the GAR processes discussed in [8]. In 1981, [4] Chernick has introduced the UAR(1) process as the strictly stationary AR(1) process with uniform marginal distribution U[0,1]. This class of time series is important in respect with the extreme value limit theorems. In 2002, [10], Ristic and Popovic have extended Chernick's model to UAR(2) processes, with uniform marginal distribution U[0,1], and they have estimated the parameters of the process by means of the method of ratios [6].

In this paper we investigate the autocorrelation structure of UAR(2) processes, both theoretical and by simulation techniques. We discuss the speed of convergence to zero of the autocorrelation function within the stationarity domain and near its border.

Definition. A stochastic process $\{X_n,\, n \in Z\}$ is called an **UAR(2)** time series if it is defined by the following expression

$$X_n = \begin{cases} \alpha X_{n-1} + \varepsilon_n w.p. \frac{\alpha}{\alpha-\beta} \\ \beta X_{n-2} + \varepsilon_n w.p. -\frac{\beta}{\alpha-\beta} \end{cases},$$

where α and β are parameters $\alpha \in [0,1)$ and $\beta \in (-1,0]$ such that $\alpha - \beta > 0$, $\{\varepsilon_n\}$ is a sequence of independent, identical distributed random variables with the distribution $P\{\varepsilon_n = s(\alpha-\beta) - \beta\} = \alpha - \beta$, $s = 0, 1, \ldots, k-1$, $k = \frac{1}{\alpha-\beta}$ and the random variables X_m and ε_n are independent iff $m <$n.

2. AUTOCORRELATION STRUCTURE

Property 1. If $\{X_n,\, n \in Z\}$ is a strictly stationary time series UAR(2), then the sequence of innovations $\{\varepsilon_n,\, n \in Z\}$ observes a discrete uniform distribution $\varepsilon_n \sim \mathit{Uniform}\{-\beta, \alpha - 2\beta, \ldots, 1-\alpha\}$, with

$$E(\varepsilon_n) = \frac{1-\alpha-\beta}{2}, \quad V(\varepsilon_n) = \frac{1-(\alpha-\beta)^2}{12}.$$

The proof follows from a direct characteristic function argument.

Property 2. The autocovariance function of an UAR(2) process satisfies the following relations

$$\gamma(0) = cov(X_n, X_n) = \frac{1}{12}$$
$$\gamma(1) = cov(X_n, X_{n-1}) = \gamma(0) \cdot \frac{\alpha^2}{\alpha - \beta + \beta^2}$$
$$\gamma(t) = cov(X_n, X_{n-t}) = \frac{\alpha^2}{\alpha-\beta} \cdot \gamma(t-1) - \frac{\beta^2}{\alpha-\beta} \cdot \gamma(t-2).$$

Proof.

$$\gamma(0) = V(X_n) = \frac{1}{12}$$
$$E\left(X_n^2\right) = \gamma(0) + \left(\frac{1}{2}\right)^2 = \frac{1}{12} + \frac{1}{4} = \frac{1}{3}$$
$$\gamma(1) = E(X_n X_{n-1}) - \left(\frac{1}{2}\right)^2$$
$$E(X_n X_{n-1}) = \gamma(1) + \frac{1}{4}$$
$$E(X_n X_{n-1}) = \frac{\alpha}{\alpha-\beta} E\left((\alpha X_{n-1} + \varepsilon_n) X_{n-1}\right) +$$
$$+ \left(-\frac{\beta}{\alpha-\beta}\right) E\left((\beta X_{n-2} + \varepsilon_n) X_{n-1}\right) =$$
$$= \frac{\alpha^2}{\alpha-\beta} \cdot \frac{1}{3} + \frac{\alpha}{\alpha-\beta} \cdot \frac{1}{2} \cdot \left(\frac{1-\alpha-\beta}{2}\right) -$$

$$-\frac{\beta^2}{\alpha-\beta}\cdot\left(\gamma(1)+\frac{1}{4}\right)-\frac{\beta}{\alpha-\beta}\cdot\frac{1}{2}\cdot\left(\frac{1-\alpha-\beta}{2}\right).$$

So,

$$\gamma(1)\cdot\left(1+\frac{\beta^2}{\alpha-\beta}\right)=\frac{\alpha^2}{3\left(\alpha-\beta\right)}+\frac{\alpha\left(1-\alpha-\beta\right)}{4\left(\alpha-\beta\right)}-$$

$$-\frac{\beta\left(1-\alpha-\beta\right)}{4\left(\alpha-\beta\right)}-\frac{1}{4}-\frac{\beta^2}{4\left(\alpha-\beta\right)}$$

$$\gamma(1)=\gamma(0)\cdot\frac{\alpha^2}{\alpha-\beta+\beta^2}$$

$$\gamma\left(2\right)=E\left(X_nX_{n-2}\right)-\left(\frac{1}{2}\right)^2$$

$$E\left(X_nX_{n-2}\right)=\gamma\left(2\right)+\frac{1}{4}$$

$$E\left(X_nX_{n-2}\right)=\frac{\alpha}{\alpha-\beta}E\left(\left(\alpha X_{n-1}+\varepsilon_n\right)X_{n-2}\right)+$$

$$+\left(-\frac{\beta}{\alpha-\beta}\right)E\left(\left(\beta X_{n-2}+\varepsilon_n\right)X_{n-2}\right)=$$

$$=\frac{\alpha^2}{\alpha-\beta}\cdot\left(\gamma(1)+\frac{1}{4}\right)+\frac{\alpha}{\alpha-\beta}\cdot\frac{1}{2}\cdot\left(\frac{1-\alpha-\beta}{2}\right)-$$

$$-\frac{\beta^2}{3\left(\alpha-\beta\right)}-\frac{\beta}{\alpha-\beta}\cdot\frac{1}{2}\cdot\left(\frac{1-\alpha-\beta}{2}\right)$$

$$\gamma\left(2\right)=\frac{\alpha^2}{\alpha-\beta}\gamma(1)+\frac{\alpha^2}{4\left(\alpha-\beta\right)}+\frac{\alpha\left(1-\alpha-\beta\right)}{4\left(\alpha-\beta\right)}-$$

$$-\frac{\beta^2}{3\left(\alpha-\beta\right)}--\frac{\beta\left(1-\alpha-\beta\right)}{4\left(\alpha-\beta\right)}-\frac{1}{4}$$

$$\gamma\left(2\right)=cov\left(X_n,X_{n-2}\right)=\frac{\alpha^2}{\alpha-\beta}\cdot\gamma(1)-\frac{\beta^2}{\alpha-\beta}\cdot\gamma(0).$$

So,

$$\gamma(t)=cov\left(X_n,X_{n-t}\right)=\frac{\alpha^2}{\alpha-\beta}\cdot\gamma(t-1)-\frac{\beta^2}{\alpha-\beta}\cdot\gamma\left(t-2\right).$$

By induction, we get

$$\gamma(t+1)=cov\left(X_n,X_{n-t-1}\right)=\frac{\alpha^2}{\alpha-\beta}\cdot\gamma(t)-\frac{\beta^2}{\alpha-\beta}\cdot\gamma(t-1)$$

$$E\left(X_nX_{n-t-1}\right)=\frac{\alpha}{\alpha-\beta}E\left(\left(\alpha X_{n-1}+\varepsilon_n\right)X_{n-t-1}\right)+$$

$$+\left(-\frac{\beta}{\alpha-\beta}\right)E\left(\left(\beta X_{n-2}+\varepsilon_n\right)X_{n-t-1}\right)=$$

$$= \frac{\alpha^2}{\alpha-\beta}\left(\gamma(t)+\frac{1}{4}\right)+\frac{\alpha}{\alpha-\beta}\cdot\frac{1}{2}\cdot\left(\frac{1-\alpha-\beta}{2}\right)+$$

$$+\left(-\frac{\beta}{\alpha-\beta}\right)\cdot\beta\left(\gamma(t-1)+\frac{1}{4}\right)+\left(-\frac{\beta}{\alpha-\beta}\right)\cdot\frac{1}{2}\cdot\left(\frac{1-\alpha-\beta}{2}\right)$$

$$\gamma(t+1)=\frac{\alpha^2}{\alpha-\beta}\gamma(t)-\frac{\beta^2}{\alpha-\beta}\gamma(t-1)+$$

$$+\frac{\alpha^2}{4(\alpha-\beta)}+\frac{\alpha(1-\alpha-\beta)}{4(\alpha-\beta)}-\frac{\beta^2}{4(\alpha-\beta)}-\frac{\beta(1-\alpha-\beta)}{4(\alpha-\beta)}-\frac{1}{4}.$$

So, we have

$$\gamma(t+1)=cov\left(X_n,X_{n-t-1}\right)=\frac{\alpha^2}{\alpha-\beta}\cdot\gamma(t)-\frac{\beta^2}{\alpha-\beta}\cdot\gamma(t-1). \quad \square$$

Property 3. The autocovariance function of an UAR(2) process satisfies the following exponential weighted equation:

$$\gamma(t)=G_1^tA_1+G_2^tA_2,$$

where G_1^{-1} and G_2^{-2} are the roots of the equation $1-\frac{\alpha^2}{\alpha-\beta}G+\frac{\beta^2}{\alpha-\beta}G^2=0$,

$$\gamma(0)=A_1+A_2$$

$$\gamma(1)=G_1A_1+G_2A_2.$$

So, we have

$$\gamma(t)=G_1^t\frac{\gamma(0)G_2-\gamma(1)}{G_2-G_1}+G_2^t\frac{\gamma(1)-G_1\gamma(0)}{G_2-G_1}.$$

3. A SIMULATION STUDY

The stationarity domain of an UAR(2) process is:

$$D_j=\left\{(\alpha,\beta)\,\middle|\,\alpha\in[0,1)\,,\,\beta\in(-1,0]\,,\,\frac{1}{\alpha-\beta}=j\in N^*,\,j\geq 2\right\}$$

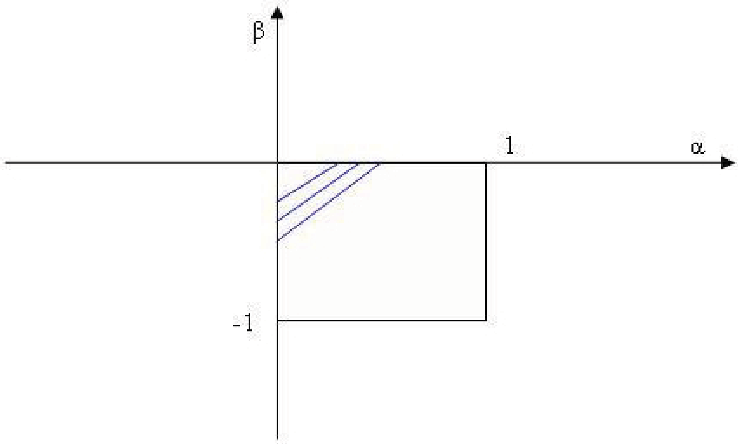
$$D=\bigcup_{j\geq 2,\,j\in N}D_j.$$

a) The aim of our simulation study consists of:

– investigation of the (weak) stationarity of the process in terms of its mean and autocovariance functions;

– the characterization of the behaviour of the autocorrelation function, both within the same domain and near its border;

– investigation of the speed of convergence to zero of the autocovariance function, both within the stationarity domain and near its border.



b) The algorithms:

Algorithm S.D. (Stationarity in the Domain):

Step 1. Generate N independent trajectories of length n

$$X_t^i, \quad t = \overline{0,\, n-1}, \quad i = \overline{1,\, N}.$$

Step 2. Calculate

$$\widehat{\mu}^i = \frac{1}{n}\sum_{t=0}^{n-1} X_t^i, \quad i = \overline{1, N}$$

$$\gamma_j\left(k\right) = cov\left(X_t, X_{t+k}\right)$$

$$\rho_j\left(k\right) = corr\left(X_t, X_{t+k}\right)$$

$$\widehat{\gamma}_j\left(k\right) = c_k$$

$$\widehat{\rho}_j\left(k\right) = r_k$$

$$r_k^i = \frac{c_k^i}{c_0^i}$$

$$r_k = \frac{1}{n}\sum_{i=1}^{N} r_k^i \quad \text{(the “mean autocorrelation”)}.$$

Step 3. Perform descriptive statistics for $\left\{\widehat{\mu}^1, \ldots, \widehat{\mu}^N\right\}$ and evaluate $\mathrm{E}(\widehat{\mu})$ and $\mathrm{Std}(\widehat{\mu})$.

Step 4. Plot (k, r_k).

Step 5. Investigate the convergence speed of r_k towards zero by choosing a threshold and identify k^*such that $|c_k| \leq$ threshold for any $k \geq k^*$.

Identify the order statistics:

$$\min_{t=0,\dots,(n-1)-k^*} \widehat{\gamma}_j\,(t+k^*,t)$$

$$\max_{t=0,\dots,(n-1)-k^*} \widehat{\gamma}_j\,(t+k^*,t)\,.$$

Algorithm GEN:

Step 1. We generate the first three values: x_0, x_1, x_2 and a value u in $(0,1)$.
Step 2. Let $u < \frac{\alpha}{\alpha-\beta}$ then $eps = x_2 - \alpha x_1$, otherwise $eps = x_2 - \beta x_0$.
Step 3. We generate $x_3, x_4, \dots, x_n$.

c) Results:

The behaviour of the **UAR(2)** process is investigated in the following points from the stationarity domain:

$$\left(\frac{1}{4},-\frac{1}{4}\right),\left(\frac{2}{5},-\frac{1}{10}\right),\left(\frac{1}{10},-\frac{2}{5}\right) \quad \text{for } j=2$$

$$\left(\frac{1}{6},-\frac{1}{6}\right),\left(\frac{1}{10},-\frac{7}{30}\right),\left(\frac{7}{30},-\frac{1}{10}\right) \quad \text{for } j=3$$

$$\left(\frac{1}{8},-\frac{1}{8}\right),\left(\frac{1}{10},-\frac{3}{20}\right),\left(\frac{3}{20},-\frac{1}{10}\right) \quad \text{for } j=4.$$

The simulation study is based on $\mathbf{N=1000}$ trajectories of length $\mathbf{n=1000}$ generated for each case considered above.

1. The behaviour of the mean function:

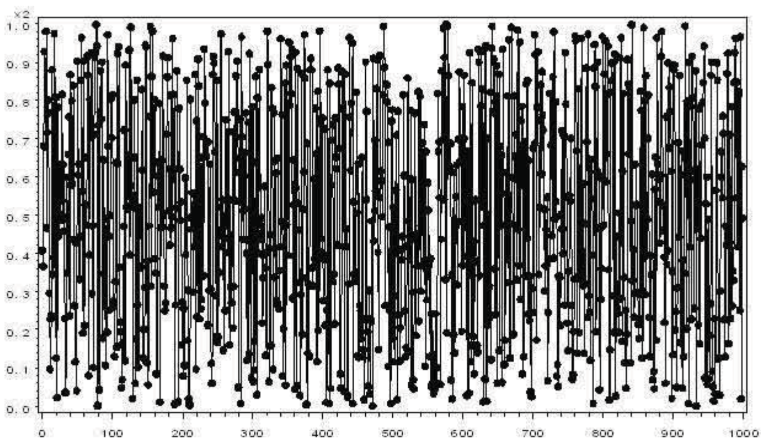


Fig. 1

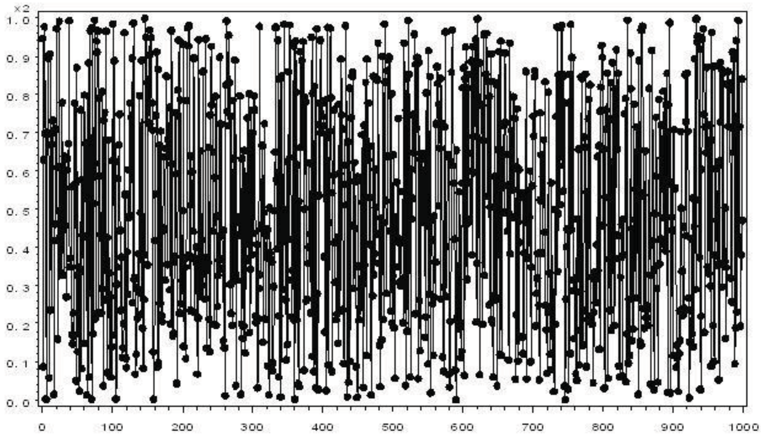


Fig. 2

In the following tables we present the descriptive statistics for the estimator $\widehat{\mu}$:

α	β	$E(\widehat{\mu})$	$Std(\widehat{\mu})$
1/4	-1/4	0.5000686	0.2915992
2/5	-1/10	0.5087181	0.2848133
1/10	-2/5	0.5083805	0.2849858

α	β	$E(\widehat{\mu})$	$Std(\widehat{\mu})$
1/6	-1/6	0.5035765	0.2907252
1/10	-7/30	0.4948670	0.2924257
7/30	-1/10	0.4967081	0.2888967

α	β	$E(\widehat{\mu})$	$Std(\widehat{\mu})$
1/8	-1/8	0.499516	0.290832
1/10	-3/20	0.5005123	0.286903
3/20	-1/10	0.5139405	0.2841330

We notice that the estimator $\widehat{\mu}$ based on trajectories of length n = 1000 has good properties in all considered points, either within the stationarity domain, or near the border.

2. The behaviour of the autocovariance function:

For $\mathbf{j} = \mathbf{2}$, the stationarity domain is

$$D_2 = \left\{ (\alpha, \beta) \;\middle|\; \alpha \in [0,1) \, , \, \beta \in (-1, 0] \, , \, \frac{1}{\alpha - \beta} = 2 \right\}.$$

In the point $\left(\frac{1}{4}, -\frac{1}{4}\right)$, the process becomes stationary relatively quickly, lag greater than 50.

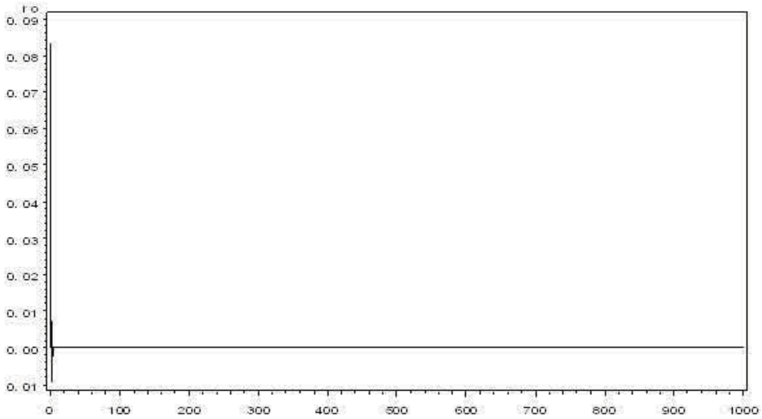


Fig. 3

At the border point $\left(\frac{2}{5}, -\frac{1}{10}\right)$, the process becomes stationary lag greater than 220.

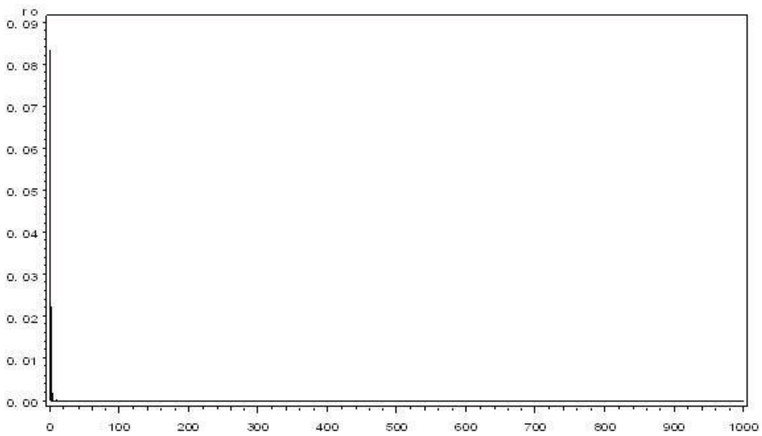


Fig. 4

At the border point $\left(\frac{1}{10}, -\frac{2}{5}\right)$, the process becomes stationary lag greater than 230.

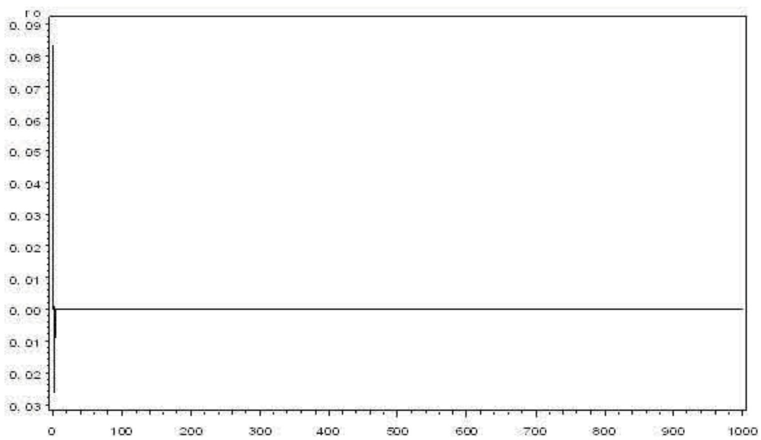


Fig. 5

For $\mathbf{j=3}$, the stationarity domain is

$$D_3 = \left\{ (\alpha, \beta) \;\middle|\; \alpha \in [0,1)\,,\ \beta \in (-1,0]\,,\ \frac{1}{\alpha - \beta} = 3 \right\}.$$

At the point $\left(\frac{1}{6}, -\frac{1}{6}\right)$, the process becomes stationary lag greater than 47.

At the border point $\left(\frac{1}{10}, -\frac{7}{30}\right)$, the process becomes stationary lag greater than 215.

At the border point $\left(\frac{7}{30}, -\frac{1}{10}\right)$, the process becomes stationary lag greater than 219.

For $\mathbf{j=4}$, the stationarity domain is

$$D_4 = \left\{ (\alpha, \beta) \;\middle|\; \alpha \in [0,1)\,,\ \beta \in (-1,0]\,,\ \frac{1}{\alpha - \beta} = 4 \right\}.$$

At the point $\left(\frac{1}{8}, -\frac{1}{8}\right)$, the process becomes stationary lag greater than 45.

At the point $\left(\frac{1}{10}, -\frac{3}{20}\right)$, the process becomes stationary lag greater than 208.

At the point $\left(\frac{3}{20}, -\frac{1}{10}\right)$, the process becomes stationary lag greater than 210.

We have recorded the estimated autocovariances:

$$\left\| \widehat{\gamma}_j\left(t+k, t\right) \right\|_{t=0,\ldots,(n-1);\, k=0,\ldots,(n-1)}$$

and we have considered a threshold for autocovariances equal to 10^{-17}.

In the following tables we present the lag corresponding to this threshold and the order statistics for the studied cases:

j = 2

Parameter	lag k^*	$\min_{t=0,\ldots,(n-1)-k^*} \widehat{\gamma}_j(t+k^*,t)$	$\max_{t=0,\ldots,(n-1)-k^*} \widehat{\gamma}_j(t+k^*,t)$
$\left(\frac{1}{4},-\frac{1}{4}\right)$	50	3.46×10^{-17}	0.11
$\left(\frac{2}{5},-\frac{1}{10}\right)$	220	2.63×10^{-17}	0.09
$\left(\frac{1}{10},-\frac{2}{5}\right)$	230	2.91×10^{-17}	0.13

j = 3

Parameter	lag k^*	$\min_{t=0,\ldots,(n-1)-k^*} \widehat{\gamma}_j(t+k^*,t)$	$\max_{t=0,\ldots,(n-1)-k^*} \widehat{\gamma}_j(t+k^*,t)$
$\left(\frac{1}{6},-\frac{1}{6}\right)$	47	3.17×10^{-17}	0.16
$\left(\frac{1}{10},-\frac{7}{30}\right)$	215	2.97×10^{-17}	0.08
$\left(\frac{7}{30},-\frac{1}{10}\right)$	219	4.13×10^{-17}	0.1

j = 4

Parameter	lag k^*	$\min_{t=0,\ldots,(n-1)-k^*} \widehat{\gamma}_j(t+k^*,t)$	$\max_{t=0,\ldots,(n-1)-k^*} \widehat{\gamma}_j(t+k^*,t)$
$\left(\frac{1}{8},-\frac{1}{8}\right)$	45	4.56×10^{-17}	0.1
$\left(\frac{1}{10},-\frac{3}{20}\right)$	208	3.91×10^{-17}	0.14
$\left(\frac{3}{20},-\frac{1}{10}\right)$	210	4.23×10^{-17}	0.09

The sample mean is a very good estimator of the expected value of the process. The autocovariance or autocorrelation functions are influenced by the position of the parameters. As expected, the "most stationary" point is the center of the domain. The autocovariance function decreases quicker to zero as j increases. The simulation study has been performed using the software SAS, version 8.1.

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