ON THE HYBRID MEAN VALUE OF COCHRANE SUMS AND TWO-TERM EXPONENTIAL SUMS

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In this paper, we use the elementary and analytic methods to study the mean value properties of the Cochrane sums weighted by two-term exponential sums, and give two exact computational formulae for them.

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1. INTRODUCTION

Let $q$ be a natural number and $h$ an integer prime to $q$. The Cochrane sums $C(h, q)$ is defined by

$$C(h, q) = \sum_{a=1}^{q} \left( \left( \frac{a}{q} \right) \left( \frac{ah}{q} \right) \right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \text{ is not an integer,} \\ 0 & \text{if } x \text{ is an integer,} \end{cases}$$

$ar{a}$ is defined by $a\bar{a} \equiv 1 \mod q$ and $\sum_{a=1}^{q}'$ denotes the summation over all $1 \leq a \leq q$ such that $(a, q) = 1$.

About the arithmetical properties of the Cochrane sums and related sums, some authors have studied it, and obtained many interesting results, see for example [1], [3], [4] and [5].

In this paper, we consider the computational problem of the mean value

$$\sum_{m=1}^{q} \sum_{n=1}^{q}' |C(m, n, k, h; q)|^2 \cdot C(mn, q),$$

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where the two-term exponential sums $C(m, n, k, h; q)$ is defined as

$$C(m, n, k, h; q) = \sum_{a=1}^{q} e\left(\frac{ma^k + na^h}{q}\right), \quad e(y) = e^{2\pi i y}.$$ 

Some results related to $C(m, n, k, h; q)$ can be found in [6]–[7].

But for mean value (1), it seems that none has studied it yet, at least we have not seen any related results before. This sum is interesting, because it has close relations with the class number $h_p$ of the quadratic field $\mathbb{Q}(\sqrt{-p})$, so we can give a new expression for $h_p$. In this paper, we use the elementary and analytic methods to study this problem, and give two exact computational formulae for (1). That is, we shall prove the following two conclusions:

**Theorem 1.** Let $p$ be an odd prime with $p \equiv 1 \pmod{4}$, then we have the computational formulae

$$\sum_{m=1}^{p-1} \sum_{n=1}^{p-1} |C(m, n, 3, 1; p)|^2 \cdot C(mn, p) =$$

$$= \begin{cases} 0 & \text{if } p \equiv 1 \pmod{8}, \\ -\left(\frac{3}{p}\right) \cdot \frac{p^3}{\pi^2} \left\{ \tau^2(\chi_4)L^2(1, \chi_4) + \tau^2(\overline{\chi_4})L^2(1, \chi_4) \right\} & \text{if } p \equiv 5 \pmod{8}, \end{cases}$$

where $\chi_4$ is an odd character mod $p$ such that $\chi_4^2$ is the Legendre’s symbol, $L(1, \chi_4)$ denotes the Dirichlet L-function corresponding to $\chi_4$, and $\tau(\chi) = \sum_{a=1}^{p-1} \chi(a)e\left(\frac{a}{p}\right)$ denotes the classical Gauss sums.

**Theorem 2.** Let $p > 3$ be an odd prime with $p \equiv 3 \pmod{4}$, then we have

$$\sum_{m=1}^{p-1} \sum_{n=1}^{p-1} |C(m, n, 3, 1; p)|^2 \cdot C(mn, p) = \begin{cases} -p \cdot h_p^2 & \text{if } p \equiv 7 \pmod{12}, \\ p \cdot h_p^2 & \text{if } p \equiv 11 \pmod{12}. \end{cases}$$

**Corollary.** For any prime $p > 3$ with $p \equiv 3 \pmod{4}$, we have

$$h_p^2 = \left(\frac{3}{p}\right) \cdot \frac{1}{p} \cdot \sum_{m=1}^{p-1} \sum_{n=1}^{p-1} |C(m, n, 3, 1; p)|^2 \cdot C(mn, p),$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre’s symbol.

For some special positive integers $k$ and $h$, we can also give an identity for

$$\sum_{m=1}^{p-1} \sum_{n=1}^{p-1} |C(m, n, k, h; p)|^2 \cdot C(mn, p).$$
But for general positive integer $q$, whether there exists a computational formula for (1) is an open problem, where $k \geq 3$ and $h \geq 1$ are two integers.

2. SEVERAL LEMMAS

In this section, we shall give several lemmas, which are necessary in the proof of our theorems. First we have the following:

**Lemma 1.** Let $p > 3$ be a prime, then for any integer $n$ with $(n, p) = 1$, we have the identity

$$
\sum_{a=1}^{p} \left( \frac{a^2 + n}{p} \right) = -1,
$$

where $\left( \frac{\cdot}{p} \right)$ denotes the Legendre’s symbol.

**Proof.** This is a well known result, here we give a proof for the sake of completeness. Since $\left( \frac{\cdot}{p} \right) \equiv \chi_2$ is a primitive character mod $p$, so from the properties of Gauss sums $\tau(\chi) = \sum_{a=1}^{p-1} \chi(a)e\left( \frac{a}{p} \right)$ we know that

$$
\sum_{a=1}^{p} \left( \frac{a^2 + n}{p} \right) = \frac{1}{\tau(\chi_2)} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left( \frac{b}{p} \right) e\left( \frac{b(a^2 + n)}{p} \right)
$$

$$
= \frac{1}{\tau(\chi_2)} \sum_{b=1}^{p-1} \left( \frac{b}{p} \right) e\left( \frac{nb}{p} \right) \sum_{a=1}^{p} e\left( \frac{ba^2}{p} \right).
$$

From Theorem 7.5.4 of [2] we know that for any integer $u$ with $(u, p) = 1$, we have

$$
\sum_{a=1}^{p} e\left( \frac{ua^2}{p} \right) = \left( \frac{u}{p} \right) \tau(\chi_2) = \left( \frac{u}{p} \right) \sum_{a=1}^{p-1} \left( \frac{a}{p} \right) e\left( \frac{a}{p} \right).
$$

Then from (2) and (3) we have

$$
\sum_{a=1}^{p} \left( \frac{a^2 + n}{p} \right) = \frac{1}{\tau(\chi_2)} \sum_{b=1}^{p-1} \left( \frac{b}{p} \right) e\left( \frac{nb}{p} \right) \left( \frac{b}{p} \right) \tau(\chi_2) = \sum_{b=1}^{p-1} e\left( \frac{nb}{p} \right) = -1.
$$

This proves Lemma 1. □

**Lemma 2.** Let $p$ be an odd prime with $p \equiv 5 \mod 8$, $\chi_4$ denotes the odd character mod $p$ such that $\chi_4^2 = \left( \frac{\cdot}{p} \right)$. Then we have the identity

$$
\sum_{a=1}^{p} \sum_{b=1}^{p} \chi_4 ((a^3 - b^3)(a - b)) = -\left( \frac{3}{p} \right) \cdot (p - 1) \cdot \sqrt{p} \cdot \frac{\tau(\chi_4)}{\tau(\chi_2)}.
$$
Proof. In fact \(\chi_4(n) = e\left(\frac{ind_g(n)}{4}\right)\), where \(g\) is a primitive root mod \(p\), and \(\text{ind}_g(n)\) denotes the index of \(n\) in primitive root \(g\) mod \(p\) (related contents can be found in [8]). Note that \(\left(\frac{2}{p}\right) = -1\), from the properties of complete residue system mod \(p\) we have

\[
\begin{align*}
4 \sum_{a=1}^{p} \sum_{b=1}^{p} \chi_4 ((a^3 - b^3)(a - b)) &= \\
&= \sum_{a=1}^{p} \chi_4(a^4) + \sum_{a=1}^{p} \sum_{b=1}^{p-1} \chi_4 ((a^3b^3 - b^3)(ab - b)) \\
&= \sum_{a=1}^{p} \chi_4(a^4) + \left(\sum_{b=1}^{p-1} \chi_4(b^4)\right) \left(\sum_{a=1}^{p} \chi_4 ((a^3 - 1)(a - 1))\right) \\
&= (p - 1) \left(1 + \sum_{a=1}^{p} \chi_4 ((a^3 - 1)(a - 1))\right) \\
&= (p - 1) \left(1 + \sum_{a=0}^{p-1} \chi_4 (a^2(a^2 + 3a + 3))\right) \\
&= (p - 1) \left(1 + \sum_{a=1}^{p-1} \chi_4 (a^4) \cdot \chi_4 (1 + 3\tau + 3\tau^2)\right) \\
&= (p - 1) \left(1 + \sum_{a=1}^{p-1} \chi_4 (3a^2 + 3a + 1)\right) \\
&= (p - 1) \left(1 + \frac{2}{p} \sum_{a=1}^{p-1} \chi_4 (12a^2 + 12a + 4)\right) \\
&= (p - 1) \left(1 + \frac{2}{p} \sum_{a=1}^{p-1} \chi_4 (3(2a + 1)^2 + 1)\right) \\
&= (p - 1) \left(\frac{2}{p}\right) \sum_{a=1}^{p} \chi_4 (3(2a + 1)^2 + 1) = -(p - 1) \sum_{a=1}^{p} \chi_4 (3a^2 + 1).
\end{align*}
\]

From the properties of Gauss sums we also have

\[
\begin{align*}
5 \sum_{a=1}^{p} \chi_4 (3a^2 + 1) &= \frac{1}{\tau (\chi_4)} \sum_{a=1}^{p} \sum_{b=1}^{p-1} \chi_4(b) e\left(\frac{b(3a^2 + 1)}{p}\right) \\
&= \frac{1}{\tau (\chi_4)} \sum_{b=1}^{p-1} \chi_4(b) \sum_{a=1}^{p} e\left(\frac{3ba^2}{p}\right)
\end{align*}
\]
\[
= \frac{\sqrt{p}}{\tau(\chi_4)} \left( \frac{3}{p} \right) \sum_{b=1}^{p-1} \chi_4(b) \left( \frac{b}{p} \right) e \left( \frac{b}{p} \right)
\]
\[
= \frac{\sqrt{p}}{\tau(\chi_4)} \left( \frac{3}{p} \right) \sum_{b=1}^{p-1} \chi_4(b) e \left( \frac{b}{p} \right) = \left( \frac{3}{p} \right) \cdot \sqrt{p} \cdot \frac{\tau(\chi_4)}{\tau(\chi_4)},
\]
where we have used the fact that \(\chi_4^2(b) = \left( \frac{b}{q} \right)\). Combining (4) and (5) we have
\[
\sum_{a=1}^{p} \sum_{b=1}^{p} \chi_4((a^3 - b^3)(a - b)) = - \left( \frac{3}{p} \right) \cdot (p - 1) \cdot \sqrt{p} \cdot \frac{\tau(\chi_4)}{\tau(\chi_4)}.
\]
This proves Lemma 2. □

**Lemma 3.** Let \(a, q\) are two integers with \(q \geq 3\) and \((a, q) = 1\). Then we have
\[
C(a, q) = - \frac{1}{\pi^2 \phi(q)} \sum_{\chi \mod q} \chi(-1 = -1) \left( \sum_{n=1}^{\infty} \frac{G(\chi, n)}{n} \right)^2,
\]
where \(\chi\) runs through the Dirichlet characters mod \(q\) with \(\chi(-1) = -1\), and
\[
G(\chi, n) = \sum_{a=1}^{q} \chi(a) e \left( \frac{an}{q} \right)
\]
denotes the Gauss sums corresponding to \(\chi\).

**Proof.** See [5, Lemma 1]. □

### 3. PROOF OF THE THEOREMS

In this section, we shall use the lemmas proved in section two to complete the proof of our theorems. First we prove Theorem 1. If \(p\) is an odd prime then all non-principal characters \(\chi \mod p\) are primitive. Hence by Lemma 3 and the property \(G(\chi, n) = \overline{\chi}(n) \tau(\chi)\) of the Gauss sums associated to primitive characters we have
\[
C(a, p) = - \frac{1}{\pi^2 (p - 1)} \sum_{\chi \mod p} \overline{\chi}(a) \cdot \tau^2(\chi) \cdot L^2(1, \overline{\chi}),
\]
where \(\tau(\chi) = \sum_{b=1}^{p-1} \chi(b) e \left( \frac{b}{p} \right)\).
By this identity and the definition of $C(m, n, k, h; q)$ we have

\begin{equation}
(7) \sum_{m=1}^{p-1} \sum_{n=1}^{p-1} |C(m, n, 3, 1; p)|^2 \cdot C(mn, p) = -\frac{\pi^2}{p-1} \sum_{\chi \mod p \chi(-1) = -1} \tau^2(\chi) \cdot L^2(1, \overline{\chi}).
\end{equation}

\begin{equation}
\cdot \frac{p}{p-1} \sum_{a=1}^{p} \sum_{b=1}^{p-1} \sum_{m=1}^{p-1} \sum_{n=1}^{p-1} \chi(mn) e \left( \frac{m(a^3 - b^3) + n(a-b)}{p} \right)
\end{equation}

\begin{equation}
= \frac{-\pi^2}{p-1} \sum_{\chi \mod p \chi(-1) = -1} L^2(1, \overline{\chi}) \tau^2(\chi) \tau^2(\overline{\chi}) \sum_{a=1}^{p} \sum_{b=1}^{p} \chi(a^3 - b^3) \chi(a-b)
\end{equation}

\begin{equation}
= \frac{-p^2 \cdot \pi^2}{p-1} \sum_{\chi \mod p \chi(-1) = -1} L^2(1, \overline{\chi}) \sum_{a=1}^{p} \sum_{b=1}^{p} \chi(a^3 - b^3) \chi(a-b)
\end{equation}

and

\begin{equation}
(8) \sum_{a=1}^{p} \sum_{b=1}^{p} \chi(a^3 - b^3) \chi(a-b)
= \left( \sum_{b=1}^{p-1} \chi^4(b) \right) \left( 1 + \sum_{a=1}^{p-1} \chi(a^2 + a + 1) \chi^2(a-1) \right)
= \left( \sum_{b=1}^{p-1} \chi^4(b) \right) \left( \sum_{a=1}^{p} \chi(a^2 + a + 1) \chi^2(a-1) \right),
\end{equation}

where we have used the identity $\tau(\chi)\tau(\overline{\chi}) = \tau(\chi)\overline{\tau(\chi)}\overline{\chi(-1)} = -p$ in (7).

If $p = 8k + 1$, then for any character $\chi \mod p$ with $\chi(-1) = -1$, $\chi^4$ is not a principal character mod $p$, so we have the identity

\begin{equation}
(9) \sum_{b=1}^{p-1} \chi^4(b) = 0.
\end{equation}

If $p = 8k + 5$, then there exist two and only two characters $\chi_4$ and $\overline{\chi}_4$ mod $p$ with $\chi_4(-1) = \overline{\chi}_4(-1) = -1$ such that $\chi^4_4$ is a principal character mod $p$. If $\chi = \chi_4$ or $\overline{\chi}_4$, then

\begin{equation}
(10) \sum_{b=1}^{p-1} \chi^4(b) = p - 1.
\end{equation}

For any other character $\chi \mod p$ with $\chi(-1) = -1$, we have

\begin{equation}
(11) \sum_{b=1}^{p-1} \chi^4(b) = 0.
\end{equation}
If \( p > 3 \) and \( p \equiv 3 \mod 4 \), then for any character \( \chi \mod p \) with \( \chi(-1) = -1 \), \( \chi^4 \) is also not a principal character \( \mod p \), except \( \chi = \left( \frac{\cdot}{p} \right) \), the Legendre’s symbol. In this case, from the properties of the complete residue system \( \mod p \) and Lemma 1 we have the identities

\[
\sum_{b=1}^{p-1} \left( \frac{b}{p} \right)^4 = p - 1
\]

and

\[
\sum_{a=1}^{p} \left( \frac{a^2 + a + 1}{p} \right) \left( \frac{a - 1}{p} \right)^2 = \sum_{a=1}^{p} \left( \frac{a^2 + a + 1}{p} \right) - \left( \frac{3}{p} \right)
\]
\[
= \sum_{a=1}^{p} \left( \frac{4a^2 + 4a + 4}{p} \right) - \left( \frac{3}{p} \right) = \sum_{a=1}^{p} \left( \frac{(2a + 1)^2 + 3}{p} \right) - \left( \frac{3}{p} \right)
\]
\[
= \sum_{a=1}^{p} \left( \frac{a^2 + 3}{p} \right) - \left( \frac{3}{p} \right) = -1 - \left( \frac{3}{p} \right).
\]

Now if \( p = 8k + 1 \), then from (7), (8) and (9) we have

\[
\sum_{m=1}^{p-1} \sum_{n=1}^{p-1} |C(m, n, 3, 1; p)|^2 \cdot C(mn, p) = 0.
\]

If \( p = 8k + 5 \), then from (7), (10), (11) and Lemma 2 we have

\[
\sum_{m=1}^{p-1} \sum_{n=1}^{p-1} |C(m, n, 3, 1; p)|^2 \cdot C(mn, p)
\]
\[
= - \left( \frac{3}{p} \right) \cdot \frac{p^2}{\pi^2} \left\{ \tau^2(\chi_4)L^2(1, \chi_4) + \tau^2(\chi_4)L^2(1, \chi_4) \right\}.
\]

Now, our Theorem 1 follows from (14) and (15).

If \( p > 3 \) and \( p \equiv 3 \mod 4 \), then note that \( L(1, \chi_2) = \pi h_p / \sqrt{p} \), from (7), (8), (12) and (13) we have the identity

\[
\sum_{m=1}^{p-1} \sum_{n=1}^{p-1} |C(m, n, 3, 1; p)|^2 \cdot C(mn, p)
\]
\[
= \frac{-p^2}{\pi^2(p - 1)} \cdot L^2(1, \chi_2) \cdot (p - 1) \cdot \left( 1 - \frac{3}{p} \right) = \left( \frac{3}{p} \right) p \cdot h_p^2
\]
\[
= \begin{cases} 
-p \cdot h_p^2 & \text{if } p = 12k + 7, \\
 p \cdot h_p^2 & \text{if } p = 12k + 11.
\end{cases}
\]

This completes the proof of Theorem 2. \( \square \)
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