AN ALMOST PARACONTACT RIEMANNIAN STRUCTURE ON INDICATRIX BUNDLE OF A FINSLER MANIFOLD

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We consider a Finsler manifold. Then the slit tangent bundle of it is endowed with the Cartan nonlinear connection and with various metrics of Sasaki type. It has also two remarkable vector fields: the Liouville vector field and the geodesic vector field. Using these elements we deform the almost product structure defined by the Cartan nonlinear connection to a framed f-structure that in some conditions reduces to a 2-paracontact structure. We show that its restriction to the indicatrix bundle is an almost paracontact structure.

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1. INTRODUCTION

A Finsler structure on a manifold M defines a Riemannian structure in the vertical tangent bundle and a natural supplement of the vertical tangent bundle called the horizontal vector bundle or the Cartan nonlinear connection [6], [2]. Using the horizontal tangent bundle, the Riemannian structure from the vertical tangent bundle is prolonged to a Riemannian structure on the whole tangent bundle TM. Such a Riemannian metric was proposed by S. Sasaki for Riemannian manifolds and by M. Matsumoto [3], for Finsler manifolds. It will be called the Sasaki-Matsumoto metric. While it is very natural defined, the Sasaki-Matsumoto metric is very rigid, i.e., if certain usual conditions are imposed on it (to be flat, to be symmetric ...) it comes out that the Finslerian structure reduces to a particular Riemannian one. On the other hand, it has not a Finslerian meaning since it is not homogeneous with respect to the variables from fibres and so it is not defined on the projectivized tangent bundle as all the other Finslerian objects. This fact makes it inappropriate in treating global problems in Finsler geometry as for instance the Gauss-Bonnet theorem [2]. Moreover, its form as a sum of two terms of different physical dimensions produces some difficulties when it is used for applications in gauge

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theories. For these reasons, R. Miron in [5] has introduced a simple version of it (see formula (2) below) that is homogeneous of degree zero with respect to variables from fibres. Starting with a different point of view, M. Anastasiei and H. Shimada [1], arrived to a more general and more flexible Riemannian metric of Sasaki-Matsumoto type that contains two arbitrary functions (see formula (3) below). In this paper we consider an almost product structure that paired with $\widetilde{G}$ given by (3) provide an almost Riemannian product structure on $\widetilde{TM} = TM \setminus \{0\}$. Using the directions defined by the Liouville vector field $\mathbf{C}$ and by the geodesic spray $\mathbf{S}$ we define by (14) a structure of corank 2 denoted by $\widetilde{p}$ and we show that in certain conditions this defines a framed $f(3,-1)$-structure and paired with $\widetilde{G}$ it defines a 2-paracontact structure. In the last Section, we show that restricting this 2-paracontact structure one gets an almost paracontact structure on the indicatrix bundle.

2. PRELIMINARIES

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \bigcup_{x\in M} T_xM$ the tangent bundle of M. A Finsler metric on M is a function $F : TM \to [0,\infty)$ which has the following properties:

(i) F is C^∞ on $\widetilde{TM}$;

(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;

(iii) for each $y \in T_xM$, the symmetric bilinear form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ given by $\mathbf{g}_y(u,v) := g_{ij}(y)u^iv^j$ where

$$g_{ij}(y) := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j},$$

and $u = u^i\frac{\partial}{\partial x^i}|_x$ and $v = v^i\frac{\partial}{\partial x^j}|_x$, is positive defined.

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by $\mathbf{C}_y(u,v,w) := C_{ijk}(y)u^iv^jw^k$ where

$$C_{ijk}(y) := \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}.$$

The family $\mathbf{C} := \{\mathbf{C_y}\}_{y\in\widetilde{TM}}$ is called the Cartan torsion. By using the notion of Cartan torsion, for $y \in T_x \setminus 0$ one defines the mean Cartan torsion $\mathbf{I}_y : T_xM \to \mathbb{R}$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i(y) := g^{jk}C_{ijk}(y)$. It is well known that $\mathbf{I} = 0$ if and only if F is Riemannian [2].

Put $C^i{}_{jk} := g^{is}C_{sjk}$ where $g^{ij} = (g_{ij})^{-1}$. The formal Christofell symbols of the second kind are

$$\gamma^k{}_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right).$$

They are functions on $\widetilde{TM}$. We can also define some other quantities on $\widetilde{TM}$ by

$$N^i_j(x,y) := \gamma^i{}_{jk}y^k - C^i{}_{jk}\gamma^k{}_{rs}y^r y^s,$$

where $y \in \widetilde{TM}$. The above N^i_j are called the nonlinear connection coefficients on $\widetilde{TM}$.

3. METRICS OF SASAKI-MATSUMOTO TYPE ON $\widetilde{TM}$

Let $F^n = (M,F)$ be a Finsler manifold and let $(x,y) = (x^i, y^i)$ be the local coordinates on $\widetilde{TM}$. It is well known that the tangent space to $\widetilde{TM}$ at (x,y) splits into the direct sum of the vertical subspace $V\widetilde{TM}_{(x,y)} = span(\dot{\partial}_i) := \frac{\partial}{\partial y^i}$ and the horizontal subspace $H\widetilde{TM}_{(x,y)} = span\,\delta_i := \frac{\partial}{\partial x^i} - N^j_i(x,y)\dot{\partial}_j$:

$$T_{(x,y)}\widetilde{TM} = V\widetilde{TM}_{(x,y)} \oplus H\widetilde{TM}_{(x,y)},$$

M. Matsumoto [3], extended to Finsler spaces F^n the notion of Sasaki metric, considering the tensor field

$$\widetilde{G}_{SM}(x,y) = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)\delta y^i \otimes \delta y^j \quad \forall (x,y) \in \widetilde{TM}, \tag{1}$$

where $\delta y^i = dy^i + N^i_j(x,y)dx^j$. It easily follows that $\widetilde{G}_{SM}$ is a Riemannian metric globally defined on $\widetilde{TM}$ and depending only on the fundamental function F of the Finsler space F^n. Also, we see that the Sasaki-Matsumoto metric $\widetilde{G}_{SM}$ is not homogeneous on the fibers of the tangent bundle TM.

The Miron metric is defined as follows:

$$\widetilde{G}_M(x,y) = g_{ij}(x,y)dx^i \otimes dx^j + \frac{c^2}{F^2}g_{ij}(x,y)\delta y^i \otimes \delta y^j, \tag{2}$$

for each $(x,y) \in \widetilde{TM}$, where $c > 0$ is a constant. It is easy to check that $\widetilde{G}_M$ is 0-homogeneous on the fibers of TM. It is clear that it depends only on the fundamental function F of the Finsler space F^n. A more general metric of this type is the Riemannian metric

$$\widetilde{G}(x,y) = g_{ij}(x,y)dx^i \otimes dx^j + (a(F^2)g_{ij}(x,y) + b(F^2)y_i y_j)\delta y^i \otimes \delta y^j, \tag{3}$$

for all $(x,y) \in \widetilde{TM}$, where $a, b : [0,\infty] \to [0,\infty]$, $a > 0$ and $a + bF^2 > 0$. This is in fact a family of metrics depending on two parameters. The Sasaki-Matsumoto metric and the Miron metric are clearly entering in this family.

Now, we consider the $\mathcal{F}(\widetilde{TM})$-linear mapping $\widetilde{P}:\chi(\widetilde{TM})\to\chi(\widetilde{TM})$, defined by

$$\text{(4)}\qquad \begin{cases} \widetilde{P}(\delta_i)=\left[-\dfrac{1}{\sqrt{a}}\delta_i^k+\dfrac{1}{F^2}\left(\dfrac{1}{\sqrt{a}}+\varepsilon\dfrac{1}{\sqrt{a+bF^2}}\right)y_iy^k\right]\dot{\partial}_k,\\[2ex] \widetilde{P}(\dot{\partial}_i)=\left[-\sqrt{a}\delta_i^k+\dfrac{1}{F^2}\left(\sqrt{a}+\varepsilon\sqrt{a+bF^2}\right)y_iy^k\right]\delta_k,\end{cases}$$

where $\varepsilon=+1$ or -1. By a direct calculation one verifies that $\widetilde{P}$ is an almost product structure on $\widetilde{TM}$ which paired with $\widetilde{G}$ provide a Riemannian almost product structure. Thus $(\widetilde{TM},\widetilde{G},\widetilde{P})$ is a Riemannian almost product manifold.

4. A FRAMED $f(3,-1)$-STRUCTURE ON $\widetilde{TM}$

On $\widetilde{TM}$ there exist two remarkable vector fields: $\mathbf{C}=y^i\dot{\partial}_i$, called the Liouville vector field and $\mathbf{S}=y^i\delta_i$, which is the geodesic spray of F^n. Let us put

$$\text{(5)}\qquad \widetilde{\xi}_1:=A\mathbf{S}=Ay^i\delta_i,\quad \widetilde{\xi}_2:=B\mathbf{C}=By^i\dot{\partial}_i,$$

where A and B are no-zero functions on $\widetilde{TM}$. In fact we take into consideration the directions defined by $\mathbf{C}$ and $\mathbf{S}$. On using (4) and (5) we obtain

$$\text{(6)}\qquad \widetilde{P}(\widetilde{\xi}_1)=\frac{A\varepsilon}{B\sqrt{a+bF^2}}\widetilde{\xi}_2,\quad \widetilde{P}(\widetilde{\xi}_2)=\frac{B\varepsilon\sqrt{a+bF^2}}{A}\widetilde{\xi}_1.$$

Then we consider the following 1-forms

$$\text{(7)}\qquad \widetilde{\eta}^1=Cy_idx^i,\quad \widetilde{\eta}^2=Dy_i\delta y^i,$$

where C and D are no-zero functions on $\widetilde{TM}$.

From (4) and (7) we get

$$\text{(8)}\qquad \begin{aligned}(\widetilde{\eta}^1\circ\widetilde{P})(\dot{\partial}_i)&=\left[-\sqrt{a}\delta_i^k+\frac{1}{F^2}\left(\sqrt{a}+\varepsilon\sqrt{a+bF^2}\right)y_iy_k\right]Cy_k\\ &=C\left[-\sqrt{a}+\frac{1}{F^2}\left(\sqrt{a}+\varepsilon\sqrt{a+bF^2}\right)F^2\right]y_i\\ &=\frac{C\varepsilon\sqrt{a+bF^2}}{D}\widetilde{\eta}^2(\dot{\partial}_i),\end{aligned}$$

$$(\widetilde{\eta}^2 \circ \widetilde{P})(\delta_i) = \left[-\frac{1}{\sqrt{a}}\delta_i^k + \frac{1}{F^2}\left(\frac{1}{\sqrt{a}} + \frac{\varepsilon}{\sqrt{a+bF^2}}\right) y_i y^k\right] Dy_k$$
$$= D\left[-\frac{1}{\sqrt{a}} + \left(\frac{1}{\sqrt{a}} + \frac{\varepsilon}{\sqrt{a+bF^2}}\right)\right] y_i$$
$$= \frac{D\varepsilon}{C\sqrt{a+bF^2}}\widetilde{\eta}^1(\delta_i), \tag{9}$$

and

$$(\widetilde{\eta}^1 \circ \widetilde{P})(\delta_i) = 0 = \widetilde{\eta}^2(\delta_i), \tag{10}$$

$$(\widetilde{\eta}^2 \circ \widetilde{P})(\dot{\partial}_i) = 0 = \widetilde{\eta}^1(\dot{\partial}_i). \tag{11}$$

From (8)–(11), it follows

$$\widetilde{\eta}^1 \circ \widetilde{P} = \frac{C\varepsilon\sqrt{a+bF^2}}{D}\widetilde{\eta}^2, \quad \widetilde{\eta}^2 \circ \widetilde{P} = \frac{D\varepsilon}{C\sqrt{a+bF^2}}\widetilde{\eta}^1. \tag{12}$$

The equations (5) and (7) give

$$\widetilde{\eta}^1(\widetilde{\xi}_1) = ACF^2, \quad \widetilde{\eta}^2(\widetilde{\xi}_2) = BDF^2, \quad \widetilde{\eta}^1(\widetilde{\xi}_2) = \widetilde{\eta}^2(\widetilde{\xi}_1) = 0,$$

or

$$\widetilde{\eta}^1(\widetilde{\xi}_a) = ACF^2\delta_a^1, \quad \widetilde{\eta}^2(\widetilde{\xi}_a) = BDF^2\delta_a^2, \tag{13}$$

where $a = 1, 2$. Also by using (3) and (7) we have

Lemma 4.1. *For $X \in \chi(\widetilde{TM})$, we have the following*

$$\frac{A}{C}\widetilde{\eta}^1(X) = \widetilde{G}(X, \widetilde{\xi}_1), \quad \frac{B(a+bF^2)}{D}\widetilde{\eta}^2(X) = \widetilde{G}(X, \widetilde{\xi}_2).$$

Now we define a tensor field $\widetilde{p}$ of type $(1,1)$ on $\widetilde{TM}$ by

$$\widetilde{p}(X) = \widetilde{P}(X) - \widetilde{\eta}^1(X)\widetilde{\xi}_2 - \widetilde{\eta}^2(X)\widetilde{\xi}_1, \quad X \in \chi(\widetilde{TM}). \tag{14}$$

This can be written in a more compact form as $\widetilde{p} = \widetilde{P} - \widetilde{\eta}^1 \otimes \widetilde{\xi}_2 - \widetilde{\eta}^2 \otimes \widetilde{\xi}_1$.

Proposition 4.2. *The tensor field $\widetilde{p}$ on $\widetilde{TM}$ satisfies the following properties:*

(i) $\widetilde{p}(\widetilde{\xi}_1) = \left(\frac{A\varepsilon}{B\sqrt{a+bF^2}} - ACF^2\right)\widetilde{\xi}_2$, $\widetilde{p}(\widetilde{\xi}_2) = \left(\frac{B\varepsilon\sqrt{a+bF^2}}{A} - BDF^2\right)\widetilde{\xi}_1$;

(ii) $\widetilde{\eta}^1 \circ \widetilde{p} = \left(\frac{C\varepsilon\sqrt{a+bF^2}}{D} - ACF^2\right)\widetilde{\eta}^2$, $\widetilde{\eta}^2 \circ \widetilde{p} = \left(\frac{D\varepsilon}{C\sqrt{a+bF^2}} - BDF^2\right)\widetilde{\eta}^1$;

(iii) $\widetilde{p}^2 = I - \left(\frac{D\varepsilon}{C\sqrt{a+bF^2}} + \frac{B\varepsilon\sqrt{a+bF^2}}{A} - BDF^2\right)\widetilde{\eta}^1 \otimes \widetilde{\xi}_1 - \left(\frac{C\varepsilon\sqrt{a+bF^2}}{D} + \frac{A\varepsilon}{B\sqrt{a+bF^2}} - ACF^2\right)\widetilde{\eta}^2 \otimes \widetilde{\xi}_2$.

Proof. Equation (i) follows from (6) and (13), (ii) follows from (12) and (13) and finally (iii) follows from (i) and (12). □

Proposition 4.3. *The Riemannian metric $\widetilde{G}$ satisfies*

$$\widetilde{G}(\widetilde{p}X,\widetilde{p}Y)=\widetilde{G}(X,Y)+\left(B^2F^2(a+bF^2)-2\frac{B\varepsilon\sqrt{a+bF^2}}{C}\right)\widetilde{\eta}^1(X)\widetilde{\eta}^1(Y)+$$
$$+\left(A^2F^2-2\frac{A\varepsilon\sqrt{a+bF^2}}{D}\right)\widetilde{\eta}^2(X)\widetilde{\eta}^2(Y).$$

Proof. By direct calculation we have

$$\widetilde{G}(\widetilde{\xi}_1,\widetilde{\xi}_1)=\widetilde{G}(Ay^i\delta_i,Ay^j\delta_j)=A^2y^iy^jg_{ij}=A^2F^2, \tag{15}$$

$$\widetilde{G}(\widetilde{\xi}_2,\widetilde{\xi}_2)=\widetilde{G}(By^i\dot{\partial}_i,By^j\dot{\partial}_j)=B^2y^iy^j(ag_{ij}+by^iy^j)$$
$$=B^2F^2(a+bF^2) \tag{16}$$

and

$$\widetilde{G}(\widetilde{\xi}_1,\widetilde{\xi}_2)=\widetilde{G}(Ay^i\delta_i,By^j\dot{\partial}_j)=0. \tag{17}$$

From (12), (15)–(17) and Lemma 4.1, we get

$$\begin{aligned}\widetilde{G}(\widetilde{p}X,\widetilde{p}Y)&=\widetilde{G}(\widetilde{P}(X),\widetilde{P}(Y))-\widetilde{G}(\widetilde{P}(X),\widetilde{\xi}_2)\widetilde{\eta}^1(Y)-\widetilde{G}(\widetilde{P}(X),\widetilde{\xi}_1)\widetilde{\eta}^2(Y)\\
&\quad-\widetilde{G}(\widetilde{\xi}_2,\widetilde{P}(Y))\widetilde{\eta}^1(X)+\widetilde{\eta}^1(X)\widetilde{\eta}^1(Y)\widetilde{G}(\widetilde{\xi}_2,\widetilde{\xi}_2)\\
&\quad-\widetilde{\eta}^2(X)\widetilde{G}(\widetilde{\xi}_1,\widetilde{P}(Y))+\widetilde{\eta}^2(X)\widetilde{\eta}^2(Y)\widetilde{G}(\widetilde{\xi}_1,\widetilde{\xi}_1)\\
&=\widetilde{G}(X,Y)-\frac{B(a+bF^2)}{D}\widetilde{\eta}^2(\widetilde{P}(X))\widetilde{\eta}^1(Y)-\frac{A}{C}\widetilde{\eta}^1(\widetilde{P}(X))\widetilde{\eta}^2(Y)\\
&\quad-\frac{B(a+bF^2)}{D}\widetilde{\eta}^2(\widetilde{P}(Y))\widetilde{\eta}^1(X)+B^2F^2(a+bF^2)\widetilde{\eta}^1(X)\widetilde{\eta}^1(Y)\\
&\quad-\frac{A}{C}\widetilde{\eta}^2(X)\widetilde{\eta}^1(\widetilde{P}(Y))+A^2F^2\widetilde{\eta}^2(X)\widetilde{\eta}^2(Y)\\
&=\widetilde{G}(X,Y)+\left(B^2F^2(a+bF^2)-\frac{2B\varepsilon\sqrt{a+bF^2}}{C}\right)\widetilde{\eta}^1(X)\widetilde{\eta}^1(Y)\\
&\quad+\left(A^2F^2-\frac{2A\varepsilon\sqrt{a+bF^2}}{D}\right)\widetilde{\eta}^2(X)\widetilde{\eta}^2(Y).\end{aligned} \tag{18}$$

This completes the proof. □

Theorem 4.4. *The ensemble $(\widetilde{p},(\widetilde{\xi}_a),(\widetilde{\eta}^b))$, $a,b=1,2$ provides a framed $f(3,-1)$-structure on $\widetilde{TM}$ if and only if*

$$\begin{cases}\text{(i)} & AC=\frac{1}{F^2},\\ \text{(ii)} & BD=\frac{1}{F^2},\\ \text{(iii)} & A=B\varepsilon\sqrt{a+bF^2},\\ \text{(iv)} & D=C\varepsilon\sqrt{a+bF^2}.\end{cases} \tag{19}$$

Proof. Let $(\widetilde{p}, (\widetilde{\xi}_a), (\widetilde{\eta}^b))$ be a framed $f(3,-1)$-structure on $\widetilde{TM}$. By definition of framed f-structures, we have $\widetilde{p}(\widetilde{\xi}_b) = 0$, $\widetilde{\eta}^a \circ \widetilde{p} = 0$ and $\widetilde{\eta}^a(\widetilde{\xi}_b) = \delta^a_b$, where $a, b = 1, 2$. Thus by using part (i) of Proposition 4.2 and (13) we get $AC = \frac{1}{F^2}$, $BD = \frac{1}{F^2}$, $A = B\varepsilon\sqrt{a + bF^2}$ and $D = C\varepsilon\sqrt{a + bF^2}$. Conversely, let (i)–(iv) be hold. Then by using (13) and Proposition 4.2 we conclude that

$$\widetilde{\eta}^a(\widetilde{\xi}_b) = \delta^a_b, \quad \widetilde{p}(\widetilde{\xi}_a) = 0, \quad \widetilde{\eta}^a \circ \widetilde{p} = 0, \tag{20}$$

and

$$\widetilde{p}^2 = I - \widetilde{\eta}^1 \otimes \widetilde{\xi}_1 - \widetilde{\eta}^2 \otimes \widetilde{\xi}_2. \tag{21}$$

For completeness of proof, we should prove that $\widetilde{p}^3 - \widetilde{p} = 0$ and $\widetilde{p}$ is of rank $2n - 2$. From (20) and (21), we get $\widetilde{p}^3(X) = \widetilde{p}(X)$ for $X \in \chi(\widetilde{TM})$. Now, we show that $\mathrm{Ker}\,\widetilde{p} = span\{\widetilde{\xi}_1, \widetilde{\xi}_2\}$. From the second equations in (20), we see that $span\{\widetilde{\xi}_1, \widetilde{\xi}_2\} \subseteq \mathrm{Ker}\,\widetilde{p}$. Now, let X belongs to $\mathrm{Ker}\,\widetilde{p}$. Then $\widetilde{p}(X) = 0$ implies that

$$\widetilde{P}(X) - \widetilde{\eta}^1(X)\widetilde{\xi}_2 - \widetilde{\eta}^2(X)\widetilde{\xi}_1 = 0.$$

Thus,

$$\widetilde{P}^2(X) = \widetilde{\eta}^1(X)\widetilde{P}(\widetilde{\xi}_2) + \widetilde{\eta}^2(X)\widetilde{P}(\widetilde{\xi}_1).$$

Since $\widetilde{P}^2 = I$, it follows from (6) that

$$X = \widetilde{\eta}^1(X)\widetilde{\xi}_1 + \widetilde{\eta}^2(X)\widetilde{\xi}_2,$$

that is $X \in span\{\widetilde{\xi}_1, \widetilde{\xi}_2\}$. □

For the geometry of framed f-structures we refer to [8], [4]. For the Finslerian setting one can see [7].

THEOREM 4.5. *The ensemble* $(\widetilde{p}, (\widetilde{\xi}_a), (\widetilde{\eta}^b))$, $a, b = 1, 2$ *provides an almost 2-paracontact Riemannian structure on* $(\widetilde{TM}, \widetilde{G})$ *if and only if*

$$A = C = -\frac{1}{F}, \quad B = \frac{-1}{\varepsilon F\sqrt{a + bF^2}}, \quad D = \frac{-\varepsilon\sqrt{a + bF^2}}{F}, \tag{22}$$

or

$$A = C = \frac{1}{F}, \quad B = \frac{1}{\varepsilon F\sqrt{a + bF^2}}, \quad D = \frac{\varepsilon\sqrt{a + bF^2}}{F}. \tag{23}$$

Proof. A framed $f(3,-1)$-structure is an almost 2-paracontact Riemannian structure if and only if

$$\widetilde{G}(\widetilde{p}X, \widetilde{p}Y) = \widetilde{G}(X, Y) - \widetilde{\eta}^1(X)\widetilde{\eta}^1(Y) - \widetilde{\eta}^2(X)\widetilde{\eta}^2(Y). \tag{24}$$

Now, if (22) or (23) hold then by using Proposition 4.3 and Theorem 4.4 it results that $(\widetilde{p}, (\widetilde{\xi}_a), (\widetilde{\eta}^b))$, $a, b = 1, 2$, provides an almost 2-paracontact Riemannian structure on $(\widetilde{TM}, \widetilde{G})$. Conversely, if $(\widetilde{p}, (\widetilde{\xi}_a), (\widetilde{\eta}^b))$, $a, b = 1, 2$, is an almost 2-paracontact Riemannian structure on $(\widetilde{TM}, \widetilde{G})$ then by using (24) and Proposition 4.3 we have

$$B^2F^2(a+bF^2) - \frac{2B\varepsilon\sqrt{a+bF^2}}{C} = -1. \tag{25}$$

By using (25) and parts (i), (iii) of (19) we obtain $C = \pm\frac{1}{F}$. If $C = -\frac{1}{F}$, then from (19) we get

$$A = -\frac{1}{F}, \quad B = \frac{-1}{\varepsilon F\sqrt{a+bF^2}}, \quad D = \frac{-\varepsilon\sqrt{a+bF^2}}{F}.$$

Similarly, if $C = \frac{1}{F}$, then from (19) we get

$$A = \frac{1}{F}, \quad B = \frac{1}{\varepsilon F\sqrt{a+bF^2}}, \quad D = \frac{\varepsilon\sqrt{a+bF^2}}{F}. \quad \square$$

Now, let A, B, C, D satisfying (22) or (23). In these cases, we denote $\widetilde{P}, \widetilde{\xi}_1, \widetilde{\xi}_2, \widetilde{\eta}^1, \widetilde{\eta}^2, \widetilde{p}$ with symbols $P, \xi_1, \xi_2, \eta^1, \eta^2, p$, respectively. Thus we have $P = \widetilde{P}$ and

$$\xi_1 = -\frac{1}{F}y^i\delta_i, \quad \xi_2 = \frac{-1}{\varepsilon F\sqrt{a+bF^2}}y^i\partial_{\bar{i}}, \tag{26}$$
$$\eta^1 = \frac{-1}{F}y_i dx^i, \quad \eta^2 = \frac{-\varepsilon\sqrt{a+bF^2}}{F}y_i\delta y^i,$$

or

$$\xi_1 = \frac{1}{F}y^i\delta_i, \quad \xi_2 = \frac{1}{\varepsilon F\sqrt{a+bF^2}}y^i\partial_{\bar{i}}, \tag{27}$$
$$\eta^1 = \frac{1}{F}y_i dx^i, \quad \eta^2 = \frac{\varepsilon\sqrt{a+bF^2}}{F}y_i\delta y^i.$$

By using (26), (27) and Theorem 4.5, we get

THEOREM 4.6. *The ensemble $(p, (\xi_a), (\eta^b))$, $a, b = 1, 2$, is a an almost 2-paracontact Riemannian structure on $(\widetilde{TM}, \widetilde{G})$.*

From (14), (26) and (27) the following local expression of p is obtained

$$p(\delta_i) = -\frac{1}{\sqrt{a}}\left(\delta_i^k - \frac{1}{F^2}y_iy^k\right)\dot{\partial}_k, \tag{28}$$

$$p(\dot{\partial}_i) = -\sqrt{a}\left(\delta_i^k - \frac{1}{F^2}y_iy^k\right)\delta_k. \tag{29}$$

Let us put

$$h(X,Y) = G(pX,Y), \quad X,Y \in \chi(TM_0). \tag{30}$$

Then we get the following

THEOREM 4.7. *The map h is a bilinear and symmetric tensor field on $\widetilde{TM}$. Further, h is of $rank(2n-2)$ with the null space* $\ker p$.

Proof. By using (28), (29) and (30) we obtain

$$h(\delta_i, \dot{\partial}_j) = G(p(\delta_i), \partial_{\bar{j}}) = -\frac{1}{\sqrt{a}}\left(\delta_i^k - \frac{1}{F^2}y_i y^k\right)(a g_{kj} + b y_j y_k) = \tag{31}$$
$$= -\sqrt{a}\left(g_{ij} - \frac{1}{F^2}y_i y_j\right),$$

and

$$h(\dot{\partial}_i, \delta_j) = -\sqrt{a}(g_{ij} - \frac{1}{F^2}y_i y_j), \quad h(\delta_i, \delta_j) = h(\dot{\partial}_i, \dot{\partial}_j) = 0. \tag{32}$$

Since G is bilinear, from (31) and (32), we conclude that h is symmetric and bilinear on $\widetilde{TM}$. The null space of h is

$$\{X \mid h(X,Y) = 0,\ \forall Y \in TM_0\} = \{X \mid G(pX, Y) = 0,\ \forall Y \in TM_0\}$$
$$= \{X \mid p(X) = 0\} = \ker p.$$

Therefore, by the proof of Theorem 4.4, we get $rank\, h = 2n-2$. □

By using (31) and (32) the tensor field h can be written as follows

$$h = -2\sqrt{a}\left(g_{ij} - \frac{1}{F^2}y_i y_j\right) dx^i \delta y^j.$$

Thus it is a singular pseudo-Riemannian metric on $\widetilde{TM}$.

5. PARACONTACT STRUCTURE ON INDICATRIX BUNDLE

The set $IM = \{(x,y) \in TM_0 \mid F(x,y) = 1\}$ is called the indicatrix bundle of F^n. This is a submanifold of dimension $2n-1$ of $\widetilde{TM}$. We show that the framed $f(3,-1)$-structure on $\widetilde{TM}$, given by Theorem 4.5, induces an almost paracontact structure on IM.

Let

$$x^i = x^i(u^\alpha), \quad y^i = y^i(u^\alpha), \quad \alpha \in \{1, \ldots, 2n-1\}, \tag{33}$$

with $rank\left(\frac{\partial x^i}{\partial u^\alpha}, \frac{\partial y^i}{\partial u^\alpha}\right) = 2n-1$ be a parametrization of IM. The natural frame field on IM is represented by

$$\frac{\partial}{\partial u^\alpha} = \frac{\partial x^i}{\partial u^\alpha}\frac{\partial}{\partial x^i} + \frac{\partial y^i}{\partial u^\alpha}\frac{\partial}{\partial y^i} = \frac{\partial x^i}{\partial u^\alpha}\frac{\delta}{\delta x^i} + \left(N_i^k \frac{\partial x^i}{\partial u^\alpha} + \frac{\partial y^k}{\partial u^\alpha}\right)\frac{\partial}{\partial y^k}. \tag{34}$$

Since $F^2 = 1$ on IM, then by using (34) we obtain

$$(35)\quad 0=\frac{\partial F^2}{\partial u^\alpha}=\frac{\partial x^i}{\partial u^\alpha}\delta_i F^2+\left(N_i^k\frac{\partial x^i}{\partial u^\alpha}+\frac{\partial y^k}{\partial u^\alpha}\right)\dot{\partial}_k F^2=2\left(N_i^k\frac{\partial x^i}{\partial u^\alpha}+\frac{\partial y^k}{\partial u^\alpha}\right)y_k,$$

because for a Finsler space we have $\delta_i F^2 = 0$ and $\dot{\partial}_i F^2 = 2y_i$. By using (34) and (35) we deduce that

$$(36)\qquad G\left(\frac{\partial}{\partial u^\alpha},\xi_2\right)=\pm\frac{\sqrt{a+bF^2}}{\varepsilon F}\left(N_i^k\frac{\partial x^i}{\partial u^\alpha}+\frac{\partial y^k}{\partial u^\alpha}\right)y_k=0.$$

Thus the vector field ξ_2 restricted to IM is normal to IM. Since $G(\xi_1, \xi_2) = 0$, ξ_1 is tangent to IM.

We restrict to IM all the objects introduced above and indicate this fact by putting a bar over the letters denoting those objects. We have the following

Lemma 5.1. *On IM, for any $X \in \chi(IM)$ we have*

$$\bar{\xi}_1=\varepsilon y^i\delta_i=\varepsilon\mathbf{S},\quad \bar{\eta}^1=\varepsilon y_i dx^i,\quad \bar{\eta}^2=0,\quad \bar{p}(X)=\bar{P}(X)-\bar{\eta}^1(X)\bar{\xi}_2.$$

Proof. Since $F = 1$ on IM, by using (26) and (27), it results that $\bar{\xi}_1$ is equal to $\varepsilon\mathbf{S}$ and $\bar{\eta}^1$ is equal with $\varepsilon y_i dx^i$. From $\bar{\eta}^2(X) = G(X, \xi_2) = 0$, the other relation of lemma will follow. □

We put $\bar{\xi} = \bar{\xi}_1$, $\bar{\eta} = \bar{\eta}^1$ and $\bar{G} = \widetilde{G}|_{IM}$. Then Theorem 4.6 and Lemma 5.1, imply the following

Theorem 5.2. *$(\bar{p}, \bar{\xi}, \bar{\eta}, \bar{G})$ defines an almost paracontact Riemannian structure on IM, that is,*
(1) $\bar{\eta}(\bar{\xi}) = 1,\ \bar{p}(\bar{\xi}) = 0,\ \bar{\eta}\circ\bar{f} = 0,$
(2) $\bar{p}^2(X) = X - \bar{\eta}(X)\bar{\xi},\ X \in \chi(IM),$
(3) $\bar{p}^3 - \bar{p} = 0,\ rank\,\bar{p} = 2n - 2 = (2n - 1) - 1,$
(4) $\bar{G}(\bar{p}X, \bar{p}Y) = \bar{G}(X, Y) - \bar{\eta}(X)\bar{\eta}(Y),\ X, Y \in \chi(TM).$

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