THE LEVI PROBLEM IN THE BLOW-UP ALONG A LINEAR SUBSPACE

NATALIA GAŞIŢOI

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We prove that a locally Stein open subset of the blow-up X of $\mathbb{C}^{n+1}$ along a k-dimensional linear subspace L is Stein if and only if it does not contain an open subset of the form $U \setminus (L \times \mathbb{P}^{n-k})$, where U is an open neighborhood of $t_0 \times \mathbb{P}^{n-k}$, for some $t_0 \in L$.

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1. INTRODUCTION AND PRELIMINARIES

A complex manifold X is called Stein if it is holomorphically convex and its global holomorphic functions separate points and give local coordinates at every point.

From Oka's characterization of Stein domains in $\mathbb{C}^n$ (see [4]) follows that the Steiness of a domain $\Omega \subset \mathbb{C}^n$ is in fact a local property of its boundary. More precisely, an open domain Ω of $\mathbb{C}^n$ is Stein if and only if it is locally Stein in the sense that for each boundary point $x \in \partial\Omega$ there exists an open neighborhood $V = V(x)$ such that $V \cap \Omega$ is Stein.

In the case of a general complex space X the Levi problem asks whether a domain in X is Stein if it is locally Stein.

R. Fujita [2] and A. Takeuchi [5] showed that for complex projective space there is a similar result as in $\mathbb{C}^n$: a locally Stein domain over $\mathbb{P}^n$ either is Stein or coincides with $\mathbb{P}^n$.

Let L denote a k-dimensional linear subspace of $\mathbb{C}^{n+1}$ and let X be the blow-up of $\mathbb{C}^{n+1}$ along L. Suppose that D is a locally Stein open subset of X. The aim of this paper is to give an answer to the following question: under what additional geometrical conditions D is Stein?

The motivation of this work is the recent paper of M. Colţoiu and C. Joiţa [1] in which the authors considered this question and gave the answer for the case of open subsets of the blow-up of $\mathbb{C}^{n+1}$ at a point.

Let $A = L \times \mathbb{P}^{n-k}$ be the exceptional divisor of X. We shall say that an open subset D of the blow-up X *satisfies the condition* (P) if there exist a point t_0 in L and an open neighborhood U of $t_0 \times \mathbb{P}^{n-k}$ such that $U \setminus A$ is contained in D.

The main result of this paper is the following theorem.

THEOREM 1. *Let D be a locally Stein open subset of the blow-up X. Then D is Stein if and only if the condition (P) does not hold.*

Definition 1. Let M be a complex manifold. Suppose that $A \subset M$ is an analytic subset of M of positive codimension and D is an open subset of $M \setminus A$. A boundary point $z \in \partial D \cap A$ is called removable along A if there exists an open neighborhood U of z such that $U \setminus A$ is contained in D.

In the proof we will use the same technique as in [1]. An essential role will play the fact that if M is a Stein manifold, A is a closed analytic subset of M of positive codimension and if D is locally Stein at every point of $\partial D \setminus A$ and there are not points in $\partial D \cap A$ which are removable along A, then D is Stein (see [3], [6]).

2. PROOF OF THEOREM 1

As X is the blow-up of $\mathbb{C}^{n+1}$ along L, it is in fact a line bundle over $L \times \mathbb{P}^{n-k}$ and we will denote by $\pi : X \to L \times \mathbb{P}^{n-k}$ the corresponding vector bundle projection.

Let $t_1, t_2, \dots, t_k$ be the coordinate functions in L and denote by $t = (t_1, t_2, \dots, t_k)$. Let $z_0, z_1, \dots, z_{n-k}$ be the coordinate functions in $\mathbb{C}^{n-k+1}$ and denote by

$$z = (z_0, z_1, \dots, z_{n-k}) \in \mathbb{C}^{n-k+1} \setminus \{0\}, \quad [z] = [z_0 : z_1 : \dots : z_{n-k}] \in \mathbb{P}^{n-k}.$$

For $i = 0, 1, \dots, n-k$ we consider the sets

$$U_i = \{(t, [z]) \in L \times \mathbb{P}^{n-k} \mid z_i \neq 0\}.$$

The standard local trivializations we shall denote by

$$\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}, \quad i = 0, 1, \dots, n-k.$$

For each $(t, [z], \lambda) \in (U_j \cap U_i) \times \mathbb{C}$ we have that

$$\left(\psi_j \circ \psi_i^{-1}\right)(t, [z], \lambda) = \left(t, [z], \frac{z_j}{z_i}\lambda\right). \tag{1}$$

For $i = 0, 1, \dots, n-k$ we consider the sets

$$W_i = \{(t, z, \lambda) \in L \times (\mathbb{C}^{n-k+1} \setminus \{0\}) \times \mathbb{C} \mid z_i \neq 0\}.$$

Let us define a holomorphic mapping $F : (L \times (\mathbb{C}^{n-k+1} \setminus \{0\}) \times \mathbb{C}) \to X$, in the following way, for $(t, z, \lambda) \in W_i$ we put

$$F(t, z, \lambda) = \psi_i^{-1}(t, [z], z_i \lambda).$$

Notice that for every $(t, [z], \lambda) \in (U_j \cap U_i) \times \mathbb{C}$ it follows from (1) that

$$(\psi_j \circ \psi_i^{-1})(t, [z], z_i \lambda) = (t, [z], z_j \lambda)$$

and then, we have that

$$\psi_i^{-1}(t, [z], z_i \lambda) = \psi_j^{-1}(t, [z], z_j \lambda).$$

Thus, the mapping F is well defined. Furthermore, since the mapping

$$(t, z, \lambda) \in W_i \to (t, [z], z_i \lambda) \in U_i \times \mathbb{C}$$

is a surjection, we conclude that $F|_{W_i} : W_i \to \pi^{-1}(U_i)$, is a surjection itself.

We shall verify that F is a local trivial fibration over X with fiber $\mathbb{C}^*$ and that the transition functions are linear on each fiber.

For $i = 0, 1, \ldots, n-k$ we denote by

$$\Phi_i : W_i \to \pi^{-1}(U_i) \times \mathbb{C}^*, \quad \Phi_i(t, z, \lambda) = (F(t, z, \lambda), z_i)$$

and we claim that Φ_i are the local trivializations. In order to show that the mappings Φ_i are invertible and well defined, for every $i = 0, 1, \ldots, n-k$ we consider the mappings

$$\widetilde{\Phi}_i : W_i \to (U_i \times \mathbb{C}) \times \mathbb{C}^*, \quad \widetilde{\Phi}_i(t, z, \lambda) = ((t, [z], z_i \lambda), z_i)$$

and

$$\chi_i : (U_i \times \mathbb{C}) \times \mathbb{C}^* \to \pi^{-1}(U_i) \times \mathbb{C}^*, \quad \chi_i((t, [z], \lambda), \mu) = (\psi_i^{-1}(t, [z], \lambda), \mu).$$

Therefore, $\Phi_i = \chi_i \circ \widetilde{\Phi}_i$.

Since $\widetilde{\Phi}_i$ are invertible and for every $((t, [z], \lambda), \mu) \in (U_i \times \mathbb{C}) \times \mathbb{C}^*$ we have

$$\widetilde{\Phi}_i^{-1}((t, [z], \lambda), \mu) = \left(t, \frac{\mu}{z_i} z, \frac{\lambda}{\mu}\right), \tag{2}$$

and the mappings χ_i are invertible too and for every $(\zeta, \mu) \in \pi^{-1}(U_i) \times \mathbb{C}^*$ we have

$$\chi_i^{-1}(\zeta, \mu) = (\psi_i(\zeta), \mu), \tag{3}$$

we conclude that Φ_i are invertible and from (1), (2) and (3) it follows that for all $(\zeta, \mu) \in \pi^{-1}(U_j \cap U_i) \times \mathbb{C}^*$, if $\pi(\zeta) = (t, [z])$, then

$$\left(\Phi_j \circ \Phi_i^{-1}\right)(\zeta, \mu) = \left(\zeta, \frac{z_j}{z_i} \mu\right)$$

and our claim is proved.

Since F is a local trivial fibration over X with fiber $\mathbb{C}^*$ there exists a holomorphic line bundle $\widetilde{F} : Z \to X$, such that $Z \setminus Z_0 = L \times (\mathbb{C}^{n-k+1} \setminus \{0\}) \times \mathbb{C}$ where Z_0 is the zero section and $F = \widetilde{F}|_{L\times(\mathbb{C}^{n-k+1}\setminus\{0\})\times\mathbb{C}}$.

We consider that D is an open subset of X and is locally Stein, but not Stein. Since $F^{-1}(D)$ is an open subset of $L \times \mathbb{C}^{n-k+1} \times \mathbb{C}$, according to [1, Lemma 1], $F^{-1}(D)$ is locally Stein at every point of $(\partial F^{-1}(D)) \setminus (L \times \{0\} \times \mathbb{C})$ and is not Stein. Hence, we deduce that there exists a boundary point $P \in (\partial F^{-1}(D)) \cap (L \times \{0\} \times \mathbb{C})$ which is removable along $L \times \{0\} \times \mathbb{C}$ (see [3], [6]). Therefore, there exist $t_0 \in L$ and $\lambda_0 \in \mathbb{C}$ such that

$$(t_0, 0, \lambda_0) \in (\partial F^{-1}(D)) \cap (L \times \{0\} \times \mathbb{C}) \subset L \times \mathbb{C}^{n-k+1} \times \mathbb{C}$$

is removable along $L \times \{0\} \times \mathbb{C}$. Then, according to Definition 1 there exists $\varepsilon > 0$ such that

$$F^{-1}(D) \supset \{(t, z, \lambda) \in L \times (\mathbb{C}^{n-k+1} \setminus \{0\}) \times \mathbb{C} : \|t - t_0\| < \varepsilon, \\ |z_j| < \varepsilon,\ \forall j = \overline{0, n-k},\ |\lambda - \lambda_0| < \varepsilon\}.$$

We shall prove that D possess the property (P), i.e., D contains an open subset of the form $U \setminus A$, where A is the exceptional divisor of X and U is an open neighborhood of $t_0 \times \mathbb{P}^{n-k}$.

We denote by B the open ball $B_\varepsilon(t_0) = \{t \in L : \|t - t_0\| < \varepsilon\}$ and by Ω_δ the punctured open disk $\{\lambda \in \mathbb{C} : 0 < |\lambda| < \delta\}$. The proof will be complete if we will show that for every $[z] \in \mathbb{P}^{n-k}$ and every $i \in \{0, 1, \ldots, n-k\}$ such that $(t, [z]) \in U_i$ there exist an open set V in $\mathbb{P}^{n-k}$ and a positive number $\delta > 0$ such that $(t, [z]) \in (B \times V)$ and

$$\psi_i(D \cap \pi^{-1}(U_i)) \supset B \times V \times \Omega_\delta.$$

Let $(t, [\widetilde{z}]) \in U_i$ be an arbitrarily fixed point in U_i, and such that $t \in B$. Let choose a positive number T such that

$$T > \max\left\{\frac{|\widetilde{z}_j|}{|\widetilde{z}_i|} : j = 0, 1, \ldots, n-k\right\}.$$

We fix arbitrarily a point $\lambda_1 \in \mathbb{C}^*$ such that $|\lambda_1 - \lambda_0| < \varepsilon$ and we take $\delta \in \mathbb{R}_+$ such that $\delta < \frac{\varepsilon}{T} \cdot |\lambda_1|$. In order to show that $\psi_i(D \cap \pi^{-1}(U_i))$ contains an open set of the form $B \times V \times \Omega_\delta$, we consider an arbitrary point $(t, [w], \nu)$ in $B \times V \times \Omega_\delta$. We put $\mu = \frac{\nu}{\lambda_1}$ and observe that $\mu \neq 0$ and $|\mu| = \frac{|\nu|}{\lambda_1} < \frac{\delta}{|\lambda_1|} < \frac{\varepsilon}{T}$. Let $z = \frac{\mu}{w_i} \cdot w$. Thus, $z_i = \mu \neq 0$ and therefore, $(t, z, \lambda_1) \in W_i$.

For every $j = 0, 1, \ldots, n-k$, we have

$$|z_j| = \frac{|\mu \cdot w_j|}{|w_i|} \leq |\mu| \cdot T < \varepsilon,$$

hence, $(t, z, \lambda_1) \in F^{-1}(D) \cap W_i$. Therefore, $F(t, z, \lambda_1) = \psi_i^{-1}(t, [z], z_i\lambda_1) \in (D \cap \pi^{-1}(U_i))$. Finally, we have

$$\psi_i(F(t, z, \lambda_1)) = (t, [z], z_i\lambda_1) = (t, [z], \mu\lambda_1) = (t, [w], \nu).$$

The proof of Theorem 1 is now complete.

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"Alecu Russo" State University
Department of Mathematics
Str. Pushkin 38, MD-3121 Balti
Republic of Moldova
natalia_gasitoi@yahoo.com