STOCHASTIC INTEGRAL EQUATIONS ASSOCIATED WITH STRATONOVICH CURVELINE INTEGRAL

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There are investigated the existence and integral representation of a solution satisfying a system of curveline stochastic equations depending on two independent Itô processes $\{\eta_1(t_1), \eta_2(t_2) \in \mathbb{R}^2 : 0 \le t_1 \le T_1,\ 0 \le t_2 \le T_2\}$.

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1. INTRODUCTION

We are concerned about curveline stochastic equations driven by a fixed vector field $g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ and using a Stratonovich type differential of a smooth test function $F(\tau_1, \tau_2)$ combined with two independent Itô processes

$$\tau_i = \eta_i(t_i), \quad i \in \{1, 2\},$$
$$(t_1, t_2) \in D = \left\{(t_1, t_2) \in \mathbb{R}^2 : 0 \le t_1 \le T_1,\ 0 \le t_2 \le T_2\right\}.$$

An explicit representation formula for a solution is given in Theorem 1, when g is a bounded smooth vector field. The case of a complete vector field $g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ is analyzed in Theorem 2, introducing adequate stopping times. The main support in writing a solution comes from the solution

$$y_\lambda(\tau_1, \tau_2) = G\left(F\left(\tau_1, \tau_2\right)\right)[\lambda], \quad (\tau_1, \tau_2) \in \mathbb{R}^2,\ \lambda \in \mathbb{R}^n,$$

satisfying a deterministic gradient system

$$\begin{cases} \partial_{\tau_1} y_\lambda\left(\tau_1, \tau_2\right) = g\left(y_\lambda(\tau_1, \tau_2)\right) \partial_{\tau_1} F(\tau_1, \tau_2), \\ \partial_{\tau_2} y_\lambda\left(\tau_1, \tau_2\right) = g\left(y_\lambda(\tau_1, \tau_2)\right) \partial_{\tau_2} F(\tau_1, \tau_2),\ (\tau_1, \tau_2) \in \mathbb{R}^2, \\ y_\lambda(0, 0) = \lambda \in \mathbb{R}^n. \end{cases}$$

Here, $\{G(\tau)[\lambda] : \tau \in \mathbb{R},\ \lambda \in \mathbb{R}^n\}$ is the global flow generated by the complete vector field $g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ and $F \in \mathcal{C}^2(\mathbb{R}^2)$ fulfils a polynomial growth condition.

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When the corresponding curveline stochastic equation are written, they make sense with respect to the Stratonovich curveline integral provided the right-hand side is a semimartingale and some stopping times are used.

This subject has some roots in the reference [3], but it was inspired by the works given in [1] and [2]. More precisely, the subject analyzed here, on our point of view, presents a new angle in solving the stochastic equations problem mentioned in [1] and [2].

2. CURVELINE STOCHASTIC INTEGRAL

To begin with, we consider two independent brownian motions

$$(W_1(t_1,\omega), W_2(t_2,\omega)) : D \times \Omega \to \mathbb{R}^2,\quad D = \left\{(t_1,t_2) \in \mathbb{R}^2 : 0 \le t_1 \le T_1,\ 0 \le t_2 \le T_2\right\}, \tag{2.1}$$

over the complete filtered probability space $\{\Omega, \mathcal{F} \supseteq \{\mathcal{F}_{t_1,t_2}\}, \mathbb{P}\}$. Here, $\mathcal{F}_{t_1,t_2} \stackrel{\text{def}}{=} \mathcal{F}_{t_1} \otimes \mathcal{F}_{t_2}$ and $\mathcal{F}_{t_i}$ is the complete σ-algebra $\sigma\left(W_i\left(s,\omega\right) : 0 \le s \le t_i\right)$ generated by the brownian motion as $W_1(t_1)$, $0 \le t_1 \le T_1$ and $W_2(t_2)$, $0 \le t_2 \le T_2$.

Denote by $\widehat{\gamma}(A,B) \subseteq D(A,B \in D)$ any polygonal curve connecting two fixed points $A, B \in D$ such that any line-segment of $\widehat{\gamma}(A,B)$ is parallel to one of the coordinates axis in $\mathbb{R}^2$.

We are going to define, following a standard procedure, two types of curveline stochastic integral.

2.1. Itô curveline integral

Let $P_1, P_2 \in \mathcal{C}^2(\mathbb{R}^2)$ be given such that the following polynomial growth condition is satisfied

$$\left|P_i(\tau_1,\tau_2)\right|, \left|\partial_{\tau_i} P_i(\tau_1,\tau_2)\right| \le C\big(1 + |\tau_1|^{N_1}\,|\tau_2|^{N_2}\big), \tag{2.2}$$

where $(\tau_1,\tau_2) \in \mathbb{R}^2$ for $i \in \{1,2\}$ and $N_1, N_2 \ge 1$ are some natural numbers and $C > 0$ is a constant.

Following the standard Itô procedure, we define the *Itô curveline stochastic integral* $I_{\widehat{\gamma}}(A,B)$ by

$$I_{\widehat{\gamma}}(A,B) = \int_{\widehat{\gamma}(A,B)} P_1\left(W_1(t_1), W_2(t_2)\right) \cdot \mathrm{d}W_1(t_1) + \int_{\widehat{\gamma}(A,B)} P_2\left(W_1(t_1), W_2(t_2)\right) \cdot \mathrm{d}W_2(t_2) \tag{2.3}$$

Here, $P_1, P_2 \in \mathcal{C}^2(\mathbb{R}^2)$ satisfy the polynomial growth condition (2.2) and Itô stochastic integrals " $\cdot$ " involved in the right-hand side of (2.3) are defined as $\mathcal{F}_{(b_1,b_2)}$-martingales, $B = (b_1, b_2) \in D$.

2.2. **Stratonovich curveline integral**

With the same notations as in above subsection and using the hypothesis (2.2), it makes sense to define the following *Stratonovich curveline integral*

$$S_{\widehat{\gamma}}(A,B) = \int_{\widehat{\gamma}(A,B)} P_1\left(W_1(t_1), W_2(t_2)\right) \circ \mathrm{d}W_1(t_1) + \\ + \int_{\widehat{\gamma}(A,B)} P_2\left(W_1(t_1), W_2(t_2)\right) \circ \mathrm{d}W_2(t_2). \tag{2.4}$$

Here, the Stratonovich integral " $\circ$ " is expressed using Itô stochastic integral " $\cdot$ " by

$$P_i\left(W_1(t_1), W_2(t_2)\right) \circ \mathrm{d}W_i(t_1) = \frac{1}{2}\partial_{\tau_i} P_i\left(W_1(t_1), W_2(t_2)\right) \mathrm{d}t_i + \\ + P_i\left(W_1(t_1), W_2(t_2)\right) \cdot \mathrm{d}W_i(t_i), \tag{2.5}$$

for $i \in \{1,2\}$ and the right-hand side of (2.4) is well defined as a $\mathcal{F}_{(b_1,b_2)}$-semimartingale, $B = (b_1, b_2) \in D$. In addition, a direct computation lead us to the formula connecting the two curveline stochastic integrals

$$S_{\widehat{\gamma}}(A,B) = I_{\widehat{\gamma}}\left(A,B\right) + \frac{1}{2}\int_{\widehat{\gamma}(A,B)} F_1\left(W_1(t_1), W_2(t_2)\right) \mathrm{d}t_1 + \\ + \frac{1}{2}\int_{\widehat{\gamma}(A,B)} F_2\left(W_1(t_1), W_2(t_2)\right) \mathrm{d}t_2, \tag{2.6}$$

where $F_i(\tau_1, \tau_2) = \partial_{\tau_i} P_i(\tau_1, \tau_2)$, $i \in \{1,2\}$.

A significant property of the Stratonovich curveline integral is its independence of polygonal path $\widehat{\gamma}(A,B)$ connecting the two fixed points $A, B \in D$. In this respect, assume: *there exists* $F \in \mathcal{C}^2(\mathbb{R}^2)$ *such that*

$$P_i(\tau_1, \tau_2) = \partial_{\tau_i} F(\tau_1, \tau_2), \quad (\tau_1, \tau_2) \in \mathbb{R}^2,\ i \in \{1,2\}. \tag{2.7}$$

Lemma 2.1. *Assume that* $P_1, P_2 \in \mathcal{C}^1(\mathbb{R}^2)$ *are given such that the conditions* (2.2) *and* (2.7) *are fulfilled. Then, the Stratonovich curveline integral* $S_{\widehat{\gamma}}(A,B)$ (*see* (2.4)) *does not depend on the polygonal path* $\widehat{\gamma}(A,B)$ *and, in addition,*

$$S_{\widehat{\gamma}}(A,B) = F\left(W_1(b_1), W_2(b_2)\right) - F\left(W_1(a_1), W_2(a_2)\right), \tag{2.8}$$

where $F \in \mathcal{C}^2(\mathbb{R}^2)$ *is given in* (2.7).

Proof. Using the simplest polygonal path $\widehat{\gamma}_0(A,B)$ (composed by two line segments) and applying the standard rule of stochastic derivation with respect to each component $W_i(t_i)$, $i \in \{1,2\}$, we get the equality

$$F\left(W_1(b_1), W_2(b_2)\right) - F\left(W_1(a_1), W_2(a_2)\right) = S_{\widehat{\gamma}_0}(A,B). \tag{2.9}$$

In general, a path $\widehat{\gamma}(A,B)$ contains several couples of segments parallel to the coordinates axis and the above given formula (see (2.9)) can be applied for each subpath containing a couple of segments. By summation in both sides of (2.9), we get

$$F\left(W_1(b_1), W_2(b_2)\right) - F\left(W_1(a_1), W_2(a_2)\right) = S_{\widehat{\gamma}}(A,B), \tag{2.10}$$

for any polygonal path $\widehat{\gamma}(A,B) \subseteq D$ and the proof is complete. □

Remark 2.1. The above given computation can be restated as follows. Assume that

1. $P_1, P_2 \in \mathcal{C}^1(\mathbb{R}^2)$ are given such that the hypothesis (2.2) is fulfilled;
2. the gradient system $S\left(P_1, P_2\right)$ defined by

$$P_i(\tau_1,\tau_2) = \partial_{\tau_i} F\left(\tau_1,\tau_2\right), \quad i \in \{1,2\}, \quad F(0,0) = F_0, \quad (\tau_1,\tau_2) \in \mathbb{R}^2$$

is completely integrable $(\partial_{\tau_2} P_1\left(\tau_1,\tau_2\right) = \partial_{\tau_1} P_2(\tau_1,\tau_2))$.

Lemma 2.1 can be restated by

LEMMA 2.2. *Assume that $P_1, P_2 \in \mathcal{C}^1(\mathbb{R}^2)$ are given such that the conditions* (2.1) *and* (2.1) *of Remark* 2 *are satisfied. Let $F \in \mathcal{C}^2(\mathbb{R}^2)$ be the unique solution of* (2.1) *in Remark* 2.1. *Then*

$$F\left(W_1(t_1), W_2(t_2)\right) = F_0 + S_{\widehat{\gamma}}\left(\theta; (t_1,t_2)\right), \quad \theta \overset{not}{=} (0,0), \quad (t_1,t_2) \in D,$$

for any polygonal path $\widehat{\gamma}\left(\theta;(t_1,t_2)\right) \subseteq D$.

3. CURVELINE INTEGRAL EQUATIONS ASSOCIATED WITH STRATONOVICH CURVELINE INTEGRAL

Let $g \in \mathcal{C}^1\left(\mathbb{R}^n, \mathbb{R}^n\right)$ be a complete vector field satisfying

$$\left|\partial_{x_i} g(x)\right| \le C, \quad x \in \mathbb{R}^n,\ i \in \{1,\ldots,n\}, \tag{3.1}$$

where $C > 0$ is a constant.

Consider $P_1, P_2 \in \mathcal{C}^1(\mathbb{R}^2)$ given such that *the gradient system*

(3.2) $S\left(P_1,P_2\right)$ *is completely integrable* (see Remark 2.1, (2.1)).

We associate the gradient system $S_g(P_1,P_2)$ defined by

$$\begin{cases} \partial_{\tau_1} y(\tau_1,\tau_2) = g\left(y\left(\tau_1,\tau_2\right)\right) P_1(\tau_1,\tau_2), \\ \partial_{\tau_2} y(\tau_1,\tau_2) = g\left(y\left(\tau_1,\tau_2\right)\right) P_2(\tau_1,\tau_2), \\ (\tau_1,\tau_2) \in \mathbb{R}^2,\ y(\theta) = \lambda \in \mathbb{R}^n,\ \theta = (0,0) \in \mathbb{R}^2. \end{cases} \tag{3.3}$$

LEMMA 3.1. *Let $g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ and $P_1, P_2 \in \mathcal{C}^1(\mathbb{R}^2)$ given such that* (3.1) *and* (3.2) *are fulifilled. Let $F \in \mathcal{C}^2(\mathbb{R}^2)$ be the unique solution given by* (3.2) *with $F(\theta) = 0$. Then, the gradient system $S_g(P_1, P_2)$ defined in* (3.3) *is completely integrable and its solution $\{y_\lambda(\tau_1, \tau_2) : (\tau_1, \tau_2) \in \mathbb{R}^2\}$ can be represented by*

$$y_\lambda(\tau_1, \tau_2) = G\,(F(\tau_1, \tau_2))\,[\lambda], \quad (\tau_1, \tau_2) \in \mathbb{R}^2,\ \lambda \in \mathbb{R}^n, \tag{3.4}$$

where $\{G(\tau)[\lambda] : \tau \in \mathbb{R}, \lambda \in \mathbb{R}^n\}$ is the global flow generated by the vector field g.

Proof. Using (3.4), by direct computation, we get that the gradient system (3.3) is satisfied. □

Remark 3.1. *Define a continuous and $\mathcal{F}_{(t_1,t_2)}$-adapted process*

$$x_\lambda(t_1, t_2) = y_\lambda(W_1(t_1), W_2(t_2)), \quad (t_1, t_2) \in D,\ \lambda \in \mathbb{R}^n, \tag{3.5}$$

where $\{y_\lambda(\tau_1, \tau_2) : (\tau_1, \tau_2) \in \mathbb{R}^2\}$ is given in Lemma 3.1 (*see* (3.4)). *It can be associated with a solution of a system of stochastic integral equations as follows*

$$x_\lambda(t_1, t_2) = \lambda + \int_{\widehat{\gamma}(\theta;(t_1,t_2))} X_1\,(W_1(s_1), W_2(s_2), x_\lambda(s_1, s_2)) \circ \mathrm{d}W_1(s_1) + \\ + \int_{\widehat{\gamma}(\theta;(t_1,t_2))} X_2\,(W_1(s_1), W_2(s_2), x_\lambda(s_1, s_2)) \circ \mathrm{d}W_2(s_2), \tag{3.6}$$

where the Stratonovich curveline integral is used and

$$X_i(\tau_1, \tau_2, x) = g(x)P_i(\tau_1, \tau_2), \quad i \in \{1, 2\}.$$

THEOREM 3.1. *Let $g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ and $P_1, P_2 \in \mathcal{C}^1(\mathbb{R}^2)$ be given such that the hypothesis* (3.1) *and* (3.2) *are fullfilled. In addition, assume that g is bounded. Then, the continuous $\mathcal{F}_{(t_1,t_2)}$-adapted process*

$$x_\lambda\,(t_1, t_2) = y_\lambda(W_1(t_1), W_2(t_2)), \quad (t_1, t_2) \in D,$$

defined in (3.5) *is a solution of stochastic integral equation* (3.6).

Proof. By hypothesis, $y_\lambda(\tau_1, \tau_2)$, $(\tau_1, \tau_2) \in \mathbb{R}^2$, satisfying the gradient system $S_g(P_1, P_2)$ in (3.3) exists. Compute

$$\partial_{\tau_i} y_\lambda(\tau_1, \tau_2) = g\,(y_\lambda(\tau_1, \tau_2))\,P_i(\tau_1, \tau_2) \overset{\text{not}}{=} Y_i^\lambda(\tau_1, \tau_2), \quad i \in \{1, 2\}. \tag{3.7}$$

Using the simplest polygonal path $\widehat{\gamma}_0\,(\theta; (t_1, t_2))$ and applying the standard rule of stochastic derivation with respect to each process $\tau_i = W_i(t_i)$, $i \in \{1, 2\}$, we get

$$y_\lambda(W_1(t_1), W_2(t_2)) = \lambda + \int_{\widehat{\gamma}_0(\theta;(t_1,t_2))} Y_1^\lambda\,(W_1(s_1), W_2(s_2)) \circ \mathrm{d}W_1(s_1) + \\ + \int_{\widehat{\gamma}_0(\theta;(t_1,t_2))} Y_2^\lambda(W_1(s_1), W_2(s_2)) \circ \mathrm{d}W_2(s_2). \tag{3.8}$$

Repeating the above given computation on each subpath in the class of $\widehat{\gamma}_0$, we get that (3.8) remains valid if $\widehat{\gamma}_0\,(\theta;(t_1,t_2))$ is replaced by any polygonal path $\widehat{\gamma}\,(\theta;(t_1,t_2))$, connecting $\theta=(0,0)\in D$ and $(t_1,t_2)\in D$. On the other hand, notice that

$$X_i\left(W_1(s_1),W_2(s_2),x_\lambda(s_1,s_2)\right)=Y_i^\lambda\left(W_1(s_1),W_2(s_2)\right),\quad i\in\{1,2\}$$

and (3.8) becomes the integral equation written in (3.6). The proof is complete. □

Remark 3.2. To write a conclusion as in Theorem 3.1, whitout assuming the boundness property of $g\in\mathcal{C}^1(\mathbb{R}^n,\mathbb{R}^n)$, we need to introduce two stopping times of the form

$$\tau_i^N=\inf\left\{t_i\in[0,T_i]:|W_i(t_i)|\geq N\right\},\quad i\in\{1,2\}\,. \tag{3.9}$$

If it is the case, define a continuous and bounded process (for each $\lambda\in\mathbb{R}^n$)

$$x_\lambda^N(t_1,t_2)=y_\lambda(W_1(t_1^N),W_2(t_2^N)),\quad t_1^N=t_1\wedge\tau_1^N,\quad t_2^N=t_2\wedge\tau_2^N \tag{3.10}$$

and it will be a solution of the following integral equation

(3.11)
$$x_\lambda^N(t_1,t_2)=\lambda+\int_{\widehat{\gamma}(\theta;(t_1^N,t_2^N))}X_1(W_1(s_1),W_2(s_2),x_\lambda^N(s_1,s_2))\circ\mathrm{d}W_1(s_1)+$$
$$+\int_{\widehat{\gamma}(\theta;(t_1^N,t_2^N))}X_2(W_1(s_1),W_2(s_2),x_\lambda^N(s_1,s_2))\circ\mathrm{d}W_2(s_2),$$

where $X_i(\tau_1,\tau_2,x)$, $i\in\{1,2\}$ are given in (3.6).

Remark 3.3. In the case $W_i(t_i)$ is replaced by an Itô process

$$\eta_i(t_i)=\int_0^{t_i}u_i(s_i)\cdot\mathrm{d}W_i(s_i),\quad i\in\{1,2\} \tag{3.12}$$

(see $\{u_i(t_i):0\leq t_i\leq T_i\}$ is a bounded measurable and $\mathcal{F}_{t_i}$-adapted process, $i\in\{1,2\}$). Then both Stratonovich curveline integrals and stochastic integral equation in (3.11) can be extended accordingly. In this respect, the following are valid

$$S_{\widehat{\gamma}}^N(A,B)=\int_{\widehat{\gamma}(A,B)}P_1(\eta_1(t_1),\eta_2(t_2))\circ\mathrm{d}\eta_1(t_1)+ \tag{3.13}$$
$$+\int_{\widehat{\gamma}(A,B)}P_2(\eta_1(t_1),\eta_2(t_2))\circ\mathrm{d}\eta_2(t_2),$$

where Stratonovich integral "$\circ$" is computed using Itô integral "$\cdot$",

$$P_i(\eta_1(t_1),\eta_2(t_2))\circ\mathrm{d}\eta_i(t_i)=[P_i(\eta_1(t_1),\eta_2(t_2))u_i(t_i)]\cdot\mathrm{d}W_i(t_i)+ \tag{3.14}$$
$$+\frac{1}{2}\partial_{\tau_i}P_i(\eta_1(t_1),\eta_2(t_2))u_i^2(t_i)\mathrm{d}t_i,\quad i\in\{1,2\}.$$

The content of Remark 3.2 changes into the following

THEOREM 3.2. *Let $g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ and $P_1, P_2 \in \mathcal{C}^1(\mathbb{R}^2)$ be given such that hypothesis* (3.1) *and* (3.2) *are fullfilled. Let $\{y_\lambda(\tau_1, \tau_2) : (\tau_1, \tau_2) \in \mathbb{R}^2\}$ the unique solution of $S_g(P_1, P_2)$ given in* (3.3)*. Define $\{x_\lambda^N(t_1, t_2) : (t_1, t_2) \in D\}$ as in* (3.10)*. Then, the following integral stochastic equations*

$$x_\lambda^N(t_1, t_2) = \lambda + \int_{\widehat{\gamma}(\theta;(t_1^N, t_2^N))} X_1(\eta_1(s_1), \eta_2(s_2), x_\lambda^N(s_1, s_2)) \circ \mathrm{d}\eta_1(s_1) +$$

$$+ \int_{\widehat{\gamma}(\theta;(t_1^N, t_2^N))} X_2(\eta_1(s_1), \eta_2(s_2), x_\lambda^N(s_1, s_2)) \circ \mathrm{d}\eta_2(s_2)$$

are verified for any $(t_1, t_2) \in D$.

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