FIXED POINT THEOREMS FOR EXPANDING MAPPINGS IN CONE METRIC SPACES

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In this paper, we define expanding mappings in the setting of cone metric spaces analogous to expanding mappings in metric spaces. We also obtain some results for two mappings to the setting of cone metric spaces.

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1. INTRODUCTION AND PRELIMINARIES

The existing literature of fixed point theory contains many results enunciating fixed point theorems for self-mappings in metric and Banach spaces. Recently, Huang and Zhang [4] introduced the concept of cone metric spaces which generalized the concept of the metric spaces, replacing the set of real numbers by an ordered Banach space, and obtained some fixed point theorems for mapping satisfying different contractive conditions. The study of fixed point theorems in such spaces is followed by some other mathematicians, see [1–2, 5–6, 8–10, 12]. In 1984, Wang et. al. [11] introduced the concept of expanding mappings and proved some fixed point theorems in complete metric spaces. In 1992, Daffer and Kaneko [3] defined an expanding condition for a pair of mappings and proved some common fixed point theorems for two mappings in complete metric spaces.

In this paper, we define expanding mappings in the setting of cone metric spaces analogous to expanding mappings in complete metric spaces. We also extend a result of Daffer and Kaneko [3] for two mappings to the setting of cone metric spaces.

Consistent with Huang and Zhang [4], the following definitions and results will be needed in the sequel.

Let E be a real Banach space. A subset P of E is called a cone if and only if:

(a) P is closed, nonempty and $P \neq \{\theta\}$;

(b) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P$ implies $ax + by \in P$;

(c) $P \cap (-P) = \{\theta\}$.

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Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \text{ implies } \|x\| \leq K\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P, while $x \ll y$ stands for $y - x \in \operatorname{int} P$ (interior of P).

Definition 1.1. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d1) $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space.

Example 1.1 ([4]). Let $E = R^2$, $P = \{(x, y) \in E \mid x, y \geq 0\}$, $X = R$ and $d : X \times X \to E$ be such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 1.2 ([4]). Let (X, d) be a cone metric space. We say that $\{x_n\}$ is:

(e) a Cauchy sequence if for every $c \in E$ with $\theta \ll c$, there is an N such that for all $n, m > N$, $d(x_n, x_m) \ll c$;

(f) a convergent sequence if for every $c \in E$ with $\theta \ll c$, there is an N such that for all $n > N$, $d(x_n, x) \ll c$ for some fixed $x \in X$.

(g) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

The continuity of the self-maps in the cone metric spaces is, in fact, the sequential continuity. If $f : X \to X$, where (X, d) is a cone metric space, then f is continuous at the point $a \in X$ if, for every sequence $\{x_n\} \in X$, which converges in the cone metric d to a, the sequence fx_n converges to fa, i.e.,

$$d(x_n, a) \ll c \Rightarrow d(fx_n, fa) \ll c.$$

In the rest of this paper, we always suppose that E is a real Banach space, $P \subseteq E$ is a cone with $\operatorname{int} P \neq \emptyset$ and $\leq$ is partial ordering with respect to P. We also note that the relations $\operatorname{int} P + \operatorname{int} P \subseteq \operatorname{int} P$ and $\lambda \operatorname{int} P \subseteq \operatorname{int} P (\lambda > 0)$ always hold true.

Definition 1.3. Let (X, d) be a cone metric space and $T : X \to X$. Then T is called a expanding mapping, if for every $x, y \in X$ there exists a number $k > 1$ such that $d(Tx, Ty) \geq kd(x, y)$.

Definition 1.4 ([7]). Two self mappings f and g of a cone metric space (X, d) are said to be commuting if $fgx = gfx$ for all $x \in X$.

Definition 1.5 ([1]). Let f and g be self mappings of a set X (i.e., $f, g : X \to X$). If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. Self mappings f and g are said to be weakly compatible if they commute at their coincidence point; i.e., if $fx = gx$ for some $x \in X$, then $fgx = gfx$.

Weakly compatible mappings are more general than that of commuting but neither implication is reversible.

The following lemma and remark will be useful in what follows.

Lemma 1.1 ([8]). *Let u, v, w be vectors from Banach space E.*
(1) *If $u \leq v$ and $v \ll w$, then $u \ll w$.*
(2) *If $\theta \leq u \ll c$ for each $c \in \operatorname{int} P$ then $u = \theta$.*

Remark 1.1 ([8]). If E is a real Banach space with cone P and if $a \leq ka$ where $a \in P$ and $0 < k < 1$, then $a = \theta$.

2. MAIN RESULTS

In this section we shall prove some fixed point theorems of expanding mappings.

We start with a lemma.

Lemma 2.1. *Let (X, d) be a cone metric space and $\{x_n\}$ be a sequence in X. If there exists a number $k \in (0, 1)$ such that*

$$d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1}), \quad n = 1, 2, \dots \tag{1}$$

then $\{x_n\}$ is a Cauchy sequence in X.

Proof. By the simple induction with the condition (1), we have

$$d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1}) \leq k^2 d(x_{n-1}, x_{n-2}) \leq \dots \leq k^n d(x_1, x_0).$$

Hence for $n > m$

$$d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \dots + d(x_{m+1}, x_m) \leq$$
$$\leq (k^{n-1} + k^{n+2} + \dots + k^m) d(x_1, x_0) \leq \frac{k^m}{1+k} d(x_1, x_0).$$

Let $\theta \ll c$ be given. Chose $\delta > 0$ such that $c + N_\delta(\theta) \subseteq P$, where $N_\delta(\theta) = \{y \in E : \|y\| < \delta\}$. Also, choose a natural number N_1 such that $\frac{k^m}{1+k} d(x_1, x_0) \in N_\delta(\theta)$, for all $m \geq N_1$. Then $\frac{k^m}{1+k} d(x_1, x_0) \ll c$, for all $m \geq N_1$. Thus

$$d(x_n, x_m) \leq \frac{k^m}{1+k} d(x_1, x_0) \ll c,$$

for all $n > m$. Hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. $\square$

THEOREM 2.1. *Let (X, d) be a complete cone metric space and $T : X \to X$ be a surjection. Suppose that there exist $a_1, a_2, a_3 \geq 0$ with $a_1 + a_2 + a_3 > 1$ such that*

(2) $d(Tx, Ty) \geq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty)$, *for all* $x, y \in X$, $x \neq y$.

Then T has a fixed point in X.

Proof. Under the assumption, it is clear that T is injective. Let G be the inverse mapping of T. Choose $x_0 \in X$, set $x_1 = G(x_0)$, $x_2 = G(x_1) = G^2(x_0)$, $\ldots, x_{n+1} = G(x_n) = G^{n+1}(x_0), \ldots$.

Without loss of generality, we assume that $x_{n-1} \neq x_n$ for all $n = 1, 2, \ldots$ (otherwise, if there exists some n_0 such that $x_{n_0-1} = x_{n_0}$, then x_{n_0} is a fixed point of T).

It follows that from condition (2)

$$d(x_{n-1}, x_n) = d(TT^{-1}x_{n-1}, TT^{-1}x_n) \geq$$
$$\geq a_1 d(T^{-1}x_{n-1}, T^{-1}x_n) + a_2 d(T^{-1}x_{n-1}, TT^{-1}x_{n-1}) + a_3 d(T^{-1}x_n, TT^{-1}x_n) =$$
$$= a_1 d(Gx_{n-1}, Gx_n) + a_2 d(Gx_{n-1}, x_{n-1}) + a_3 d(Gx_n, x_n) =$$
$$= a_1 d(x_n, x_{n+1}) + a_2 d(x_n, x_{n-1}) + a_3 d(x_{n+1}, x_n)$$

or

$$(1 - a_2) d(x_{n-1}, x_n) \geq (a_1 + a_3) d(x_{n+1}, x_n).$$

If $a_1 + a_3 = 0$, then $a_2 > 1$. The above inequality implies that a negative number is greater than or equal to zero. That is impossible. So, $a_1 + a_3 \neq 0$ and $(1 - a_2) > 0$. Therefore,

$$d(x_{n+1}, x_n) \leq h d(x_{n-1}, x_n),$$

where $h = \frac{1-a_2}{a_1+a_3} < 1$. By Lemma 2.1, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since (X, d) is complete, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a point $z \in X$. Let $z = Tu$, $u \in X$, we have

$$d(x_n, z) = d(Tx_{n+1}, Tu) \geq a_1 d(x_{n+1}, u) + a_2 d(x_{n+1}, x_n) + a_3 d(u, Tu)$$

which implies that as $n \to \infty$

$$\theta \geq (a_1 + a_3) d(u, z).$$

Hence, $u = z = Tu$.

This gives that z is a fixed point of T. This completes the proof. □

Remark 2.1. Setting $a_2 = a_3 = 0$ and $a_1 = k$ in Theorem 2.1, we can obtain the following result.

COROLLARY 2.1. *Let (X, d) be a complete cone metric space and $T : X \to X$ be a surjection. Suppose that there exists a constant $k > 1$ such that*

(3) $$d(Tx, Ty) \geq k d(x, y), \quad \textit{for all } x, y \in X.$$

Then T has a unique fixed point in X.

Proof. From Theorem 2.1, it follows that T has a fixed point z in X by setting $a_2 = a_3 = 0$ and $a_1 = \lambda$ in condition (2).

Uniqueness. Suppose that $z \neq w$ is also another fixed point of T, then from condition (3), we obtain

$$d(z,w) = d(Tz,Tw) \geq \lambda d(z,w)$$

which implies $d(z,w) = \theta$, that is $z = w$. This completes the proof. □

COROLLARY 2.2. *Let (X,d) be a complete cone metric space and $T : X \to X$ be a surjection. Suppose that there exist a positive integer n and a real number $k > 1$ such that*

$$d(T^n x, T^n y) \geq kd(x,y), \quad \text{for all } x,y \in X. \tag{4}$$

Then T has a unique fixed point in X.

Proof. From Corollary 2.1, T^n has a unique fixed point z. But $T^n(Tz) = T(T^n z) = Tz$, so Tz is also a fixed point of T^n. Hence $Tz = z$, z is a fixed point of T. Since the fixed point of T is also fixed point of T^n, the fixed point of T is unique. □

THEOREM 2.2. *Let (X,d) be a complete cone metric space and $T : X \to X$ be a continuous surjection. If there exist a constant $k > 1$ such that, for any $x,y \in X$, there is*

$$u \equiv u(x,y) \in \{d(x,y), d(x,Tx), d(y,Ty)\}$$

satisfying

$$d(Tx,Ty) \geq ku, \quad \text{for all } x,y \in X. \tag{5}$$

Then T has a fixed point in X.

Proof. Similar to the proof of Theorem 2.1, we can obtain a sequence $\{x_n\}$ such that $x_{n-1} = Tx_n$.

Without loss of generality, we assume that $x_{n-1} \neq x_n$ for all $n = 1,2,\ldots$ (otherwise, if there exists some n_0 such that $x_{n_0-1} = x_{n_0}$, then x_{n_0} is a fixed point of T).

It follows that from condition (5)

$$d(x_{n-1},x_n) = d(Tx_n,Tx_{n+1}) \geq ku_n,$$

where $u_n = \{d(x_n,x_{n+1}), d(x_n,x_{n-1})\}$.

Now we have to consider the following two cases:

• If $u_n = d(x_n,x_{n-1})$, then

$$d(x_{n-1},x_n) \geq kd(x_n,x_{n-1})$$

which implies $d(x_{n-1},x_n) = \theta$ by Remark 1.1, that is $x_{n-1} = x_n$. This is a contradiction.

- If $u_n = d(x_n, x_{n+1})$, then

$$d(x_{n-1}, x_n) \geq kd(x_n, x_{n+1}).$$

By Lemma 2.1, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since (X, d) is complete, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a point $z \in X$.

Since T is continuous, it is clear that z is a fixed point of T. This completes the proof. $\square$

Now, we give a common fixed point theorem of two weakly compatible mappings in cone metric spaces.

THEOREM 2.3. *Let (X, d) be a cone metric space. Let S and T be weakly compatible self-mappings of X and $T(X) \subseteq S(X)$. Suppose that there exists $k > 1$ such that*

$$d(Sx, Sy) \geq kd(Tx, Ty), \quad \text{for all } x, y \in X. \tag{6}$$

If one of the subspaces $T(X)$ or $S(X)$ is complete, then S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Since $T(X) \subseteq S(X)$, choose x_1 such that $y_1 = Sx_1 = Tx_0$. In general, choose x_{n+1} such that $y_{n+1} = Sx_{n+1} = Tx_n$. Then from (6),

$$\begin{aligned} d(y_{n+1}, y_{n+2}) = d(Tx_n, Tx_{n+1}) &\leq \frac{1}{k}\, d(Sx_n, Sx_{n+1}) \\ &= \frac{1}{k}\, d(Tx_{n-1}, Tx_n) = \frac{1}{k}\, d(y_n, y_{n+1}). \end{aligned}$$

Thus, by Lemma 2.1, $\{y_n\}$ is a Cauchy sequence, and hence is convergent. Call the limit z, the $\lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n$. Since $T(X)$ or $S(X)$ is complete and $T(X) \subseteq S(X)$, there exists a point $p \in X$ such that $Sp = z$. Now from (6)

$$d(Tp, Tx_n) \leq \frac{1}{k} d(Sp, Sx_n).$$

Proceeding to the limit as $n \to \infty$, we have $d(Tp, z) \leq \frac{1}{k} d(Sp, z)$, which implies that $Tp = z$. Therefore, $Tp = Sp = z$. Since S and T are weakly compatible, therefore $STp = TSp$, that is $Sz = Tz$.

Now we show that z is a fixed point of S and T. From (6)

$$d(Sz, Sx_n) \geq kd(Tz, Tx_n).$$

Proceeding to the limit as $n \to \infty$, we have $d(Sz, z) \geq kd(Tz, z)$, which implies that $Sz = z$. Hence $Sz = Tz = z$.

Uniqueness. Suppose that $z \neq w$ is also another common fixed point of S and T. Then $d(Sz, Sw) \geq kd(Tz, Tw)$, this implies that $z = w$. This completes the proof. $\square$

Now we give an example illustrating Theorem 2.3.

Example 2.1. Let $E = C^1([0,1], R)$, $P = \{\varphi \in E : \varphi(t) \geq 0,\ t \in [0,1]\}$, $X = [0,1]$, and $d : X \times X \to E$ defined by $d(x,y) = |x-y|\varphi$, where $\varphi \in P$ is a fixed function, e.g., $\varphi(t) = e^t$. Then (X,d) is a complete cone metric space with a non-normal cone having the nonempty interior. Let $S(x) = \frac{x}{2}$, $T(x) = \frac{x}{6}$ for all $x, y \in X$. Then $T(X) \subseteq S(X)$ and $S(X)$ is complete. Further,

$$d(Sx, Sy) = \frac{1}{2}|x-y|\varphi \geq \frac{k}{6}d(Tx, Ty)$$

for $1 < k < 3$ and (6) is satisfied. Moreover, mappings are weakly compatible at $x = 0$ and 0 is the unique common fixed point. Thus all the conditions of Theorem 2.3 are satisfied.

Remark 2.2. In Theorem 2.3, the necessary condition of weakly compatibility cannot be removed.

Note that in Example 2.1, if we consider $S(x) = 1 - x, T(x) = 1 - \frac{x}{2}$ for all $x, y \in X$. Then $T(X) \subseteq S(X)$ and $S(X)$ is complete. Moreover,

$$d(Sx, Sy) = |x-y|\varphi \geq kd(Tx, Ty)$$

for $1 < k < 2$ and (6) is satisfied. $S0 = T0 = 1$ but $ST0 = 0$ and $TS0 = \frac{1}{2}$, so S and T are not weakly compatible. It follows that except for the weakly compatibility of S and T all other hypotheses of Theorem 2.3 are satisfied. But they do not have a common fixed point. This shows that the weakly compatibility of S and T in Theorem 2.3 is an essential condition.

Daffer and Kaneko[3] prove a fixed point theorem for a pair of mappings. We extend their result in cone metric space, thus defining an expanding condition for a pair of mappings in Corollary 2.3 below.

Corollary 2.3. *Let (X,d) be a complete cone metric space. Let $S : X \to X$ be a surjection and $T : X \to X$ be an injective. If S and T are commutative, and there exists $k > 1$ such that*

$$d(Sx, Sy) \geq kd(Tx, Ty), \quad \text{for all } x, y \in X, \tag{7}$$

then S and T have a unique common fixed point in X.

Proof. Note that mappings which commute are clearly weakly compatible and $S(X)$ is complete and $T(X) \subseteq S(X)$ in Corollary 2.3 since S is surjective. Then, we can apply Theorem 2.3 that assures the existence of a unique common fixed point of S and T in X. □

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