# ON STAR PARTITION DIMENSION OF TREES

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For a connected graph $G$ and any two vertices $u$ and $v$ in $G$, let $d(u, v)$ denote the distance between $u$ and $v$. For a subset $S$ of $V(G)$, the distance between $v$ and $S$ is $d(v, S) = \min\{d(v, x) \mid x \in S\}$. Let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be an ordered $k$-partition of $V(G)$. The representation of $v$ with respect to $\Pi$ is the $k$-vector $r(v \mid \Pi) = (d(v, S_1), d(v, S_2), \ldots d(v, S_k))$. $\Pi$ is a resolving partition for $G$ if the $k$-vectors $r(v \mid \Pi)$, $v \in V(G)$ are distinct. The minimum $k$ for which there is a resolving $k$-partition of $V(G)$ is the partition dimension of $G$, and is denoted by $pd(G)$. $\Pi = \{S_1, S_2, \ldots, S_k\}$ is a resolving star $k$-partition for $G$ if it is a resolving partition and each subgraph induced by $S_i$, $1 \leq i \leq k$, is a star. The minimum $k$ for which there exists a star resolving $k$-partition of $V(G)$ is the star partition dimension of $G$, denoted by $spd(G)$. In this paper star partition dimension of trees and the existence of graphs with given star partition, partition and metric dimension, respectively are studied.

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## 1. INTRODUCTION

As described in [2] and [7], dividing the vertex set of a graph into classes according to some prescribed rule is a fundamental process in graph theory. Perhaps the best known example of this process is graph coloring. In [3], the vertices of a connected graph are represented by other criterion, namely through partitions of vertex set and distances between each vertex and the subsets in the partition. Thus a new concept is introduced – resolving partition for a graph.

Let $G$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For any two vertices $u$ and $v$ in $G$, let $d(u, v)$ be the distance between $u$ and $v$. The diameter of $G$, denoted by $d(G)$ is the greatest distance between any two vertices of $G$. For a subset $S$ of $V(G)$ and a vertex $v$ of $G$, the distance $d(v, S)$ between $v$ and $S$ is defined as $d(v, S) = \min\{d(v, x) \mid x \in S\}$.

For an ordered $k$-partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ and a vertex $v$ of $G$, the *representation of* $v$ *with respect to* $\Pi$ is the $k$-vector

$$r(v \mid \Pi) = (d(v, S_1), d(v, S_2), \ldots d(v, S_k)).$$

$\Pi$ is called a *resolving k-partition* for $G$ if the $k$-vectors $r(v \mid \Pi)$, $v \in V(G)$ are distinct. The minimum $k$ for which there is a resolving $k$-partition of $V(G)$ is the *partition dimension* of $G$ and is denoted by $pd(G)$. A resolving partition of $V(G)$ containing $pd(G)$ classes is called a *minimum resolving partition.*

In [6] a particular case of resolving partitions is considered – connected resolving partitions. $\Pi = \{S_1, S_2, \ldots, S_k\}$ is a *connected resolving k-partition* if it is a resolving partition and each subgraph induced by class $S_i$, denoted $\langle S_i \rangle$, $1 \leq i \leq k$, is connected in $G$. The minimum $k$ for which there is a connected resolving $k$ partition of $V(G)$ is the *connected partition dimension* of $G$, denoted by $cpd(G)$.

Another type of resolving partitions, mentioned in [6] as topic for study, is resolving star partitions. $\Pi = \{S_1, S_2, \ldots, S_k\}$ is a *resolving star k-partition* if it is a resolving partition and each subgraph induced by $S_i$, for $1 \leq i \leq k$, is a star. The minimum $k$ for which there exists a resolving star $k$-partition of $V(G)$ is the *star partition dimension* of $G$, denoted by $spd(G)$. A resolving star partition of $V(G)$ containing $spd(G)$ classes is called a *minimum resolving star partition.*

Partition dimension of a graph is related to an older type of dimension of a graph, introduced by Slater in [10] and later in [9], and independently by Harary and Melter in [4] – metric dimension of a graph.

For an ordered set $W = \{w_1, w_2, \ldots w_k\}$ of vertices of $G$ and a vertex $v \in V(G)$, the *metric representation* of $v$ with respect to $W$ is the $k$-vector $r(v \mid W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. If all vertices of $G$ have distinct representations $W$ is called a *resolving set* for $G$. A resolving set containing a minimum number of vertices is called a *minimum resolving set* or a *basis* for $G$. The number of vertices in a basis for $G$ is the *metric dimension* of $G$ and is denoted by $\dim(G)$.

If $\Pi = \{S_1, S_2, \ldots, S_k\}$ is a partition of $V(G)$ and $u_1, u_2, \ldots, u_r$ are $r$ distinct vertices, we say that $u_1, u_2, \ldots, u_r$ are separated by classes $S_{i_1}, \ldots, S_{i_q}$ of partition $\Pi$ if the $q$-vectors $(d(u_p, S_{i_1}), d(u_p, S_{i_2}), \ldots, d(u_p, S_{i_q}))$, $1 \leq p \leq r$ are distinct.

A partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$ is an *induced-star partition* of $G$ if each subgraph induced by $S_i$, $1 \leq i \leq k$, is a star. Hence, a resolving star partition is an induced-star partition which is also a resolving partition. We call the minimum cardinality of an induced-star partition of $G$ the *induced-star number* of $G$ and we denote it by $sp(G)$. The idea of partitioning graphs into subgraphs belonging to a given family of graphs have been studied by many authors, results on star partitions or star packing can be found in [5], [8]. Next, we will use the term *star partition* for a tree $T$ instead of induced-star partition, since any subgraph of $T$ isomorphic to a star is an induced star in $T$.

In this paper we will study star partition dimension of trees and the existence of graphs with given star partition dimension.

## 2. STAR PARTITION DIMENSION OF TREES

First we remind some notions and notations from [6].

Let $G$ be a connected graph. A vertex of degree at least 3 of $G$ is called *major vertex* of $G$. A vertex $u$ of degree one of $G$ is called a *terminal vertex of a major vertex* $v$ of $G$ if $d(u,v) < d(u,w)$, for every major vertex $w \neq v$ of $G$ ($v$ is the closest major vertex to $u$). The *terminal degree* of a major vertex $v$, denoted by $ter_T(v)$ or, if $T$ is known, by $ter(v)$, is the number of terminal vertices of $v$. A major vertex $v$ with $ter(v) > 0$ is said to be an *exterior major vertex* of $G$. We will call an exterior major vertex $v$ with $ter(v) > 1$ a *branched major vertex* of $G$.

We denote by $\sigma(G)$ the sum of terminal degrees of the major vertices of $G$, by $\sigma_b(G)$ the sum of terminal degrees of the branched major vertices of $G$, by $ex(G)$ the number of exterior major vertices of $G$ and by $ex_b(G)$ the number of branched major vertices of $G$.

Let $T$ be a tree.

A star $S$ in $T$ is called *maximal star* if $\langle V(S) \cup \{v\} \rangle$ is not a star in $T$ for every $v \in V(T) - V(S)$. An *$n$-star* is a star with $n$ vertices.

For an exterior major vertex $v$ of $T$ a path $Q$ to one of its terminal vertices $u$ is called *terminal path* for vertex $v$. The maximal induced star of $Q$ that contains the terminal vertex $u$ is called *terminal star* of the exterior major vertex $v$. Obviously, the terminal star is isomorphic to $P_3$ if the terminal path has at least 3 vertices or is isomorphic to the terminal path otherwise.

We can extend the notion of terminal stars to paths. A *terminal star* of a path $P_n$ is a maximal induced star in $P_n$ that contains one of the extremity of the path.

Let $p = ex_b(T)$ be the number of branched major vertices of $T$. We denote by $B(T) = \{v_1, v_2, \ldots, v_p\}$ the set of branched major vertices of $T$. For $1 \leq i \leq p$ we denote by $k_i = ter(v_i)$ and by $u_{i1}, u_{i2}, \ldots, u_{ik_i}$ the terminal vertices of $v_i$, by $P_{ij}$ the path from $v_i$ to $u_{ij}$, for $1 \leq j \leq k_i$, with $x_{ij}$ the vertex of $P_{ij}$ adjacent with $v_i$ and by $Q_{ij}$ the subpath of $P_{ij}$ from $x_{ij}$ to $u_{ij}$ (i.e., $Q_{ij} = P_{ij} - v_i$). For a branched major vertex $v_i$, $1 \leq i \leq p$, the paths $Q_{ij}$, $1 \leq j \leq k_i$ are the terminal paths of vertex $v_i$.

For a branched major vertex $v_i$, $1 \leq i \leq p$, we denote by $S_{ij}^t$ the terminal star from the path $Q_{ij}$, $1 \leq j \leq ter(v_i)$.

For example, the tree from Figure 1 has 3 exterior major vertices: $v_1$, $u$ and $v_2$. Only $v_1$ and $v_2$ are branched major vertices: $ter(v_1) = ter(v_2) = 2$. The terminal path for $v_1$ are $Q_{11}$ and $Q_{12}$, for $v_2$ are $x_{21}, u_{21}$ and $x_{22}$ and for

$u$ is $u'$. Terminal stars for $v_1$ are also illustrated. For $u$ and $v_2$ the terminal path are also terminal stars.

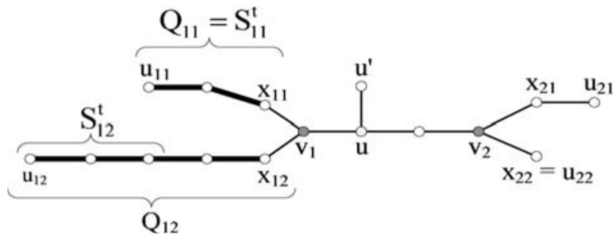


Fig. 1. $T$

Next we assume that terminal paths of a branched major vertex $v_i$ are indexed such that

$$|S^t_{i1}| \leq |S^t_{i2}| \leq \ldots \leq |S^t_{ik_i}|.$$

Lemma 2.1. *For a tree $T$ and a terminal vertex $x$ of $T$ we have*

$$sp(T - x) \leq sp(T).$$

*Proof.* Let $y$ be the only vertex adjacent to $x$ in $T$. Let $\Pi = (S_1, \ldots, S_q)$ be a minimum star partition of $T$ such that $x \in S_1$, where $q = sp(T)$ is the induced star number of $T$.

If $S_1 = \{x\}$ then partition $\Pi' = (S_2, \ldots, S_q)$ is a star partition in $T - x$ with $q - 1$ classes. Otherwise partition $\Pi' = (S_1 - \{x\}, S_2, \ldots, S_q)$ is a star partition in $T - x$ with $q$ classes.

Hence $sp(T - x) \leq sp(T)$. $\square$

Lemma 2.2. *Let $T$ be a tree and $v$ a vertex of $T$ adjacent with terminal vertices $u_1, u_2, \ldots, u_t$, $t \geq 1$. Then there exists a minimum star partition of $T$ such that vertices $v, u_1, u_2, \ldots, u_t$ belong to the same class of the partition.*

*Proof.* Let $\Pi$ be a minimum star partition of $T$. Denote by $S_1$ the class to which vertex $v$ belongs. A terminal vertex $u_i$, $1 \leq i \leq t$ either forms a separate class in $\Pi$ or belongs to class $S_1$.

If $|S_1| \leq 2$ or $v$ is center of star $\langle S_1 \rangle$, since $\Pi$ has minimum number of classes, it follows that $u_1, u_2, \ldots, u_t \in S_1$.

If $|S_1| > 2$ and $v$ is terminal vertex in $\langle S_1 \rangle$, then each vertex $u_i$, $1 \leq i \leq t$, forms a separate class in $\Pi$. Then partition

$$\Pi' = \Pi - \{S_1\} - \bigcup_{i=1}^{t} \{\{u_i\}\} \cup \{S_1 - \{v\}\} \cup \{\{v, u_1, \ldots, u_t\}\}$$

is a star partition of $T$ with $|\Pi| - t + 1$ classes. Since $\Pi$ is a minimum star partition of $T$, it follows that $t = 1$ and $\Pi'$ is a minimum star partition of $T$ such that vertices of $v, u_1, u_2, \ldots, u_t$ belong to the same class of the partition. $\square$

Lemma 2.3. *Let $T$ be a tree and $S$ a star in $T$ with center $v$ and terminal vertices $u_1, \ldots, u_t$, such that $\deg_T(u_i) = 1$, for every $1 \leq i \leq t-1$ and $\deg_T(u_t) = 2$. Then there exists a minimum star partition of $T$ such that vertices of $S$ are in the same class.*

*Proof.* By Lemma 2.2, there exists a minimum star partition $\Pi$ of $T$ such that vertices $v, u_1, \ldots, u_{t-1}$ are in the same class, denoted by $S_1$. If $u_t \in S_1$ then in partition $\Pi$ vertices of $S$ belong to the same class. Otherwise, let $S_2$ be the class of $\Pi$ to which $u_t$ belongs. Since $\deg_T(u_t) = 2$, $u_t$ is terminal in $\langle S_2 \rangle$. Then partition

$$\Pi' = \Pi - \{S_1, S_2\} \cup \{S_1 \cup \{u_t\}, S_2 - \{u_t\}\}$$

is also a minimum star partition in $T$, and vertices of $S$ belong to the same class of $\Pi'$. $\square$

Lemma 2.4. *Let $T$ be a tree and $v$ a vertex of $T$. Denote by $C_1, \ldots, C_t$ the components of the forest $T - v$. If $|V(C_i)| \geq 2$ and vertices $V(C_i) \cup \{v\}$ induce a star in $T$ for every $1 \leq i \leq t-1$, then there exists a minimum star partition of $T$ such that vertex set $V(C_1) \cup \{v\}$ is a class of the partition.*

*Proof.* Since $|V(C_i)| \geq 2$, $\langle V(C_i) \cup \{v\} \rangle$ is a maximal star in $T$, for every $1 \leq i \leq t-1$.

By Lemma 2.3, there exists a minimum star partition $\Pi$ of $T$ such that vertices from $V(C_i)$ belong to the same class of the partition, for every $1 \leq i \leq t-1$.

Let $S_1$ be the class of $\Pi$ to which $v$ belongs.

If there exists a component $C_i$, $1 \leq i \leq t-1$ such that $V(C_i) \subseteq S_1$, then, since $\langle V(C_i) \cup \{v\} \rangle$ is a maximal star, $S_1 = V(C_i) \cup \{v\}$ and each set $V(C_k)$, $1 \leq k \leq t-1$, $k \neq i$ is a class in $\Pi$. We can assume $i = 1$ and the result follows.

If $S_1 - \{v\} \subseteq V(C_t)$, then $v$ is a terminal vertex of the induced star $\langle S_1 \rangle$ and each set $V(C_i)$, $1 \leq 1 \leq t-1$ is a class in $\Pi$. Since $V(C_1) \cup \{v\}$ induces a star in $T$, the partition

$$\Pi' = \Pi - \{S_1, V(C_1)\} \cup \{S_1 - \{v\}, V(C_1) \cup \{v\}\}$$

is a minimum star partition having as class $V(C_1) \cup \{v\}$. $\square$

Lemma 2.5. *Let $T$ be a tree which is not isomorphic to a path and $v$ a branched major vertex of $T$. If $S_1^t$ and $S_2^t$ are terminal stars for $v$ such that*

$|S_1^t| \geq |S_2^t|$, *then*

$$sp(T - S_1^t) \leq sp(T - S_2^t).$$

*Proof.* If $|S_1^t| = |S_2^t|$ then $T - S_1^t$ is isomorphic to $T - S_2^t$, hence $sp(T - S_1^t) = sp(T - S_2^t)$.

Assume $|S_1^t| > |S_2^t|$. By Lemma 2.3, there exists a minimum star partition of $T$ such that vertices of the star $S_2^t$ belong to the same class of the partition. It follows that

$$sp(T - S_2^t) \geq sp(T) - 1.$$

If $|S_1^t| = 3$, by Lemma 2.4, there exists a minimum star partition of $T$ such that $V(S_1^t)$ is a class of the partition, hence in this case

$$sp(T - S_1^t) = sp(T) - 1 \leq sp(T - S_2^t).$$

If $|S_1^t| = 2$, then $|S_2^t| = 1$ and by Lemma 2.3 there exists a minimum star partition $\Pi$ of $T$ such that the only vertex of the star $S_2^t$ and vertex $v$ are in the same class. Since $\Pi$ has a minimum number of classes and star $S_1^t$ can be extended only through vertex $v$, it follows that the set $V(S_1^t)$ is a class of $\Pi$, hence in this case we also have

$$sp(T - S_1^t) = sp(T) - 1 \leq sp(T - S_2^t). \quad \square$$

We denote by $T_{j_1 \ldots j_p}$, where $1 \leq j_i \leq k_i$, for every $1 \leq i \leq p$ the tree obtained from $T$ by removing all its terminal stars excepting $S_{ij_i}^t$, $1 \leq i \leq p$.

Theorem 2.6. *For $n \geq 1$ we have*

$$sp(P_n) = spd(P_n) = \left\lceil \frac{n}{3} \right\rceil.$$

*Proof.* Since an induced-star of a path can have at most 3 vertices, the result follows. $\square$

Theorem 2.7. *For a tree $T$ which is not isomorphic to a path we have*

$$spd(T) = \sigma_b(T) - ex_b(T) + sp(T_{\underbrace{1 \ldots 1}_{p}}).$$

*Proof.* Let $T$ be a tree which is not isomorphic to a path.

Since a terminal path $Q_{ij}$ can be extended only through vertex $v_i$, vertices from a set $V(Q_{ij})$, $1 \leq i \leq p$, $1 \leq j \leq k_i$ have distinct distances to any other fixed vertex of $T$. Moreover, with the above notation, any two distinct vertices $x_{ij_1}$ and $x_{ij_2}$, adjacent to vertex $v_i$ have equal distances to any fixed vertex from $V(T) - (V(Q_{ij_1}) \cup V(Q_{ij_2}))$.

Hence, if $\Pi$ is a minimum resolving star partition, there exists an induced star $S_{ij}$ in each terminal path $Q_{ij}$ such that $V(S_{ij})$ is a class of $\Pi$, with at most one exception for every branched major vertex $v_i$, $1 \leq i \leq p$. Since $u_{ij}$ is a terminal vertex, we can assume the induced star $S_{ij}$ contains $u_{ij}$. We have

$$spd(T) \geq \sum_{v \in B(T)} (ter(v) - 1) + \min \Bigg\{ sp\Bigg( T - \bigcup_{i=1}^{p} \bigcup_{\substack{j=1 \\ j \neq j_i}}^{k_i} S_{ij} \Bigg) \mid \text{for every } 1 \leq i \leq p,\ j_i \in \{1, \ldots, k_i\},\ S_{ij} \text{ is an induced star in } Q_{ij} \text{ such that } u_{ij} \in S_{ij}, \text{ for } 1 \leq j \leq k_i,\ j \neq j_i \Bigg\}.$$

Denote by $m$ the minimum from the above formula.

By Lemmas 2.1 and 2.5, it follows that the minimum $m$ is obtained by removing terminal stars with maximum cardinalities, and since

$$|S^t_{i1}| \leq |S^t_{i1}| \ldots \leq |S^t_{ik_i}|, \text{ for every } 1 \leq i \leq p$$

we have

$$m = \min \Bigg\{ sp\Bigg( T - \bigcup_{i=1}^{p} \bigcup_{\substack{j=1 \\ j \neq j_i}}^{k_i} S^t_{ij} \Bigg) \mid j_i \in \{1, \ldots, k_i\},\ 1 \leq i \leq p \Bigg\} =$$

$$= sp\Bigg( T - \bigcup_{i=1}^{p} \bigcup_{j=2}^{k_i} S^t_{ij} \Bigg) = sp(T_{\underbrace{1 \ldots 1}_{p}}).$$

Hence

$$spd(T) \geq \sum_{v \in B(T)} (ter(v) - 1) + sp(T_{\underbrace{1 \ldots 1}_{p}}) = \sigma_b(T) - ex_b(T) + sp(T_{\underbrace{1 \ldots 1}_{p}}).$$

Let $\Pi_1$ be a minimum star partition in $T_{\underbrace{1 \ldots 1}_{p}}$ and

$$\Pi = \bigcup_{i=1}^{p} \bigcup_{j=2}^{k_i} \{V(S^t_{ij})\} \cup \Pi_1 \tag{1}$$

a star partition in $T$. $\Pi$ has $\sigma_b(T) - ex_b(T) + sp(T_{\underbrace{1 \ldots 1}_{p}})$ classes. We will prove that $\Pi$ is a resolving partition.

Indeed, vertices from a class $V(S^t_{ij})$, $1 \leq i \leq p$, $1 \leq j \leq k_i$ have distinct distances to all other vertices.

Let $S$ be an induced star in $T_{\underbrace{1 \ldots 1}_{p}}$, $|V(S)| \geq 2$.

If $V(S) = \{x, y\}$ then, since $T$ is not a path, at least one of vertices $x$ and $y$, assume $x$, is not terminal in $T$. Then there exists a unique path $P$ from $x$ to

$u_{12}$. If $y \in V(P)$ then $d(y, V(S^t_{12})) = d(x, V(S^t_{12})) - 1$, otherwise $d(y, V(S^t_{12})) = d(x, V(S^t_{12})) + 1$, hence $x$ and $y$ are separated by the class $V(S^t_{12})$.

If $|V(S)| \geq 3$, denote by $c$ the center of the star $S$. Let $x$, $y$ be two terminal vertices of $S$.

If both vertices $x$ and $y$ belong to terminal paths in $T$, then $c$ is a branched major vertex. Assume $c = v_{i_0}$, $1 \leq i_0 \leq p$ and $x \in V(Q_{i_0 j_0})$, $2 \leq j_0 \leq ter(v_{i_0})$. Vertices $x, c, y$ are separated by the class $V(S^t_{i_0 j_0})$, since $d(y, S^t_{i_0 j_0}) = d(c, S^t_{i_0 j_0}) + 1$ and $d(c, S^t_{i_0 j_0}) = d(x, S^t_{i_0 j_0}) + 1$.

If $x$ does not belong to a terminal path of $T$, then there exists a path $P$ from $x$ to a branched major vertex $v_{i_1}$, $1 \leq i_1 \leq p$, such that $y \notin V(P)$. Vertices $x, c, y$ are separated by the class $V(S^t_{i_1 2})$.

Hence all vertices of a star $S$ in $T_{\underbrace{1\ldots1}_{p}}$ are separated by classes of $\Pi$.

It follows that $\Pi$ is a resolving partition and we have

$$spd(T) = \sigma_b(T) - ex_b(T) + sp(T_{\underbrace{1\ldots1}_{p}}). \quad \square$$

Next, we present a linear time algorithm for finding a minimum resolving star partition for a tree.

The problem of finding a minimum resolving star partition for a tree $T$ is reduced by Theorem 2.7 to the problem of finding a minimum star partition for one particular subtree of $T$. Therefore, we will first propose a linear time algorithm for building a minimum star partition $\Pi$ for a tree $T$ with $n$ vertices. The algorithm is based on Lemmas 2.3 and 2.4.

Thus, at one step of the algorithm it suffices to consider only terminal stars in the current tree $T$.

A terminal 3-star is a class in the partition $\Pi$ build by the algorithm (according to Lemma 2.3).

Moreover, if $v$ is a branched major vertex such that all the components $C_1, \ldots, C_t$ in $T - v$ with at most one exception, $C_t$, are terminal stars in $T$, we can build classes of the partition according to Lemmas 2.3 and 2.4 as follows. Let $y$ be the vertex from $C_t$ adjacent to $v$ and $S \in \{C_1, \ldots, C_{t-1}\}$ be a terminal star. We have the following cases:

(1) If $S$ is a 3-star, then vertices of $S$ form a distinct class in $\Pi$.

(2) If $S$ is a 1-star, then $v$ is in the same class with all its terminal 1-stars. To this class only vertex $y$ can be eventually added, if the algorithm does not place it in another class after partitioning $\deg_T(y) - 2$ components of $T - y$.

(3) If $S$ is a 2-star, then vertices from $S$ are placed in the same class of $\Pi$. To this class only vertex $v$ can be added, if it has no terminal 1-stars.

Next, we will call the number of terminal stars adjacent to a branched major vertex $v$ the *terminal star degree* of $v$ and we will denote it by $ter_s(v)$.

The algorithm will associate each vertex a color, such that vertices with the same color are in the same class. The color of a vertex can be changed only if a terminal 1-star of this vertex is found. Also, a vertex adjacent with a terminal 1-star (case (2)) is marked by algorithm, since in this class all terminal 1-stars adjacent of $v$ should be added.

Denote by $deg$ the vector of degrees, $ters$ the vector of terminal star degrees, $color$ the vector with the colors of each vertex and $mark$ a vector for marking the vertices that have terminal 1-stars.

We assume the tree is represented using adjacency lists.
Then the algorithms has the following steps:

1. Initialize

$$ters[u] := 0,\ color[u] := 0,\ mark[u] := 0 \text{ for every } u \in V(T);$$

2. Calculate the degree $deg[u]$ for every $u \in V(T)$.
3. if $|V(T)| \leq 3$ then
   let $c$ be a new color;
   color vertices of $T$ with $c$;
   form classes of partition $\Pi$ according to colors and STOP.
4. Build all terminal stars for $T$ and update the terminal star degrees for vertices.
5. Repeat steps 6-9 while $|V(T)| \geq 3$
   6. for each terminal 3-star $S$ do
      let $c$ be a new color;
      $color[s] := c$ for every $s \in V(S)$;
      $T := T - S$;
      let $v$ be the vertex from $T$ that was adjacent to a vertex of $S$;
      $ters[v] := ters[v] - 1$;
   7. for each vertex $v$ with $deg[v] \leq ter[v] + 1$ (equvalent to $deg[v] = ters[v] + 1$ or $deg[v] = ters[v]$)
      for each terminal star $S$ of $v$ do
         if $|V(S)| = 2$ then
            let $c$ be a new color;
            $color[s] := c$ for every $s \in V(S)$;
            if $color[v] = 0$ then
               $color[v] := c$;
         if $|V(S)| = 1$ then
            if $mark[v] = 1$ then
               $color[s] := color[v]$ for every $s \in V(S)$;
            else
               $mark[v] := 1$

let $c$ be a new color;
$color[v] := c$;
$color[s] := c$ for every $s \in V(S)$;
$T := T - S$;
$ters[v] := ters[v] - 1$;
if $deg[v] = 1$ and $color[v] \neq 0$ then
if $mark[v] = 1$ then
let $y$ be the vertex from $T$ adjacent to $v$;
if $color[y] = 0$ then
$color[y] := color[v]$;
$T := T - v$;
8. extend the terminal stars of vertices that are no longer exterior in the current tree $T$ to terminal stars in $T$.
9. for every new terminal vertex $u$ in $T$ obtained at step 7 do
if $color[u] \neq 0$ then
$T := T - u$;
else
build the terminal star that contains $u$
10. if $T$ has uncolored vertices then
let $c$ be a new color;
color vertices of $T$ previous uncolored with $c$;
form classes of partition $\Pi$ according to colors and STOP.

Obviously, the above algorithm has the complexity $O(n)$, since the operations of building or removing a terminal star have the complexity equal to the number of edges of the star and the stars considered are edge-disjoint.
Then a linear time algorithm for finding a minimum resolving star partition $\Pi$ of a tree $T$ is the following:

1. If $T$ is a path then find a minimum star partition $\Pi$ with the previous algorithm and STOP.

2. Find all terminal stars of branched major vertices of $T$ and form classes according to relation (1). Remove from $T$ the terminal stars added as classes to the partition.

3. Find a minimum star partition $\Pi_1$ for the remaining tree and add its classes to $\Pi$. STOP.

## 3. EXISTENCE OF GRAPHS WITH GIVEN STAR PARTITION DIMENSION

THEOREM 3.1. a) *For any two integers $a$ and $b$ such that $3 \leq a \leq b$ there exists a connected graph $G$ such that $pd(G) = a$ and $spd(G) = b$.*

b) *For any two integers* $a$ *and* $b$ *such that* $3 \leq a \leq b$ *there exists a connected graph* $G$ *such that* $\dim(G) = a$ *and* $spd(G) = b$.

*Proof.*

**Case 1.** $a < b$.

a) Denote $n = 3(b-a)+2$. Let $G$ be the graph obtained from path $P_n$ by attaching $a$ new terminal vertices $x_1, \dots, x_a$ to one of the two terminal vertices of the path, denoted by $y$. Let $z$ be the vertex from initial path $P_n$ adjacent to $y$ (Fig. 2).

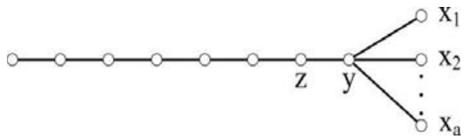


Fig. 2

Vertices $x_1, \dots, x_a$ have equal distances to any other vertex of $G$, hence they belong to different classes in a resolving partition of $G$. It follows that

$$pd(G) \geq a.$$

Moreover, partition with classes $\{x_1, y, z\}$, $\{x_2\} \cup V(P_n - \{y, z\})$, $\{x_i\}$, $3 \leq i \leq a$ is a resolving partition, hence $pd(G) = a$.

By Theorems 2.7 and 2.6, since $y$ is the only exterior major vertex in $G$ and $ter(y) = a+1$ we have

$$spd(G) = a + 1 - 1 + sp(P_{n-3} + yx_1) = a + sp(P_{3(b-a)}) = a + b - a = b.$$

By [1], for a tree $T$ which is not isomorphic to a path we have

$$\dim(T) = \sigma(T) - ex(T),$$

hence

$$\dim(T) = a + 1 - 1 = a.$$

**Case 2.** $a = b$.

a) Let $G$ be the star $S_a$. By Theorem 2.7 and [3] we have

$$pd(G) = spd(G) = a.$$

b) Let $G$ be the graph obtained from the cycle with 4 vertices, denoted by $x, y, w, z$, by attaching $a-1$ terminal vertices $x_1, \dots, x_{a-1}$ to $x$ (Fig. 3).

Since vertices $x_1, \dots, x_{a-1}$ have equal distances to any other vertex of $G$, it follows that a basis of $G$ must contain all these vertices with at most one exception, say $x_{a-1}$, and vertices $x_1, \dots, x_{a-1}$ belong to different classes in a resolving partition of $G$. Moreover, vertices $y$ and $z$ have equal distances to vertices $x_1, \dots, x_{a-2}$, hence a basis of $G$ must also contain at least one

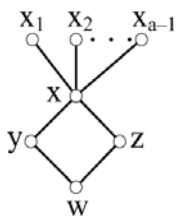


Fig. 3

of vertices $y$, $z$. But vertices $z$ and $x_{a-1}$ have the same distances to vertices $x_1, \ldots, x_{a-2}, y$ and, by symmetry, vertices $y$ and $x_{a-1}$ have the same distances to vertices $x_1, \ldots, x_{a-2}, z$. It follows that

$$\dim(G) \geq a.$$

It is easy to verify that $\{x_1, \ldots, x_{a-2}, z, y\}$ is a resolving set in $G$, hence

$$\dim(G) = a.$$

The minimum number of stars in which $V(G)$ can be partitioned such that $x_1, \ldots, x_{a-1}$ belong to different stars is $a$. Also, the star partition with classes $\{x_1, x, y\}$, $\{w, z\}$, $\{x_i\}$, $2 \leq i \leq a-1$ is a resolving partition, hence

$$spd(G) = a. \quad \square$$

THEOREM 3.2. *For any two integers $a$ and $b$ such that $3 \leq a \leq b$ there exists a connected graph $G$ such that $sp(G) = a$ and $spd(G) = b$.*

*Proof.* Let $G$ be the graph obtained from path $P_{a+2}$ with vertices $x_1, \ldots, x_{a+2}$ by attaching one new terminal vertex $y_i$ to each of vertices $x_i$, $3 \leq i \leq a$, and another $b-a$ terminal vertices $z_1, \ldots, z_{b-a}$ to vertex $x_a$ (Fig. 4).

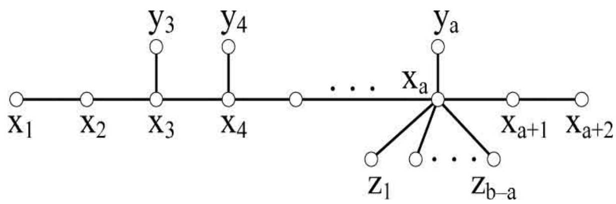


Fig. 4

Using the algorithm from previous section for building a minimum star partition for $G$, we obtain that $sp(G) = a$, a minimum star partition for $G$ having classes $\{x_{a+1}, x_{a+2}\}$, $\{y_a, x_a, z_1, \ldots, z_{b-a}\}$, $\{x_1, x_2\}$, $\{x_i, y_i\}$, for $3 \leq i \leq a-1$.

By Theorem 2.7, since $ter(x_3) = 2$ and $ter(x_a) = b - a + 2$, we have

$$spd(G) = 2 + b - a + sp(G'),$$

where $G'$ is the graph obtained from $G$ by removing terminal stars $\{x_1, x_2\}$, $\{x_{a+1}, x_{a+2}\}$, $\{z_i\}$, $1 \leq i \leq b-a$. $G'$ has $a-4$ vertices of degree greater than 3 and the induced-star number $a-2$, a minimum star partition for $G'$ build by the algorithm form previous section being $(\{x_i, y_i\},\ 3 \leq i \leq a)$. It follows that

$$spd(G) = 2 + b - a + a - 2 = b. \quad \square$$

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