HERMITE-HADAMARD TYPE INEQUALITIES OF CONVEX FUNCTIONS WITH RESPECT TO A PAIR OF QUASI-ARITHMETIC MEANS

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In this paper, we establish some integral inequalities of Hermite-Hadamard type, in the framework of Borel probability measures.

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The Hermite-Hadamard inequality asserts that for every continuous convex function $f$ defined on an interval $[a, b]$ and every Borel probability measure $\mu$ on $[a, b]$ we have

\[
\begin{align*}
(HH) \quad f(b) - f(a) & \leq \int_a^b f(x) \, d\mu(x) \leq \frac{b - a}{b - a} f(a) + \frac{b - a}{b - a} f(b),
\end{align*}
\]

where

\[
b_\mu = \int_a^b x \, d\mu(x)
\]

is the barycenter of $\mu$. See [3] for details.

The aim of this paper is to prove an analogue of Hermite-Hadamard inequality in the framework of quasi-arithmetic means.

Let $I$ be an interval and $\varphi : I \to \mathbb{R}$ a continuous increasing function. The weighted quasi-arithmetic mean associated to $\varphi$ is defined by the formula

\[
M_{[\varphi]}(a, b; 1 - \lambda, \lambda) = \varphi^{-1}((1 - \lambda) \varphi(a) + \lambda \varphi(b)),
\]

for $a, b \in I$ and $\lambda \in [0, 1]$.

The weighted arithmetic mean

\[
A(a, b; 1 - \lambda, \lambda) = (1 - \lambda)a + \lambda b
\]

corresponds to $\varphi(x) = x$, and the weighted geometric mean

\[
G(a, b; 1 - \lambda, \lambda) = a^{1 - \lambda}b^\lambda
\]

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corresponds to $\varphi(x) = \log x$.

Given a pair of continuous increasing functions $\varphi : [a, b] \to \mathbb{R}$ and $\psi : [c, d] \to \mathbb{R}$, a function $f : [a, b] \to [c, d]$ is called $(M_{[\varphi]}, M_{[\psi]})$-convex if
\[
f \left( M_{[\varphi]}(x, y; 1 - \lambda, \lambda) \right) \leq M_{[\psi]}(f(x), f(y); 1 - \lambda, \lambda)
\]
for every $x, y \in [a, b]$ and $\lambda \in [0, 1]$.

The theory of $(M_{[\varphi]}, M_{[\psi]})$-convex functions can be deduced from the theory of usual convex functions. Indeed, $f$ is a $(M_{[\varphi]}, M_{[\psi]})$-convex function if and only if $\psi \circ f \circ \varphi^{-1}$ is convex. This fact allows us to translate results known for convex functions into their counterparts for $(M_{[\varphi]}, M_{[\psi]})$-convex functions. We will next consider the case of Hermite-Hadamard inequality. Our approach is based on the concept of push-forward measure.

Given a Borel probability measure $\mu$ (on an interval $[a, b]$), the push-forward of $\mu$ through a continuous map $\varphi : [a, b] \to \mathbb{R}$ is defined by
\[
(\varphi \# \mu)(A) = \mu(\varphi^{-1}(A))
\]
for every Borel subset $A$ of $[\varphi(a), \varphi(b)]$. This measure allows the following change of variable formula
\[
\int_a^b f(\varphi(x)) \, d\mu(x) = \int_{\varphi(a)}^{\varphi(b)} f(x) \, d\mu(\varphi^{-1}(x)).
\]

The barycenter of $\varphi \# \mu$ is
\[
b_{\varphi \# \mu} = \int_{\varphi(a)}^{\varphi(b)} x \, d\mu(\varphi^{-1}(x)) = \int_a^b \varphi(x) \, d\mu(x),
\]
so if we put
\[
\xi = \varphi^{-1}(b_{\varphi \# \mu})
\]
and
\[
\mathcal{M}(\xi) = \frac{b(\varphi(b) - \varphi(\xi)) - a(\varphi(a) - \varphi(\xi))}{\varphi(b) - \varphi(a)},
\]
we obtain the identity
\[
\varphi(\xi) - \varphi(a) = \frac{b - \mathcal{M}(\xi)}{b - a} \cdot (\varphi(b) - \varphi(a))
\]
(1)

\textbf{Lemma 1.} The barycenter of $\varphi \# \mu$ verifies the formula
\[
b_{\varphi \# \mu} = \mathcal{M}(\xi) - a \cdot \varphi(a) + \frac{b - \mathcal{M}(\xi)}{b - a} \cdot \varphi(b).
\]
(2)
Proof. In fact
\[ b_{\varphi} = \frac{\varphi(b) - \varphi(a)}{a - b} - b_{\varphi#\mu} = \frac{b_{\varphi#\mu} - \varphi(a)}{a - b} \varphi(b) - \varphi(a) \varphi(b) \]
\[ = \frac{\mathcal{M}(\xi) - a \varphi(a) - b - \mathcal{M}(\xi) b - a \varphi(a)}{b - a} \varphi(b) - \varphi(a) \varphi(b) - \varphi(a). \]
due to the identity (1).

Theorem 1 [The Hermite-Hadamard inequality for \((M_\varphi, M_\psi)-\)convex functions]. Let \(f : [a, b] \rightarrow [c, d]\) be a continuous \((M_\varphi, M_\psi)-\)convex function and \(\mu\) be a Borel probability measure on \([a, b]\). Then

\begin{align*}
\text{(RHH)} & \quad f(\xi) \leq \psi^{-1} \left( \int_a^b \psi(f(x)) \, d\mu(x) \right) \\
\text{(LHH)} & \quad \leq M_\psi \left( f(a), f(b); \frac{\varphi(b) - \varphi(\xi)}{\varphi(b) - \varphi(a)}, \frac{\varphi(\xi) - \varphi(a)}{\varphi(b) - \varphi(a)} \right),
\end{align*}

where \(\xi = \varphi^{-1}(b_{\varphi#\mu}).\)

Proof. We apply the inequality (HH) to \(\psi \circ f \circ \varphi^{-1}\). As we have seen,
\[(\psi \circ f)(\xi) = (\psi \circ f \circ \varphi^{-1})(b_{\varphi#\mu}) \leq \int_{\varphi(a)}^{\varphi(b)} \psi(f(\varphi^{-1}(x))) \, d\mu(\varphi^{-1}(x)) = \int_a^b \psi(f(x)) \, d\mu(x) \]
\[ \leq \frac{\varphi(b) - b_{\varphi#\mu}}{\varphi(b) - \varphi(a)} \psi(f(a)) + \frac{b_{\varphi#\mu} - \varphi(a)}{\varphi(b) - \varphi(a)} \psi(f(b)) \]
and the conclusion follows.

Remark 1. Theorem 1 was proved for \((A, M_\psi)-\)convex functions in [1, Theorem 3.3], under more restrictive conditions. The particular case of \((G, A)-\)convex functions was proved in [5], while the case of \((G, G)-\)convex functions appeared in [2] and [4].

We will call the function \(\Phi\) a support of \(f\) if \(\psi \circ \Phi \circ \varphi^{-1} = \Psi\), where \(\Psi\) is a support line of the convex function \(\psi \circ f \circ \varphi^{-1}\).

Theorem 2. Let \(f : [a, b] \rightarrow [c, d]\) be a continuous \((M_\varphi, M_\psi)-\)convex function, \(\psi\) concave and \(\mu\) be a Borel probability measure on \([a, b]\). Then
\[ \int_a^b f(x) \, d\mu(x) \geq f\left( \varphi^{-1}(b_{\varphi#\mu}) \right) \]
\[ = \sup_{\Phi \text{ is a support of } f} \left\{ \psi^{-1} \left( \int_a^b \psi(\Phi(x)) \, d\mu(x) \right) \right\}. \]
Proof. The proof is similar to [2, Theorem 3]. Details are left to the reader.

The Hermite-Hadamard type inequalities proved in Theorem 1 are not just consequences of \((M_{[\varphi]}, M_{[\psi]})\)-convexity, but also characterize it. The converse of Hermite-Hadamard inequality for \((M_{[\varphi]}, M_{[\psi]})\)-convex functions reads as follows:

**Theorem 3.** Let \(I, J\) be two intervals and \(f : I \to J\) a continuous function. Assume that \(\varphi : I \to \mathbb{R}\) and \(\psi : J \to \mathbb{R}\) are continuous increasing functions. If for every compact subinterval \([a, b]\) of \(I\) and for every atomless Borel probability measure \(\mu\) on \([a, b]\) the function \(f\) satisfies either the inequality (RHH) or (LHH) then \(f\) is \((M_{[\varphi]}, M_{[\psi]})\)-convex.

Proof. If (RHH) holds, by Jensen’s inequality we conclude that \(\psi \circ f \circ \varphi^{-1}\) is convex, hence \(f\) is \((M_{[\varphi]}, M_{[\psi]})\)-convex.

It remains to consider that (LHH) holds. We proceed by reductio ad absurdum. Assume that \(f\) is not \((M_{[\varphi]}, M_{[\psi]})\)-convex. Then there exists a subinterval \([x, y] \subset I\) and a number \(\varepsilon \in (0, 1)\) such that

\[
(3) \quad f(M_{[\varphi]}(x, y; 1 - \varepsilon, \varepsilon)) > M_{[\psi]}(f(x), f(y); 1 - \varepsilon, \varepsilon).
\]

Since \(f\) is continuous, the inequality (3) holds on an entire neighbourhood \((\varepsilon_1, \varepsilon_2)\) of \(\varepsilon\). We choose \((\varepsilon_1, \varepsilon_2)\) the biggest neighbourhood with this property.

Put \(a = M_{[\varphi]}(x, y; 1 - \varepsilon_1, \varepsilon_1)\) and \(b = M_{[\varphi]}(x, y; 1 - \varepsilon_2, \varepsilon_2)\) \((a < b)\). The continuity of \(f\) ensures that

\[
f(a) = M_{[\varphi]}(f(x), f(y); 1 - \varepsilon_1, \varepsilon_1)
\]

and

\[
f(b) = M_{[\varphi]}(f(x), f(y); 1 - \varepsilon_2, \varepsilon_2).
\]

Since we have \((1 - t)\varepsilon_1 + t\varepsilon_2 \in (\varepsilon_1, \varepsilon_2)\) for every \(t\) in \((0, 1)\), we infer from (3) that

\[
f(M_{[\varphi]}(a, b; 1 - t, t))
\]

\[
= f(M_{[\varphi]}(M_{[\varphi]}(x, y; 1 - \varepsilon_1, \varepsilon_1), M_{[\varphi]}(x, y; 1 - \varepsilon_2, \varepsilon_2); 1 - t, t))
\]

\[
= f(M_{[\varphi]}(x, y; 1 - (1 - t)\varepsilon_1 - t\varepsilon_2, (1 - t)\varepsilon_1 + t\varepsilon_2))
\]

\[
> M_{[\varphi]}(f(x), f(y); 1 - (1 - t)\varepsilon_1 - t\varepsilon_2, (1 - t)\varepsilon_1 + t\varepsilon_2)
\]

\[
= M_{[\varphi]}(M_{[\varphi]}(f(x), f(y); 1 - \varepsilon_1, \varepsilon_1), M_{[\varphi]}(f(x), f(y); 1 - \varepsilon_2, \varepsilon_2); 1 - t, t)
\]

\[
= M_{[\varphi]}(f(a), f(b); 1 - t, t).
\]
Thus, it follows
\[
\int_a^b \psi(f(x)) \, d\mu(x) \\
= \int_a^b \psi \left( f \left( M_{[\varphi]} \left( a, b; \frac{\varphi(b) - \varphi(x)}{\varphi(b) - \varphi(a)}, \frac{\varphi(x) - \varphi(a)}{\varphi(b) - \varphi(a)} \right) \right) \right) \, d\mu(x) \\
> \int_a^b \psi \left( M_{[\psi]} \left( f(a), f(b); \frac{\varphi(b) - \varphi(x)}{\varphi(b) - \varphi(a)}, \frac{\varphi(x) - \varphi(a)}{\varphi(b) - \varphi(a)} \right) \right) \, d\mu(x) \\
= \frac{\varphi(b) - \varphi(\xi)}{\varphi(b) - \varphi(a)} \psi(f(a)) + \frac{\varphi(\xi) - \varphi(a)}{\varphi(b) - \varphi(a)} \psi(f(b)).
\]
This is a contradiction, completing the reductio ad absurdum. □

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REFERENCES