ON SOME SINGULAR INTEGRAL EQUATIONS IN ASYMMETRIC ELASTICITY

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We construct the matrix of fundamental solutions for the equations of theory of asymmetric elasticity in the two dimensional case. In this context, we obtain the analogous of simple and double layers potentials as well as those of volume potential, as in the classical theory of singular integral equations. The system of singular integral equations for the first and the second principal boundary value problems are obtained. It is proved that for every system of singular integral equations the index is equal to zero.

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1. INTRODUCTION

There are a lot of papers dedicated to the theory of asymmetric elasticity. Classical continuum mechanics considers material continua as simple point-continua with points having three displacement-degrees of freedom, and the response of a material to the displacement of its points is characterized by a symmetric Cauchy stress tensor presupposing that the transmission of loads through surface elements is uniquely determined by a force vector, neglecting couples. Such a model may be insufficient for the description of certain physical phenomena. Non-classical behaviour due to microstructural effects is observed most in regions of high strain gradients, e.g. at notches, holes or cracks.

The model proposed by the Cosserat brothers is one of the most prominent extended continuum models. Postulating the invariance of energy under Euclidean transformations they were able to derive the equations of balance of forces and balance of angular momentum in a geometrically exact format. However, they never wrote down any constitutive equations.

Compared to classical linear elasticity the model features three additional, independent degrees of freedom, related to the rotation of each particle which need not coincide with the macroscopic rotation of the continuum at the same point. One of the essential features of polar continua is that the stress tensor

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is not necessarily symmetric, and the balance of angular momentum equation has to be modified accordingly. All theories in which the stress tensor is not symmetric can be regarded as polar-continua. The non-symmetry of the stress tensor appears also if higher order deformation gradients are included in the free energy, instead of only the first order gradients. Both such theories typically predict a size-effect, meaning that smaller samples of the same material behave relatively stiffer than larger samples. This is an experimental fact, but completely neglected in the classical approach. It implies that some of the additional parameters in the Cosserat model define a length-scale present in the material.

A.C. Eringen (see, for instance, [1]) has complemented the theory by introducing micro-inertia and renamed it subsequently micropolar theory. In static elasticity, Cosserat and micropolar may be used interchangeably.

The so-called indeterminate couple stress model (Koiter-Mindlin model) appears formally by setting the Cosserat couple modulus to infinity and is, in fact, a special higher order gradient continuum where the higher derivatives act only on the continuum rotation. In general, the higher the value of the modulus, the closer is the Cosserat rotation to the continuum (macroscopic) rotation.

The mathematical analysis of linear micropolar models is fairly well established with a wealth of analytical solutions for boundary value problems, existence and uniqueness theorems and continuous dependence results. It is usually based on a uniform positivity assumption on the free energy which sets it apart from linear elasticity in that Korns inequality is traditionally not needed. D. Iesan (see, for instance, [5]), has contributed greatly to this field.

In the paper [3] of Gorbachev, several new integral representations of the solutions to some problems of the moment and nonmoment theories of elasticity for heterogeneous bodies are proposed in terms of the solutions to the same problems for homogeneous bodies. In particular, these integral representations can be used to substantiate the homogenization procedure for composite mechanics problems.

A general solution of the homogeneous static relations of the theory of asymmetric elasticity is constructed in the paper [9] of Olifer. The passage to the solution of the classical (symmetric) theory of elasticity is shown and the form of the general solution for the palne problem is derived. Certain modifications to the general solution of the equations of equilibrium in the theory of elasticity serve as a basis for formulating various different expressions for the Castigliano functional in the stress functions.

In the paper [11] of Yanqi and Zhida, based on the finite deformation decomposition theorem, the definition of the body moment is renewed as the sum of its internal and external. The expression of the increment rate of the deformation energy is derived and the physical meaning is clarified. The power variational principle and the complementary power variational principle for finite deformation mechanics are supplemented and perfected.

In the paper [10] of Tianmin, the equations of motion and all boundary conditions as well as the energy equation for non-local asymmetric elasticity are derived together from the complete principles of virtual work and virtual power as well as the generalized Piola theorem. Adding the boundary conditions presented here to a previous its result, the mixed boundary-value problem of the non-local asymmetric linear elasticity are formulated.

Some basic results regarding microstretch thermoelastic bodies are deduced in our paper [6], by using the Lagrange Identity method.

2. BASIC EQUATIONS

The equations of assymetrical Elasticity, written in displacements and couple, in two-dimensional space, have the following form (see [8], [4]):

$$(1)\qquad (\mu+\alpha)\Delta u_1 + (\lambda+\mu-\alpha)\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right) + 2\alpha\frac{\partial u_3}{\partial x_2} = 0 \quad (f_1)$$

$$A\left(\frac{\partial}{\partial x}\right)U \equiv (\mu+\alpha)\Delta u_2 + (\lambda+\mu-\alpha)\frac{\partial}{\partial x_2}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right) - 2\alpha\frac{\partial u_3}{\partial x_1} = 0 \quad (f_2)$$

$$(\mu+\varepsilon)\Delta u_3 + 2\alpha\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right) - 4\,\alpha\,u_3 = 0 \quad (f_3)$$

Here $A\left(\frac{\partial}{\partial x}\right)U$ contains all three lines of the formula (1). By $U(u_1,u_2,u_3)$ we denote the displacement (u_1,u_2) and the rotation $\omega = u_3$. Also, $f(f_1,f_2,f_3)$ represents the force (f_1,f_2) and the moment f_3.

Consider the operator of the efforts, defined by means of the matrix 3×3-dimensional matrix

$$(2)\qquad T\left(\frac{\partial}{\partial x},n\right)\equiv$$

$$\equiv\begin{pmatrix} (\lambda+2\mu)n_1\frac{\partial}{\partial x_1}+(\mu+\alpha)n_2\frac{\partial}{\partial x_2} & (\mu-\alpha)n_2\frac{\partial}{\partial x_1}+\lambda\,n_1\frac{\partial}{\partial x_2} & 2\alpha n_2 \\ (\mu-\alpha)n_1\frac{\partial}{\partial x_2}+\lambda\,n_2\frac{\partial}{\partial x_1} & (\lambda+2\mu)n_2\frac{\partial}{\partial x_2}+(\mu+\alpha)n_1\frac{\partial}{\partial x_1} & -2\alpha n_1 \\ 0 & 0 & (\gamma+\varepsilon)\frac{\partial}{\partial n} \end{pmatrix}.$$

The corresponding determinant of the matrix

$$A\left(\frac{\partial}{\partial x}\right) = \left[A_{ij}\left(\frac{\partial}{\partial x}\right)\right]$$

can be rewritten in the form

$$\det A\left(\frac{\partial}{\partial x}\right) = (\lambda+2\mu)\Delta^2[(\alpha+\mu)(\gamma+\varepsilon)\Delta - 4\alpha\mu].$$

The matrix

$$E(x) = (E_{ij}(x)) = ({}^*A_{ij}(x))\, F$$

is the matrix of the fundamental solutions of the equation (1) if F satisfies the equation

$$(\lambda+2\mu)\Delta^2[(\alpha+\mu)(\gamma+\varepsilon)\Delta - 4\alpha\mu]F = \delta(x), \tag{3}$$

where δ is the well known Dirac distribution.

3. BASIC RESULTS

From the equation (3) we can write

$$\begin{aligned}
\Delta^2 F &= -\frac{1}{2\pi(\lambda+2\mu)(\lambda+\mu)(\gamma+\varepsilon)}\ K_0(l\varrho)\\
\Delta F &= -\frac{1}{8\pi\ \mu\ \alpha\ (\lambda+2\mu)}\ [K_0(l\varrho)+\ln\varrho]\\
F &= -\frac{(\alpha+\mu)(\gamma+\varepsilon)}{32\mu^2\alpha^2\pi(\lambda+2\mu)}\ [K_0(l\varrho)+\ln\varrho] - \frac{1}{32\pi\ \mu\ \alpha\ (\lambda+2\mu)}\ \varrho^2(\ln\ \varrho - 1).
\end{aligned}$$

In the above relations $K_0(l\varrho)$ represents the modified Bessel's functions of order zero (see [7]). Also, we used the following notations

$$l^2 = \frac{4\alpha\mu}{(\alpha+\mu)(\gamma+\varepsilon)}, \quad \varrho^2 = x_1^2 + x_2^2.$$

Taking into account the matrix $({}^*A_{ij}(x))$, we have

$$\begin{aligned}
{}^*A_{ij}\left(\frac{\partial}{\partial x}\right) &= (\lambda+2\mu)\left[(\gamma+\varepsilon)\Delta^2 - 4\alpha\Delta\right]\delta_{ij} - \\
&\quad - \frac{\partial^2}{\partial x_i\,\partial x_j}\,[(\lambda+\mu-\alpha)(\gamma+\varepsilon)\Delta - 4\alpha(\lambda+\mu)]\\
{}^*A_{ij}\left(\frac{\partial}{\partial x}\right) &= 2\alpha(\lambda+2\mu)\varepsilon_{ijk}\frac{\partial\Delta}{\partial x_k}\\
{}^*A_{33}\left(\frac{\partial}{\partial x}\right) &= (\lambda+2\mu)(\gamma+\varepsilon)\Delta^2, \quad i,j,k = 1,2.
\end{aligned}$$

As a consequence, for the matrix of fundamental solutions E_{ij} we obtain

$$E_{ij}(x) = \left[\frac{\alpha}{2\pi\mu(\alpha+\mu)}K_0 + \frac{\lambda+3\mu}{4\pi\mu(\alpha+2\mu)}\ln\varrho\right]\delta_{ij} - \\ - \frac{\lambda+\mu}{4\pi\mu(\alpha+2\mu)}\frac{x_i x_j}{\varrho^2} - \frac{\gamma+\varepsilon}{8\pi\mu^2}\left[K_0+\ln\varrho\right]x_i x_j, \quad i,j=1,2$$

$$E_{ij}(x) = -\frac{1}{4\pi\mu}\varepsilon_{ijk}\left(K_0+\ln\varrho\right)x_k, \quad k=1,2$$

$$E_{33}(x) = -\frac{1}{2\pi(\gamma+\varepsilon)}\,K_0,$$

where δ_{ij} and ε_{ijk} represent the Kronecker symbol and the Ricci's tensor, respectivelly. The above relations can be rewritten in the following form

$$E_{ij}(x) = \left[\frac{\alpha}{2\pi\mu(\alpha+\mu)}K_0 + \frac{\lambda+3\mu}{4\pi\mu(\alpha+2\mu)}\ln\varrho - \frac{\gamma+\varepsilon}{8\pi\mu^2}\,\frac{1-l\varrho K_1}{\varrho^2}\right]\delta_{ij} - \\ - \frac{\lambda+\mu}{4\pi\mu(\alpha+2\mu)}\,\frac{x_i x_j}{\varrho^2} - \frac{\gamma+\varepsilon}{8\pi\mu^2}\,\frac{l^2\varrho^2K_0+2l\varrho K_1-2}{\varrho^2}\,\frac{x_i x_j}{\varrho^2},\ i,j=1,2$$

$$E_{ij}(x) = -\frac{1}{4\pi\mu}\varepsilon_{ijk}\,\frac{1-l\varrho K_1}{\varrho^2}x_k, \quad k=1,2$$

$$E_{33}(x) = -\frac{1}{2\pi(\gamma+\varepsilon)}K_0.$$

Let us remark that

$$\lim_{\varrho\to 0}\frac{1-l\varrho K_1}{\varrho^2} = \frac{l^2}{2}\lim_{\varrho\to 0}K_0(l\varrho), \quad \lim_{\varrho\to 0}\frac{2-2l\varrho K_1-l^2\varrho^2K_0}{\varrho^2} = 0,$$

and, for $\varrho\to 0$ we have the approximation

$$E_{ij} \approx A\delta_{ij}\ln\varrho + \mathcal{O}(\varrho).$$

Let D be the domain having the boundary $\partial D = \Gamma$. Consider two vector fields U and V which satisfy the properties:

– the partial derivative of these functions of order I are continuous on $\bar{D} = D\cup\Gamma$;

– the partial derivative of these functions of second order are continuous on D.

According to the Green's formula, we have

$$\int_D\left[V.A\left(\frac{\partial}{\partial x}\right)U - U.A\left(\frac{\partial}{\partial x}\right)V\right]\mathrm{d}\omega = \\ = \int_\Gamma\left[U.T\left(\frac{\partial}{\partial x},n\right)V - V.T\left(\frac{\partial}{\partial x},n\right)U\right]\mathrm{d}s. \tag{4}$$

We shall substitute, in (4) the vector field V by

$$V \equiv E_i\left(E_{1i}(x-y),\, E_{2i}(x-y),\, E_{3i}(x-y)\right),$$

such that we deduce

$$u_i(y)=\int_D E_i(x-y)A\left(\frac{\partial}{\partial x}\right)U(x)\mathrm{d}\omega_x-$$
$$-\int_\Gamma\left[U(x)T\left(\frac{\partial}{\partial x},n\right)E_i(x-y)-E_i(x-y)\,T\left(\frac{\partial}{\partial x},n\right)U(x)\right]\mathrm{d}_xS.$$

If we take into account the fact that

$$E^*(y-x)=E(x-y),$$

we obtain

$$U(x)=\int_D E(x-y)A\left(\frac{\partial}{\partial y}\right)U(y)\mathrm{d}\omega_y- \tag{5}$$
$$-\int_\Gamma\left[T\left(\frac{\partial}{\partial y},n\right)E(y-x)\right]^*U(y)\mathrm{d}_yS+\int_\Gamma E(x-y)T\left(\frac{\partial}{\partial y},n\right)U(y)\mathrm{d}_yS.$$

If we use the notation

$$\mathcal{E}(x,y)=\left[T\left(\frac{\partial}{\partial y},n(y)\right)E(y-x)\right]^*$$

then we deduce

$$A\left(\frac{\partial}{\partial x}\right)\mathcal{E}(x,y)=T\left(\frac{\partial}{\partial y},n\right)\left[A\left(\frac{\partial}{\partial y}\right)E(x-y)\right]^*=0.$$

The result from formula (5) is applicable for $x\in D$. If $x\in\Gamma$ then the left part of (5) becomes $1/2U(x)$. If $x\notin\bar{D}$, where $\bar{D}=D\cup\Gamma$, then the left part of (5) becomes zero. In the particular case when in (5) we take $U\equiv u_0=$ constant, then

$$\int_\Gamma\mathcal{E}(x,y)u_0\mathrm{d}_yS=\begin{cases}-u_0+\text{a continuous function}, & \text{for } x\in D,\\ -u_0/2+\text{a continuous function}, & \text{for } x\in\Gamma,\\ \text{a continuous function}, & \text{for } x\notin\bar{D}.\end{cases} \tag{6}$$

The formula (6) suggests us to introduce the following potentials

$$H(x)=\int_D E(x-y)f(y)\mathrm{d}_y\omega, \tag{7}$$

$$V(x)=\int_\Gamma E(x-y)\varphi(y)\mathrm{d}_yS, \tag{8}$$

$$W(x)=\int_\Gamma\mathcal{E}(x,y)h(y)\mathrm{d}_yS, \tag{9}$$

in which we have the following three quantities:

– $H(x)$ is the volume potential and satisfies the following equation

$$A\left(\frac{\partial}{\partial x}\right)H(x)=f(x);$$

– $V(x)$ is the surface potential of a simple layer;

– $W(x)$ is the surface potential of a double layer.

Regarding the surface potential of a double layer, $W(x)$, we have the following result.

THEOREM 1. *If the boundary Γ is a Lyapunov closed curve and h is a function that satifies the Hőlder's condition on Γ, then we have*

$$(10)\qquad \begin{aligned} W_i(x_0)&=-\frac{h(x_0)}{2}+\int_\Gamma \mathcal{E}(x_0,y)h(y)\mathrm{d}_yS,\\ W_e(x_0)&=\frac{h(x_0)}{2}+\int_\Gamma \mathcal{E}(x_0,y)h(y)\mathrm{d}_yS,\end{aligned}$$

where the indices i and e designate the limit value of the potential W by means of points from inside of the domain D and the limit value of the potential W by means of points from outside of the domain D, respectively.

The values $W_i(x_0)$ and $W_e(x_0)$ are calculated by passing to the limit in $x_0\in\Gamma$. Using $(10)_1$ and $(10)_2$ we can write

$$(11)\qquad W(x)=\int_\Gamma \mathcal{E}(x,y)\left[h(y)-h(x_0)\right]\mathrm{d}_yS+\int_\Gamma \mathcal{E}(x,y)h(x_0)\mathrm{d}_yS.$$

First integral in the right side of (11) is a continuous function when $x\to x_0$. As regards the second integral in the right side of (11), its signification can be determined form formula (6).

Regarding the surface potential of a simple layer, $V(x)$, we have the following result.

THEOREM 2. *Suppose the conditions of Theorem* 1 *are satisfied. Then, we have*

$$(12)\qquad \begin{aligned}\left[T\left(\frac{\partial}{\partial x},n\right)V(x_0)\right]_i&=\frac{\varphi(x_0)}{2}+\int_\Gamma\left[T\left(\frac{\partial}{\partial x},n\right)E(x_0-y)\right]\varphi(y)\mathrm{d}_yS,\\ \left[T\left(\frac{\partial}{\partial x},n\right)V(x_0)\right]_e&=-\frac{\varphi(x_0)}{2}+\int_\Gamma\left[T\left(\frac{\partial}{\partial x},n\right)E(x_0-y)\right]\varphi(y)\mathrm{d}_yS.\end{aligned}$$

The relations (10) and (12) give us the possibility to assert/express first and second boundary value problems by means of singular integral equations.

We shall find the solution of first boundary value problem under the form of a surface potential of a double layer and the solution of second boundary value problem under the form of a surface potential of a simple layer. The boundary conditions are taken in the following form

$$U_i(x_0) = g(x_0), \quad x_0 \in \Gamma$$

in the case of first inside problem;

$$U_e(x_0) = g(x_0), \quad x_0 \in \Gamma$$

in the case of first outside problem;

$$\left[T\left(\frac{\partial}{\partial x}, n\right) V(x_0)\right]_i = p(x_0), \quad x_0 \in \Gamma$$

in the case of second inside problem;

$$\left[T\left(\frac{\partial}{\partial x}, n\right) V(x_0)\right]_e = p(x_0), \quad x_0 \in \Gamma$$

in the case of second outside problem.

By virtue of Theorem 1 and Theorem 2, in order to determine the unknown density h, we obtain the following singular integral equations

$$h(x_0) - 2\int_\Gamma \mathcal{E}(x,y)h(y)\, \mathrm{d}_y S = -2g(x_0), \tag{13}$$

for the first inside problem, and,

$$h(x_0) + 2\int_\Gamma \mathcal{E}(x,y)h(y)\, \mathrm{d}_y S = 2g(x_0), \tag{14}$$

for the first outside problem. For the second inside problem, we obtain the following integral equation, having as unknown density the function φ

$$\varphi(x_0) + 2\int_\Gamma \mathcal{E}^*(x,y)\varphi(y)\, \mathrm{d}_y S = 2p(x_0), \tag{15}$$

and, for the second outside problem,

$$\varphi(x_0) - 2\int_\Gamma \mathcal{E}^*(x,y)\varphi(y)\, \mathrm{d}_y S = -2p(x_0). \tag{16}$$

Each of the above equation represents, in fact, a system of three equations.

In what follows we wish to study *the index* of these systems. Let us consider, for instance, the system

$$h(x_0) + 2\int_\Gamma \mathcal{E}(x,y)h(y)\, \mathrm{d}_y S = 2g(x_0).$$

We have

$$2\mathcal{E}(x_0, y) = \tag{17}$$

$$= \frac{1}{\pi}\begin{bmatrix} -an_1\xi_1/\varrho^2 & -bn_2\xi_1/\varrho^2 + cn_1\xi_2/\varrho^2 & 0 \\ -bn_1\xi_2/\varrho^2 + cn_2\xi_1/\varrho^2 & -an_2\xi_2/\varrho^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \bar{K}(x_0, y),$$

where we use the notation

- $\xi = y_i - x_i$;
- $n\,(n_1,\ n_2)$ is the outward unit normal in the point y;

$$-a = \frac{\alpha(\lambda+2\mu)}{3\mu(\alpha+\mu)},\quad b = \frac{3\mu^3+\alpha\lambda^2-\alpha\mu^2}{3\mu(\alpha+\mu)(\lambda+2\mu)},\quad c = \frac{3\mu^3-4\alpha\lambda\mu-10\alpha\mu^2}{3\mu(\alpha+\mu)(\lambda+2\mu)}. \tag{18}$$

Also, in (17) $\bar{K}(x_0, y)$ represents a fixed regular kernel which is a negligible quatity. If we take

$$t - t_0 = \xi_1 + i\,\xi_2 = \varrho e^{i(\theta-\theta_0)}$$

then we have

$$\mathrm{d}\xi_1 = \cos(\theta-\theta_0)\,\mathrm{d}\varrho - \varrho\sin(\theta-\theta_0)\,\mathrm{d}\theta,$$
$$\mathrm{d}\xi_2 = \sin(\theta-\theta_0)\,\mathrm{d}\varrho + \varrho\cos(\theta-\theta_0)\,\mathrm{d}\theta.$$

Also, by direct calculations we obtain

$$n_1 = -\frac{\mathrm{d}\xi_2}{\mathrm{d}_yS},\quad n_2 = \frac{\mathrm{d}\xi_1}{\mathrm{d}_yS},$$
$$\frac{\xi_1n_2-\xi_2n_1}{\varrho^2}\,\mathrm{d}_yS = \frac{\mathrm{d}\varrho}{\varrho} = \frac{\mathrm{d}t}{t-t_0} - i\mathrm{d}\theta,$$
$$\frac{n_2\xi_1}{\varrho^2}\,\mathrm{d}_yS = -\cos^2(\theta-\theta_0)\,\frac{\mathrm{d}\varrho}{\varrho} + \sin(\theta+\theta_0)\cos(\theta-\theta_0)\,\mathrm{d}\theta,$$
$$\frac{n_1\xi_2}{\varrho^2}\,\mathrm{d}_yS = \sin^2(\theta-\theta_0)\,\frac{\mathrm{d}\varrho}{\varrho} + \sin(\theta-\theta_0)\cos(\theta-\theta_0)\,\mathrm{d}\theta,$$
$$\frac{n_1\xi_1}{\varrho^2}\,\mathrm{d}_yS = \sin(\theta-\theta_0)\cos(\theta-\theta_0)\,\frac{\mathrm{d}\varrho}{\varrho} + \cos^2(\theta-\theta_0)\,\mathrm{d}\theta,$$
$$\frac{n_2\xi_2}{\varrho^2}\,\mathrm{d}_yS = -\sin(\theta-\theta_0)\cos(\theta-\theta_0)\,\frac{\mathrm{d}\varrho}{\varrho} + \sin^2(\theta-\theta_0)\,\mathrm{d}\theta.$$

We can transform the equation (14) such that it receives the following form

$$h(t_0) + \frac{1}{\pi}\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}\int_\Gamma \frac{\varphi(t)}{t-t_0}\mathrm{d}t + \int_\Gamma K\,(x_0,\ y)\,\varphi(y)\mathrm{d}_yS = 2g(x_0), \tag{19}$$

where $K\,(x_0,\ y)$ is the regularized kernel.

Following [2] and [7], the index can be determined by means of formula

$$\chi = \frac{1}{2\pi}\left[\arg\frac{\det(A-B)}{\det(A+B)}\right]_{\Gamma}.$$

We have

$$\det(A-B) = \det(A+B) = \det\begin{bmatrix} 1 & -ib \\ -ic & 1 \end{bmatrix} = 1 + bc.$$

But, taking into account the notations (18) we deduce

$$1 + bc = \frac{9\mu(\alpha+\mu)^2(\lambda+2\mu)^2 + \left(3\mu^2+\alpha\lambda^2-\alpha\mu^2\right)\left(4\alpha\lambda+5\alpha\mu-3\mu^2\right)}{9\mu(\alpha+\mu)^2(\lambda+2\mu)^2}.$$

After simple calculations on the numerator of the fraction in the right side of the above relation, we find

$$9\mu(\alpha+\mu)^2(\lambda+2\mu)^2 + \left(3\mu^2+\alpha\lambda^2-\alpha\mu^2\right)\left(4\alpha\lambda+5\alpha\mu-3\mu^2\right) =$$
$$= 3\mu^5 + 42\alpha\mu^4 + \frac{34}{3}\mu\alpha^2\lambda^2 + \frac{29}{3}\alpha^2\mu^3 + 15\alpha\lambda^2\mu^2 + 9\mu^3\lambda^2 +$$
$$+(3\lambda+2\mu)\left(24\alpha\mu^3 + 12\mu^4 + \frac{4}{3}\alpha^2\lambda^2 + \frac{32}{3}\mu^2\alpha^2\right).$$

Taking into account the conditions

$$3\lambda + 2\mu > 0,\ \alpha > 0,\ \mu > 0$$

we obtain

$$\det(A-B) \neq 0.$$

4. CONCLUSION

We conclude that the index of the system (14) becomes zero. As a consequence, for the system (14) the Fredholm's Theorems is still valid. Analagous considerations we can make with regard to the systems (13), (15) and (16).

5. ANNEX

We wish to outline some notions regarding the index of a function and the index of a singular equation.

Let us consider Γ a closed and smooth curve and $G(\xi)$ a continuous function on the curve Γ.

It is called **the index** *of the function $G(\xi)$ on the curve Γ, the variation of the argument of $G(\xi)$, when the curve Γ is being covered twice in direct way.*

$$\chi = \frac{1}{\pi}\,\mathrm{var}_{\xi\in\Gamma}\arg G(\xi).$$

It is known that

$$\ln G(\xi) = \ln |G(\xi)| + i \arg G(\xi).$$

If ξ goes allong the closed curve Γ, then the quantity is added to its initial value, such that,

$$\operatorname{var}_{\xi\in\Gamma} \ln |G(\xi)| = 0$$

such that we can write the index in the form

$$\chi = \frac{1}{\pi} \operatorname{var}_{\xi\in\Gamma} \ln G(\xi).$$

If the function $G(\xi)$ is of the class C^1 on Γ, then we can represent the index of $G(\xi)$ in the form

$$\chi = \frac{1}{\pi}\int_\Gamma \mathrm{d}\left[\arg G(\xi)\right] = \frac{1}{\pi}\int_\Gamma \mathrm{d}\left[\ln G(\xi)\right]. \tag{20}$$

Taking into account the fact that $G(\xi)$ is a continuous function, we deduce that the variation of the argument of $G(\xi)$, when the curve Γ is covered one time, must be a multiple of the number 2π. As a consequence, the index χ of a continuous function, on a closed curve Γ ($G(\xi) \neq 0$, $\forall\xi \in \Gamma$) is an integer number or zero.

From the definition of the index, it is easy to prove that:

– the index of a product of two functions is equal to the sum of the index of the factors

$$\chi(f.g) = \chi(f) + \chi(g);$$

– the index of a quotient of two functions is equal to the difference of the index of the factors

$$\chi\left(\frac{f}{g}\right) = \chi(f) - \chi(g).$$

If G is a holomorphyc function on the inside domain of the path Γ, it is continuous on Γ and $G(\xi) \neq 0$, $\forall\xi \in \Gamma$, then the index of G is equal to the difference betwen the number of zeroes and the number of poles of the function G inside of the path Γ.

Let us consider the singular integral equation

$$a(z)\varphi(z) + \frac{b(z)}{i\pi}\int_\Gamma \frac{\varphi(\xi)}{\xi - z}\,\mathrm{d}\xi = f(z), \quad z \in C. \tag{21}$$

It is called **the index of the singular integral equation** (21)*, the index of the function*

$$\frac{a(z) - b(z)}{a(z) + b(z)}.$$

Taking into account the expression (20) of the index of a function, we obtain the following expression of the index of a singular integral equation

$$\chi = \frac{1}{2\pi} \int_\Gamma \mathrm{d} \left[\arg \frac{a(\xi) - b(\xi)}{a(\xi) + b(\xi)} \right].$$

The adjunct equation of the singular integral equation (21) is

$$a(z)\psi(z) - \frac{1}{i\pi} \int_\Gamma \frac{b(\xi)\psi(\xi)}{\xi - z} \mathrm{d}\xi = h(z), \quad z \in C \tag{22}$$

and it is obtained by permutation of the variables ξ and z in the kernel of the equation (21). It is easy to prove that the index χ^* of the singular integral equation (22) is

$$\chi^* = -\chi.$$

It is well known that, for the singular integral equations, instead of the Fredholm's Theorems, are still valid Theorems of F. Noether. But, for the singular integral equation, having a null index, the Fredholm's Theorems are valid.

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