A $k$-bridge hypergraph is an $h$-uniform linear hypergraph consisting of $k$ linear paths having common ends. In this note it is shown that every two chromatically equivalent $k$-bridge hypergraphs are isomorphic if $k \geq 3$. This solves in affirmative an open question raised by Bokhary et al. [2], where a supplementary condition on the multiplicities of path lengths was imposed.

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1. NOTATION AND PRELIMINARY RESULTS

An $h$-uniform hypergraph $(h \geq 2)$ $H = (V, \mathcal{E})$ of order $n = |V|$ and size $m = |\mathcal{E}|$, consists of a vertex set $\mathcal{V}(H) = V$ and edge set $\mathcal{E}(H) = \mathcal{E}$, where $E \subset \mathcal{V}$ and $|E| = h$ for each edge $E$ in $\mathcal{E}$. $H$ is said to be linear if $0 \leq |E \cap F| \leq 1$ for any two distinct edges $E, F \in \mathcal{E}(H)$ [1].

Let $P_{p}^{h,1}$ denote the linear path consisting of $p \geq 1$ edges $E_1, \ldots, E_p$ such that $|E_1| = \ldots = |E_p| = h$, $|E_k \cap E_l| = 1$ if $\{k, l\} = \{i, i+1\}$ for any $1 \leq i \leq p - 1$ and 0 otherwise.

Each vertex from $E_1 \setminus E_2$ or from $E_p \setminus E_{p-1}$ will be called an end vertex of $P_{p}^{h,1}$.

For any positive integers $a_1, \ldots, a_k \in \mathbb{N}$ and $h \geq 2$ we denote by $\theta(h; a_1, \ldots, a_k)$ the $h$-uniform linear hypergraph consisting of $k$ linear paths $P_{a_1}^{h,1}, P_{a_2}^{h,1}, \ldots, P_{a_k}^{h,1}$ of lengths $a_1, a_2, \ldots, a_k$ respectively, having in common only two fixed ends. It is a parallel hypergraph [4] of order $(h - 1)(a_1 + \ldots + a_k) - k + 2$. For example, $\theta(3; 4, 3, 5)$ is depicted in Fig. 1 with $x$ and $y$ common ends of the paths. This linear hypergraph will be called a $k$-bridge hypergraph; $\theta(2; a_1, \ldots, a_k)$ is a notation for a $k$-bridge graph (see [3]).

A $\lambda$-coloring of a hypergraph $H$ is a function $f : \mathcal{V}(H) \to \{1, \ldots, \lambda\}$ such that each edge $E \in \mathcal{E}(H)$ contains two vertices $x$ and $y$ having different colors $f(x) \neq f(y)$. The number of $\lambda$-colorings of $H$ is given by a polynomial $P(H, \lambda)$.
in \( \lambda \), called the chromatic polynomial of \( H \), of degree equal to \(|V(H)|\). Two hypergraphs \( H \) and \( G \) are said to be chromatically equivalent if they have the same chromatic polynomial, i.e., \( P(H, \lambda) = P(G, \lambda) \).

The chromatic polynomial of \( k \)-bridge hypergraphs was deduced in [2]:

**Lemma 1.1.** For every \( h \geq 2 \) and \( k, a_1, \ldots, a_k \geq 1 \) we have

\[
P(\theta(h; a_1, \ldots, a_k), \lambda) = \frac{1}{\lambda^{k-1}} \prod_{i=1}^{k} ((\lambda^{h-1} - 1)^{a_i} + (-1)^{a_i}(\lambda - 1))
+ \frac{\lambda - 1}{\lambda^{k-1}} \prod_{i=1}^{k} ((\lambda^{h-1} - 1)^{a_i} - (-1)^{a_i}).
\]

In [2] it was proved that if \( k, h \geq 3; 2 \leq a_1 \leq \ldots \leq a_k; 2 \leq b_1 \leq \ldots \leq b_k \) and any number in the multisets \( \{a_1, \ldots, a_k\} \) and \( \{b_1, \ldots, b_k\} \) has a multiplicity less than \( 2^{h-1} - 1 \), then chromatic equivalence between \( \theta(h; a_1, \ldots, a_k) \) and \( \theta(h; b_1, \ldots, b_k) \) implies \( a_i = b_i \) for all \( i = 1, \ldots, k \). This means that \( \theta(h; a_1, \ldots, a_k) \) and \( \theta(h; b_1, \ldots, b_k) \) are isomorphic hypergraphs.

An open question raised in [2] was to decide whether the condition on the multiplicities of path lengths can be removed, since this can be done at least for \( h = 3 \) and a similar property also holds for \( k \)-bridge graphs \( (h = 2) \) [3].

In the next section, we shall prove that this can be done for all \( k, h \geq 3 \).

**2. MAIN RESULT**

First, we need a lemma concerning a non-divisibility property in a polynomial ring.

**Lemma 2.1.** Let \( m, n \geq 2 \) be natural numbers. The polynomial \( \lambda^n - 1 \) does not divide \( (\lambda - 1)^m + (-1)^m(\lambda - 1) \).

**Proof.** Suppose that \( \lambda^n - 1 \) divides \( (\lambda - 1)^m + (-1)^m(\lambda - 1) \). It follows that \( m \geq n \) and \( P(\lambda) = \lambda^{n-1} + \lambda^{n-2} + \ldots + 1 \) divides \( Q(\lambda) = (\lambda - 1)^{m-1} + (-1)^m \).

This means that all roots of \( P(\lambda) \) are also roots of \( Q(\lambda) \). The roots of \( P(\lambda) \) are all roots of order \( n \) of unity which are different from 1, i.e., complex
numbers \( \cos \frac{2\pi t}{n} + i\sin \frac{2\pi t}{n} \), where \( t = 1, \ldots, n - 1 \). The roots of \( Q(\lambda) \) can be easily obtained from the roots of unity by the substitution \( \lambda - 1 = \mu \) and they are numbers \( 1 + \cos \frac{2\pi t}{m-1} + i\sin \frac{2\pi t}{m-1} \) for \( m \) odd and \( 1 + \cos \frac{\pi(2t+1)}{m-1} + i\sin \frac{\pi(2t+1)}{m-1} \)

for \( m \) even (\( 0 \leq t \leq m - 2 \)).

It is necessary to see if there exist \( \alpha, \beta \) such that \( 0 \leq \alpha, \beta < 2\pi \) and \( 1 + \cos \beta + i\sin \beta = \cos \alpha + i\sin \alpha \). We deduce \( \cos \alpha = 1 + \cos \beta \) and \( \sin \alpha = \sin \beta \), hence \( (1 + \cos \beta)^2 + \sin^2 \beta = 1 \), which implies \( \cos \beta = -\frac{1}{2} \). We get \( \beta_1 = \frac{2\pi}{3} \) and \( \beta_2 = \frac{4\pi}{3} \) and corresponding values \( \alpha_1 = \frac{\pi}{3} \) and \( \alpha_2 = \frac{5\pi}{3} \).

Consequently, only two roots of \( P(\lambda) \) can possibly be roots of \( Q(\lambda) \). It follows that \( P(\lambda) \) does not divide \( Q(\lambda) \) for \( n \geq 4 \).

It remains to verify this property for \( n = 2 \) and \( n = 3 \). For \( n = 2 \) we have \( P(\lambda) = \lambda + 1 \) and \( Q(-1) \neq 0 \). If \( n = 3 \) the roots of \( P(\lambda) \) are \( \varepsilon \) and \( \varepsilon^2 \), where \( \varepsilon = \frac{-1+i\sqrt{3}}{2} \) is a cubic root of unity. Suppose that \( Q(\varepsilon) = 0 \). This would imply \( |\varepsilon - 1| = 1 \). But \( |\varepsilon - 1| = \sqrt{3} \), a contradiction. \( \square \)

**Theorem 2.2.** Let \( k, h \geq 3 \). If \( k \)-bridge hypergraphs \( H_1 \) and \( H_2 \) are chromatically equivalent, then they are isomorphic hypergraphs.

**Proof.** Suppose that \( H_1 = \theta(h; a_1, \ldots, a_k) \) and \( H_2 = \theta(h; b_1, \ldots, b_k) \) with \( 1 \leq a_1 \leq \ldots \leq a_k \) and \( 1 \leq b_1 \leq \ldots \leq b_k \).

By hypothesis we have \( P(H_1, \lambda) = P(H_2, \lambda) \). It is necessary to prove that \( a_i = b_i \) for all \( i = 1, \ldots, k \).

By Lemma 1.1 we get after reductions

\[
(1) \quad \lambda^{k-1} P(H_1, \lambda) = \lambda^{h-1} - 1)^a_1 + \ldots + a_k + \sum_{i=2}^{k} \sum_{K \subset \{1, \ldots, k\}} (\lambda - 1)^i (\lambda - 1) \times (-1)^{a_K} (\lambda^{h-1} - 1)^a_K,
\]

where we have denoted \( K = \{1, \ldots, k\} \setminus K \), \( a_k = \sum_{j \in K} a_j \) and \( a_\emptyset = 0 \). Since \( P(H_1, \lambda) = P(H_2, \lambda) \) it follows that \( \sum_{i=1}^{k} a_i = \sum_{i=1}^{k} b_i \) because the terms of the highest degree in both polynomials must be equal.

We denote

\[
S_1 = \sum_{i=2}^{k-1} \sum_{|K|=i} (\lambda - 1)^i (\lambda - 1) (-1)^{a_K} (\lambda^{h-1} - 1)^a_K
\]

and let \( S_2 \) be the corresponding sum for \( b_1, \ldots, b_k \).

By equating \( \lambda^{k-1} P(H_1, \lambda) \) and \( \lambda^{k-1} P(H_2, \lambda) \), from (1) we get \( S_1 = S_2 \) since the free terms in (1) coincide for \( H_1 \) and \( H_2 \).
Suppose that $a_1 \neq b_1$. Without loss of generality we can consider $a_1 < b_1$. Let $p$ denote the multiplicity of $a_1 (1 \leq p \leq k)$. If $p = k$ it follows that $a_1 = \ldots = a_k$. We deduce that $\sum_{i=1}^{k} a_i = ka_1 < kb_1 \leq \sum_{i=1}^{k} b_i$, a contradiction.

Consequently, $p \leq k - 1$ hence, $a_1 = \ldots = a_p < a_{p+1} \leq \ldots \leq a_k$. It can be seen that the term of $S_1$ containing the smallest power of $\lambda^{h-1} - 1$ corresponds to $i = k - 1$ and choices $\overline{K} = \{1\}, \{2\}, \ldots, \{p\}$ and is equal to

$$R(\lambda) = p((\lambda - 1)^{k-1} + (-1)^{k-1}(\lambda - 1))(-1)^{\sum_{i=1}^{k} a_i - a_1}(\lambda^{h-1} - 1)^{a_1}.$$ 

Since $b_1 > a_1$, or $b_1 \geq a_1 + 1$, the expression $S_2$ is divisible by $(\lambda^{h-1} - 1)^{a_1+1}$. Also, $S_1 - R(\lambda)$ is divisible by $(\lambda^{h-1} - 1)^{a_1+1}$ since $a_p+1, \ldots, a_k \geq a_1+1$.

It follows that $R(\lambda) = S_2 - (S_1 - R(\lambda))$ is also divisible by $(\lambda^{h-1} - 1)^{a_1+1}$, which means that $(\lambda - 1)^{k-1} + (-1)^{k-1}(\lambda - 1)$ is divisible by $\lambda^{h-1} - 1$. This is impossible by Lemma 2.1 since $h, k \geq 3$. It follows that $a_1 = b_1$.

Let $m$ be such that $2 \leq m \leq k - 1$. Suppose we have proved that $a_i = b_i$ for $i = 1, \ldots, m - 1$ and $a_m < b_m$ holds.

By canceling all equal terms from both sides of the equation $S_1 = S_2$ we get another equation, denoted by $S_1^1 = S_2^1$.

In $S_1^1$ and $S_2^1$ the second sum is over all subsets $K \subset \{1, \ldots, k\}$ with $|K| = i$ such that $\overline{K} \not\subset \{1, \ldots, m - 1\}$, since the corresponding terms in $S_1$ and $S_2$ have been canceled because $\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} b_i$. In this case, the minimum of $a_{\overline{K}}$ is reached only for $i = k - 1$ and $\overline{K} = \{m\}, \{m+1\}, \ldots,$ or $\{m + q - 1\}$, and the corresponding term of $S_1^1$ equals $q((\lambda - 1)^{k-1} + (-1)^{k-1}(\lambda - 1))(-1)^{\sum_{i=1}^{k} a_i - a_m}(\lambda^{h-1} - 1)^{a_m}$ if the multiplicity of $a_m$ is $q \geq 1$.

A similar argument as above shows that $a_m = b_m$.

If $m = k$, then $a_k = \sum_{i=1}^{k} a_i - \sum_{i=1}^{k-1} a_i = \sum_{i=1}^{k} b_i - \sum_{i=1}^{k-1} b_i = b_k$. Therefore, $a_i = b_i$ for $i = 1, \ldots, k$, which concludes the proof. □

Note that for $k = 2$ the property is not true for $a_1 + a_2 \geq 4$ since all 2-bridge hypergraphs $\theta(h; a_1, a_2)$ having $a_1 + a_2 = m \geq 4$ represent the same linear cycle with $m$ edges.

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