DUALITY FOR MULTIOBJECTIVE PROGRAMMING INVOLVING SECOND ORDER GENERALIZED TYPE I FUNCTIONS

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In this paper we consider a new class of (F, α, ρ, d)-type I functions to second order and relative to this class we establish several sufficient optimality conditions and mixed duality results for a multiobjective programming problem.

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1. **INTRODUCTION**

The study of higher-order duality is significant due to the computational advantage over first-order duality as it provides higher bounds for the value of the objective function when approximations are used (Mangasarian [4], Yang [10]). Mangasarian [4] and Hason [3] had also indicated, by example, that one advantage of second-order duality when applicable, is that if a feasible solution in the primal problem is given and first-order duality conditions do not apply, then we can use second-order duality to provide a lower bound of the value in the primal problem (see also Yang [10] for new and interesting results in this sense). In [11], Zhang and Mond introduced a concept of second order (F, ρ)-convexity and gave some duality results. Recently, Yang and all [9], Aghezzaf and Hachimi [2], Preda [5, 7] had obtained results for higher-order duality in multiobjective mathematical programming. The class of (F, α, ρ, d)-type I functions considered in this paper is more general than the classes presented in [9–11].

Consider the following nonlinear multiobjective programming problem:

(MOP)   minimize $f(x) = (f_1(x) , \ldots , f_m(x))$
            subject to $x \in A = \{x \in X \; / \; g(x) \leqq 0\}$,

where $X \subset R^n$, X open, and $f : X \to R^m$, $g : X \to R^q$ are twice differentiable functions on A.

This paper is organized as follows. In Section 2 we defined a new class of functionals (F, α, ρ, d)-type I and some concepts of generalized convexity type. Thus, in Section 3, relative to the dual of Mond-Weir type, some duality results are given.

2. SOME PRELIMINARIES

In this section, we introduce the notion of function q-sublinear and give some definitions of second order quasi and pseudoquasi.

Definition 2.1 ([2]). We say that y∈A is an efficient solution for problem (MOP) if and only if there exists no x∈A such that f(x) $\leq$ f(y).

Definition 2.2 ([2]). We say that y∈A is a weak efficient solution for problem (MOP) if and only if there exists no x∈A such that f(x) $<$ f(y).

Definition 2.3. A functional F : X $\times$ X $\times$ R^n $\to$ R is q-sublinear if

$$(1a) \quad F(x, y \; ; \sum_{i=1}^{k} a_i) \leq \max_{i=\overline{1,k}} F(x, y \; ; a_i) \; , \; \forall \; x, y \in X, a_i \in R^n \; , \; i \in \overline{1,k} \; , \; k \in N^*$$

and

$$(1b) \qquad F(x, y \; ; \alpha a) = \alpha \, F(x, y \; ; a) \; , \; \forall \; x, y \in X, \alpha \in R, \alpha \geqq 0, a \in R^n \; .$$

We see that the class of F-sublinear functionals is more large then the class of sublinear functionals considered, for example, in Preda [7].

Let F be a q-sublinear functional and suppose the functions f = (f_1, ..., f_m) : X $\to$ R^m and h = (h_1, ... , h_r) : X $\to$ R^r are twice differentiable at y∈X. Let ρ = (ρ^1, ρ^2), where ρ^1 = (ρ_1, ..., ρ_m)∈R^m, ρ^2 = (ρ_{m+1}, ..., ρ_{m+r})∈R^r. Let α = (α^1, α^2) where α^1: X $\times$ X $\to$ R$_+^*$, α^2 : X $\times$ X $\to$ R$_+^*$ and let d($\cdot$, $\cdot$) : X $\times$ X $\to$ [0, $+\infty$).

For the sake of simplicity, we will use the following notations

if f : X $\to$ R and β : X $\times$ X $\to$ R$_+^*$, then Δ(f, x, y, β, p) = β(x , y)[∇f(y) + ∇^2f(y)p]

and

if f : X $\to$R^m then F(x, y ; Δ(f, x, y, β, p)) = (F(x, y ; Δ(f$_1$, x, y, β, p), ... , F(x, y ; Δ(f$_m$, x, y, β, p)).

Definition 2.4 ([2]). (f , h) is said to be second order (F, α, ρ, p, d)-type I at y, if for all x∈A we have

$$(2a) \quad f(x) - f(y) + \tfrac{1}{2}p\nabla^2 f(y)p \geqq F(x, y \; ; \Delta(f, x, y, \alpha^1, p)) + \rho^1 d^2(x \; , \; y)$$

$$(2b) \quad \text{-} \, h(y) + \tfrac{1}{2}p\nabla^2 \, h(y)p \geqq F(x, y \; ; \Delta(h, x, y, \alpha^2, p)) + \rho^2 d^2(x \; , \; y).$$

Definition 2.5 ([2]). (f , h) is said to be second order quasi (F, α, ρ, p, d)-type I at y, if for all x$\in$A we have

(3a) $f(x) \leqq f(y) - \frac{1}{2}p \nabla^2 f(y)p \implies F(x, y ; \Delta(f, x, y, \alpha^1, p)) \leqq -\rho^1 d^2 (x , y)$

(3b) $- h(y) + \frac{1}{2}p\nabla^2 h(y)p \leqq 0 \implies F(x, y ; \Delta(h, x, y, \alpha^2, p)) \leqq -\rho^2 d^2(x , y)$.

If in the above definition, x $\neq$ y and inequality (3b) is satisfied as

(3c) $- h(y) + \frac{1}{2}p\nabla^2 h(y)p \leqq 0 \implies F(x, y ; \Delta(h, x, y, \alpha^2, p)) < -\rho^2 d^2(x , y)$

then we say that (f , h) is second order quasistrictly-pseudo (F, α, ρ, p, d)-type I at y.

Definition 2.6 ([2]). (f , h) is said to be second order pseudoquasi (F, α, ρ, p, d)-type I at y, if for all x$\in$A we have:

(4a) $f(x) < f(y) - \frac{1}{2}p\nabla^2 f(y)p \implies F(x, y ; \Delta(f, x, y, \alpha^1, p)) < - \rho^1 d^2(x , y)$

(4b) $- h(y) + \frac{1}{2}p\nabla^2 h(y)p \leqq 0 \implies F(x, y ; \Delta(h, x, y, \alpha^2, p)) \leqq -\rho^2 d^2(x , y)$.

If in the above definition, x $\neq$ y and inequality (4a) is satisfied as

(4c) $f(x) \leqq f(y) - \frac{1}{2}p\nabla^2 f(y)p \implies F(x, y ; \Delta(f, x, y, \alpha^1, p)) < - \rho^1 d^2(x , y)$

then we say that (f , h) is second order strictly pseudoquasi (F, α, ρ, p, d)-type I at y.

Definition 2.7 ([2]). (f , h) is said to be second order weak strictly-pseudoquasi (F, α, ρ, p, d)-type I at y, if for all x$\in$A we have:

(5a) $f(x) \leq f(y) - \frac{1}{2}p\nabla^2 f(y)p \implies F(x, y ; \Delta(f, x, y, \alpha^1, p)) < -\rho^1 d^2(x , y)$

(5b) $- h(y) + \frac{1}{2}p\nabla^2 h(y)p \leqq 0 \implies F(x, y ; \Delta(h, x, y, \alpha^2, p)) \leqq -\rho^2 d^2 (x , y)$.

Definition 2.8 ([2]). (f , h) is said to be second order strong pseudoquasi (F, α, ρ, p, d)-type I at y, if for all x$\in$A we have:

(6a) $f(x) \leq f(y) - \frac{1}{2}p\nabla^2 f(y)p \implies F(x, y ; \Delta(f, x, y, \alpha^1, p)) \leq - \rho^1 d^2(x , y)$

(6b) $- h(y) + \frac{1}{2}p\nabla^2 h(y)p \leqq 0 \implies F(x, y ; \Delta(h, x, y, \alpha^2, p)) \leqq - \rho^2 d^2(x, y)$.

If in the above definition, inequality (6a) is satisfied as

(6c) $f(x) < f(y) - \frac{1}{2}p\nabla^2 f(y)p \implies F(x, y ; \Delta(f, x, y, \alpha^1, p)) \leq - \rho^1 d^2(x , y)$

then we say that (f , h) is second order weak pseudoquasi (F, α, ρ, p, d)-type I at y.

Remark ([2]). Note that for the scalar objective functions the class of second order pseudoquasi (F, α, ρ, p, d)-type I, the class of second order weak

strictly-pseudoquasi (F, α, ρ, p, d)-type I, and the class of second order strong pseudoquasi (F, α, ρ, p,d)-type I functions coincide.

Definition 2.9 ([2]). (f , h) is said to be second order sub-strictly-pseudo-quasi (F, α, ρ, p, d)-type I at y, if for all x$\in$A, x $\neq$ y, we have:

(7a) $f(x) \leqq f(y) - \frac{1}{2}p\nabla^2 f(y)p \Longrightarrow F(x, y ; \Delta(f, x, y, \alpha^1, p)) \leq -\rho^1 d^2(x , y)$

(7b) $- h(y) + \frac{1}{2}p\nabla^2 h(y)p \leqq 0 \Longrightarrow F(x, y ; \Delta(h, x, y, \alpha^2, p)) \leqq -\rho^2 d^2(x , y).$

Definition 2.10 ([2]). (f , h) is said to be second order weak quasistrictly-pseudo (F, α, ρ, p, d)-type I at y, if for all x$\in$A we have:

(8a) $f(x) \leq f(y) - \frac{1}{2}p\nabla^2 f(y)p \Longrightarrow F(x, y ; \Delta(f, x, y, \alpha^1, p)) \leqq -\rho^1 d^2(x , y)$

(8b) $- h(y) + \frac{1}{2}p\nabla^2 h(y)p \leqq 0 \Longrightarrow F(x, y ; \Delta(h, x, y, \alpha^2, p)) \leq -\rho^2 d^2(x , y).$

Definition 2.11 ([2]). (f , h) is said to be second order weak quasisemi-pseudo (F, α, ρ, p, d)-type I at y, if for all x$\in$A we have

(9a) $f(x) \leq f(y) - \frac{1}{2}p\nabla^2 f(y)p \Longrightarrow F(x, y ; \Delta(f, x, y, \alpha^1, p)) \leqq -\rho^1 d^2(x , y)$

(9b) $- h(y) + \frac{1}{2}p\nabla^2 h(y)p \leqq 0 \Longrightarrow F(x, y ; \Delta(h, x, y, \alpha^2, p)) < -\rho^2 d^2(x , y).$

Definition 2.12 ([2]). (f , h) is said to be second order weak strictly-pseudo (F, α, ρ, p, d)-type I at y, if for all x$\in$A, x $\neq$ y, we have

(10a) $f(x) \leq f(y) - \frac{1}{2}p\nabla^2 f(y)p \Longrightarrow F(x, y ; \Delta(f, x, y, \alpha^1, p)) < -\rho^1 d^2(x , y)$

(10b) $- h(y) + \frac{1}{2}p\nabla^2 h(y)p \leqq 0 \Longrightarrow F(x, y ; \Delta(h, x, y, \alpha^2, p)) < -\rho^2 d^2(x , y).$

3. GENERALIZED MOND-WEIR TYPE DUALITY

Consider the following Mond-Weir type dual of (MOP)

maximize $f(y) + v_{J_0}g_{J_0}(y)e - \frac{1}{2}p\nabla^2[f(y) + v_{J_0}g_{J_0}(y)e]p,$

subject to $u\nabla f(y) + u \nabla^2 f(y)p + v\nabla g(y) + v \nabla^2 g(y)p = 0,$

(GMWD) $v_{J_k}g_{J_k}(y) - \frac{1}{2}p\nabla^2 v_{J_k}g_{J_k}(y)p \geqq 0, k = 1, 2, \dots , \nu,$

$v \geqq 0,$

$u \geqq 0, \ u^t e = 1,$

where $\nu \geq 1$, $J_s \cap J_t = \varnothing$ for s $\neq$ t and $\bigcup_{s=0}^{\nu} J_s = \{1, 2, \dots , q\} = Q$.

Relative to (MOP) and (GMWD) we give some duality results.

THEOREM 3.1 (Weak Duality). *Assume that for all feasible x for (MOP) and all feasible (y, u, v, p) for (GMWD), any of the following holds:*

(a) $u > 0$, and $(f(\cdot) + v_{J_0}g_{J_0}(\cdot)e , v_{J_k}g_{J_k}(\cdot))$, $k = \overline{1,\nu}$, *is second order strong pseudoquasi (F, α, ρ, p, d)-type I at y with*
$$min\{u\rho^0\alpha^0(\cdot , y)^{-1}, \min_{k=\overline{1,\nu}}[\rho^k\alpha^k(\cdot , y)^{-1}]\} \geqq 0;$$

(b) $u > 0$, and $(uf(\cdot) + v_{J_0}g_{J_0}(\cdot), v_{J_k}g_{J_k}(\cdot))$, $k = \overline{1,\nu}$, *is second order pseudoquasi (F, α, ρ, p,d)-type I at y with*
$$min\{\rho^0\alpha^0(\cdot , y)^{-1}, \min_{k=\overline{1,\nu}}[\rho^k\alpha^k(\cdot , y)^{-1}]\} \geqq 0.$$

Then, $f(x) \not< f(y) + v_{J_0}g_{J_0}(y)e -\frac{1}{2}p\nabla^2[f(y) + v_{J_0}g_{J_0}(y)e]p$.

Proof. Suppose contrary to the result of the theorem that $f(x) < f(y) + v_{J_0}g_{J_0}(y)e - \frac{1}{2}p\nabla^2[f(y) + v_{J_0}g_{J_0}(y)e]p$. Since x is feasible for (MOP) and $v \geqq 0$, we have $f(x) + v_{J_0}g_{J_0}(x)e \leq f(y) + v_{J_0}g_{J_0}(y)e -\frac{1}{2}p\nabla^2[f(y) + v_{J_0}g_{J_0}(y)e]p$.

Since (y, u, v, p) is feasible for (GMWD), it follows that

$$- v_{J_k}g_{J_k}(y) + \frac{1}{2}p\nabla^2 v_{J_k}g_{J_k}(y)p \leqq 0 , k = 1, 2, ... , \nu.$$

By hypothesis (a), we have

$F(x, y ; \Delta(f(\cdot) + v_{J_0}g_{J_0}(\cdot)e, x, y, \alpha^0, p)) \leq -\rho^0 d^2(x , y)$

$\alpha^k F(x, y; \Delta(v_{J_k}g_{J_k}(\cdot), x, y,1,p)) \leqq F(x,y; \Delta(v_{J_k}g_{J_k}(\cdot), x, y, \alpha^k,p)) \leqq -\rho^k d^2(x ,y)$, $k =1,...,\nu$.

Since $\alpha^0(x , y) > 0$, $\alpha^k(x , y) > 0$, $k = 1, 2, ... , \nu$ and $u > 0$, we have

$F(x, y ; \Delta(uf(\cdot) + v_{J_0}g_{J_0}(\cdot), x, y, 1, p)) \leq - \alpha^0(x, y)^{-1}u\rho^0 d^2(x , y)$

$F(x, y ; \Delta(v_{J_k}g_{J_k}(\cdot), x, y, 1, p)) \leqq -\alpha^k(x, y)^{-1}\rho^k d^2(x , y)$, $k = 1, 2, ...,\nu$.

By q-sublinearity of F , we obtain

$0 = F(x, y ; 0) = F(x, y ; u\nabla f(y) + u\nabla^2 f(y)p + v\nabla g(y) + v\nabla^2 g(y)p) =$

$= F(x, y ; \Delta(uf(\cdot) + v_{J_0}g_{J_0}(\cdot), x, y, 1, p) + \sum_{k=1}^{\nu}\Delta(v_{J_k}g_{J_k}(\cdot), x, y, 1, p)) \leq$

$\leq max\{F(x, y ; \Delta(uf(\cdot) + v_{J_0}g_{J_0}(\cdot), x, y,1, p)),$
$\max_{k=\overline{1,\nu}} [F(x, y ; \Delta(v_{J_k}g_{J_k}(\cdot), x, y,1, p))]\} \leq$

$\leq max\{- \alpha^0(x, y)^{-1}u\rho^0 d^2(x , y), \max_{k=\overline{1,\nu}} [-\alpha^k(x, y)^{-1}\rho^k d^2(x , y)]\} =$

$= - min\{ \alpha^0(x, y)^{-1}u\rho^0 d^2(x , y), \min_{k=\overline{1,\nu}} [\alpha^k(x, y)^{-1}\rho^k d^2(x , y)]\} \leqq 0,$

contradicting.

By hypothesis (b), we have

• $f(x) + v_{J_0}g_{J_0}(x)e \leq f(y) + v_{J_0}g_{J_0}(y)e - \frac{1}{2}p\nabla^2[f(y) + v_{J_0}g_{J_0}(y)e]p$,   which implies

$$uf(x) + v_{J_0}g_{J_0}(x) < uf(y) + v_{J_0}g_{J_0}(y) - \frac{1}{2}p\nabla^2[uf(y) + v_{J_0}g_{J_0}(y)]p;$$

• $- v_{J_k}g_{J_k}(y) + \frac{1}{2}p\nabla^2 v_{J_k}g_{J_k}(y)p \leqq 0$, $k = 1, 2, ... , \nu$,   which implies

$$F(x, y \ ; \ \Delta(uf(\cdot) + v_{J_0}g_{J_0}(\cdot), x, y, \alpha^0, p)) < \text{-} \rho^0 d^2(x \ , y)$$
$$F(x, y \ ; \ \Delta(v_{J_k}g_{J_k}(\cdot), x, y, \alpha^k, p)) \leqq \text{-}\rho^k d^2(x \ , y), k = 1, 2, \dots , \nu.$$

Since $\alpha^0(x \ , y) > 0, \ \alpha^k(x \ , y) > 0, k = 1, 2, \dots , \nu$ and $u > 0$, the above inequalities give

$$F(x, y \ ; \ \Delta(uf(\cdot) + v_{J_0}g_{J_0}(\cdot), x, y, 1, p)) \leq \text{-} \alpha^0(x, y)^{-1}\rho^0 d^2(x \ , y)$$
$$F(x, y \ ; \ \Delta(v_{J_k}g_{J_k}(\cdot), x, y, 1, p)) \leqq \text{-}\alpha^k(x, y)^{-1}\rho^k d^2(x \ , y), k = 1, 2, \dots , \nu.$$

By q-sublinearity of F , we obtain

$$0 = F(x, y \ ; \ 0) = F(x, y \ ; \ u\nabla f(y) + u\nabla^2 f(y)p + v\nabla g(y) + v\nabla^2 g(y)p) =$$
$$= F(x, y \ ; \ \Delta(uf(\cdot) + v_{J_0}g_{J_0}(\cdot), x, y, 1, p) + \sum_{k=1}^{\nu}\Delta(v_{J_k}g_{J_k}(\cdot), x, y, 1, p)) \leq$$
$$\leq \max\{F(x, y \ ; \ \Delta(uf(\cdot) + v_{J_0}g_{J_0}(\cdot), x, y, 1, p)),$$
$$\max_{k=\overline{1,\nu}} [F(x, y \ ; \ \Delta(v_{J_k}g_{J_k}(\cdot), x, y,1, p))]\} \leqq$$
$$\leqq \max\{\text{-} \alpha^0(x, y)^{-1}\rho^0 d^2(x \ , y), \max_{k=\overline{1,\nu}} [\text{-}\alpha^k(x, y)^{-1}\rho^k d^2(x \ , y)]\} =$$
$$= \text{-} \min\{ \alpha^0(x, y)^{-1}\rho^0 d^2(x \ , y), \min_{k=\overline{1,\nu}} [\alpha^k(x, y)^{-1}\rho^k d^2(x \ , y)]\} \leqq 0,$$

contradicting.

Hence, in both cases we have contradicting, so

$$f(x) \nless f(y) + v_{J_0}g_{J_0}(y)e - \tfrac{1}{2}p\nabla^2[f(y) + v_{J_0}g_{J_0}(y)e]p. \quad \square$$

THEOREM 3.2 (Weak Duality). *Assume that for all feasible x for (MOP) and all feasible (y, u, v, p) for (GMWD), any of the following holds:*

(a) $(f(\cdot)+v_{J_0}g_{J_0}(\cdot)e, v_{J_k}g_{J_k}(\cdot)), k=\overline{1,\nu}$, *is second order weak strictly-pseudoquasi (F, α, ρ, p, d)-type I at y with* $\min\{u\rho^0\alpha^0(\cdot,y)^{-1}, \min_{k=\overline{1,\nu}} [\rho^k\alpha^k(\cdot,y)^{-1}]\} \geqq 0$;

(b) $(uf(\cdot)+v_{J_0}g_{J_0}(\cdot), v_{J_k}g_{J_k}(\cdot)), k = \overline{1,\nu}$, *is second order strictly pseudoquasi (F, α, ρ, p,d)-type I at y with* $\min\{\rho^0\alpha^0(\cdot \ , y)^{-1}, \min_{k=\overline{1,\nu}} [\rho^k\alpha^k(\cdot \ , y)^{-1}]\} \geqq 0$;

(c) $(f(\cdot)+v_{J_0}g_{J_0}(\cdot)e, v_{J_k}g_{J_k}(\cdot)), k = \overline{1,\nu}$, *is second order weak quasisemi-pseudo (F, α, ρ, p, d)-type I at y with* $\min\{u\rho^0\alpha^0(\cdot,y)^{-1}, \min_{k=\overline{1,\nu}} [\rho^k\alpha^k(\cdot,y)^{-1}]\} \geqq 0$;

(d) $(uf(\cdot)+v_{J_0}g_{J_0}(\cdot), v_{J_k}g_{J_k}(\cdot)), k = \overline{1,\nu}$, *is second order quasistrictly-pseudo (F, α, ρ, p,d)-type I at y with* $\min\{\rho^0\alpha^0(\cdot,y)^{-1}, \min_{k=\overline{1,\nu}} [\rho^k\alpha^k(\cdot,y)^{-1}]\} \geqq 0$.

Then, $f(x) \nless f(y) +v_{J_0}g_{J_0}(y)e -\tfrac{1}{2}p\nabla^2[f(y) + v_{J_0}g_{J_0}(y)e]p$.

Proof. Suppose contrary to the result of the theorem that

$$f(x) < f(y) + v_{J_0}g_{J_0}(y)e - \tfrac{1}{2}p\nabla^2[f(y) + v_{J_0}g_{J_0}(y)e]p.$$

Since x is feasible for (MOP) and (y, u, v, p) is feasible for (GMWD), we have

• $f(x) + v_{J_0}g_{J_0}(x)e \leq f(y) + v_{J_0}g_{J_0}(y)e - \frac{1}{2}p\nabla^2[f(y) + v_{J_0}g_{J_0}(y)e]p$, which implies

$$uf(x) + v_{J_0}g_{J_0}(x) \leq uf(y) + v_{J_0}g_{J_0}(y) - \frac{1}{2}p\nabla^2[uf(y) + v_{J_0}g_{J_0}(y)]p;$$

• $-v_{J_k}g_{J_k}(y) + \frac{1}{2}p\nabla^2 v_{J_k}g_{J_k}(y)p \leqq 0$, $k = 1, 2, \dots, \nu$.

By hypothesis (a), we have

$$F(x, y \; ; \Delta(f(\cdot) + v_{J_0}g_{J_0}(\cdot)e, x, y, \alpha^0, p)) < - \rho^0 d^2(x, y)$$

$$F(x, y \; ; \Delta(v_{J_k}g_{J_k}(\cdot), x, y, \alpha^k, p)) \leqq -\rho^k d^2(x, y), k = 1, 2, \dots, \nu .$$

Since $\alpha^0(x, y) > 0$, $\alpha^k(x, y) > 0$, $k = 1, 2, \dots, \nu$ and $u \geq 0$, the above inequalities give

$$F(x, y \; ; \Delta(uf(\cdot) + v_{J_0}g_{J_0}(\cdot), x, y, 1, p)) < - \alpha^0(x, y)^{-1}u\rho^0 d^2(x, y)$$

$$F(x, y \; ; \Delta(v_{J_k}g_{J_k}(\cdot), x, y, 1, p)) \leqq -\alpha^k(x, y)^{-1}\rho^k d^2(x, y), k = 1, 2, \dots, \nu.$$

By q-sublinearity of F, we obtain

$$0 = F(x, y \; ; 0) = F(x, y \; ; u\nabla f(y) + u\nabla^2 f(y)p + v\nabla g(y) + v\nabla^2 g(y)p) =$$

$$= F(x, y \; ; \Delta(uf(\cdot) + v_{J_0}g_{J_0}(\cdot), x, y, 1, p) + \sum_{k=1}^{\nu}\Delta(v_{J_k}g_{J_k}(\cdot), x, y, 1, p)) \leq$$

$$\leq \max\{F(x, y \; ; \Delta(uf(\cdot) + v_{J_0}g_{J_0}(\cdot), x, y, 1, p)),$$

$$\max_{k=\overline{1,\nu}} [F(x, y \; ; \Delta(v_{J_k}g_{J_k}(\cdot), x, y, 1, p))]\} \leqq$$

$$\leqq \max\{- \alpha^0(x, y)^{-1}u\rho^0 d^2(x, y), \max_{k=\overline{1,\nu}} [-\alpha^k(x, y)^{-1}\rho^k d^2(x, y)]\} =$$

$$= - \min\{ \alpha^0(x, y)^{-1}u\rho^0 d^2(x, y), \min_{k=\overline{1,\nu}} [\alpha^k(x, y)^{-1}\rho^k d^2(x, y)]\} \leqq 0,$$

contradicting.

Using hypothesis (b), (c) or (d), we obtain, also, contradicting, so

$$f(x) \not< f(y) + v_{J_0}g_{J_0}(y)e - \frac{1}{2}p\nabla^2[f(y) + v_{J_0}g_{J_0}(y)e]p. \quad \square$$

THEOREM 3.3 (Weak Duality). *Assume that for all feasible x for (MOP) and all feasible (y, u, v, p) for (GMWD), any of the following holds:*

(a) $u > 0$, *and* $(f(\cdot) +v_{J_0}g_{J_0}(\cdot)e, v_{J_k}g_{J_k}(\cdot))$, $k = \overline{1,\nu}$, *is second order weak pseudoquasi* (F, α, ρ, p, d)-*type I at y with* $min\{u\rho^0\alpha^0(\cdot, y)^{-1},$ $\min_{k=\overline{1,\nu}} [\rho^k\alpha^k(\cdot, y)^{-1}]\} \geqq 0$;

(b) $(uf(\cdot) +v_{J_0}g_{J_0}(\cdot), v_{J_k}g_{J_k}(\cdot))$, $k = \overline{1,\nu}$, *is second order pseudoquasi* (F, α, ρ, p, d)-*type I at y with* $min\{\rho^0\alpha^0(\cdot, y)^{-1}, \min_{k=\overline{1,\nu}} [\rho^k\alpha^k(\cdot, y)^{-1}]\} \geqq 0$;

(c) $(f(\cdot) +v_{J_0}g_{J_0}(\cdot)e, v_{J_k}g_{J_k}(\cdot))$, $k = \overline{1,\nu}$, *is second order pseudoquasi* (F, α, ρ, p, d)-*type I at y with* $min\{u\rho^0\alpha^0(\cdot, y)^{-1}, \min_{k=\overline{1,\nu}} [\rho^k\alpha^k(\cdot, y)^{-1}]\} \geqq 0$;

(d) $(uf(\cdot) +v_{J_0}g_{J_0}(\cdot), v_{J_k}g_{J_k}(\cdot))$, $k = \overline{1,\nu}$, *is second order quasistrictly-pseudo* (F, α, ρ, p, d)-*type I at y with* $min\{\rho^0\alpha^0(\cdot, y)^{-1}, \min_{k=\overline{1,\nu}} [\rho^k\alpha^k(\cdot, y)^{-1}]\} \geqq 0$.

Then, $f(x) \not< f(y) + v_{J_0}g_{J_0}(y)e - \frac{1}{2}p\nabla^2[f(y) +v_{J_0}g_{J_0}(y)e]p.$

Proof. It follows on the lines of Theorem 3.2. $\square$

THEOREM 3.4 (Strong Duality). *Let ($\overline{y}$, $\overline{u}$, $\overline{v}$, $\overline{p}$) be feasible solution for (GMWD) such that*

$$\overline{v}_{J_0}g_{J_0}(\overline{y})e -\tfrac{1}{2}\overline{p}\nabla^2[f(\overline{y}) + \overline{v}_{J_0}g_{J_0}(\overline{y})e]\overline{p} \geqq 0$$

and assume that $\overline{y}$ is feasible for (MOP). If weak duality (any of Theorem 3.1. or 3.2.) holds between (MOP) and (GMWD), then $\overline{y}$ is efficient for (MOP) and ($\overline{y}$, $\overline{u}$, $\overline{v}$, $\overline{p}$) is efficient for (GMWD).

Proof. Suppose that $\overline{y}$ is not efficient for (MOP), then there exists a feasible x for (MOP) such that $f(x) \leq f(\overline{y})$. On using hypothesis, we have

$$f(x) \leq f(\overline{y}) + \overline{v}_{J_0}g_{J_0}(\overline{y})e -\tfrac{1}{2}\overline{p}\nabla^2[f(\overline{y}) + \overline{v}_{J_0}g_{J_0}(\overline{y})e]\overline{p},$$

contradicts weak duality (Theorem 3.1. or 3.2.), so $\overline{y}$ is efficient for (MOP).

Also, suppose that ($\overline{y}$, $\overline{u}$, $\overline{v}$, $\overline{p}$) is not efficient for (GMWD), then there exists a feasible (y, u, v, p) for (GMWD) such that

$$f(\overline{y}) + \overline{v}_{J_0}g_{J_0}(\overline{y})e -\tfrac{1}{2}\overline{p}\nabla^2[f(\overline{y}) + \overline{v}_{J_0}g_{J_0}(\overline{y})e]\overline{p} \leq$$
$$\leq f(y) + v_{J_0}g_{J_0}(y)e -\tfrac{1}{2}p\nabla^2[f(y) + v_{J_0}g_{J_0}(y)e]p.$$

Using hypothesis, we obtain $f(\overline{y}) \leq f(y) + v_{J_0}g_{J_0}(y)e -\tfrac{1}{2}p\nabla^2[f(y) + v_{J_0}g_{J_0}(y)e]p$, contradicts weak duality (Theorem 3.1. or 3.2.), since (y, u, v, p) is feasible for (GMWD) and $\overline{y}$ is feasible for (MOP). $\square$

THEOREM 3.5. *Let ($\overline{y}$, $\overline{u}$, $\overline{v}$, $\overline{p}$) be feasible solution for (GMWD) such that*
$$\overline{v}_{J_0}g_{J_0}(\overline{y})e -\tfrac{1}{2}\overline{p}\nabla^2[f(\overline{y}) + \overline{v}_{J_0}g_{J_0}(\overline{y})e]\overline{p} \geqq 0$$

and assume that $\overline{y}$ is feasible for (MOP). If weak duality (Theorem 3.3.) holds between (MOP) and (GMWD), then $\overline{y}$ is weak efficient for (MOP) and ($\overline{y}$, $\overline{u}$, $\overline{v}$, $\overline{p}$) is weak efficient for (GMWD).

Proof. It follows from Theorem 3.4. $\square$

THEOREM 3.6 (Strict converse duality). *Let $\overline{x}$ be feasible solution for (MOP) and ($\overline{y}$, $\overline{u}$, $\overline{v}$, $\overline{p}$) be feasible solution for (GMWD) such that*

$$\overline{u}f(\overline{x}) \leqq \overline{u}f(\overline{y}) + \overline{v}_{J_0}g_{J_0}(\overline{y}) -\tfrac{1}{2}\overline{p}\nabla^2[\overline{u}f(\overline{y}) + \overline{v}_{J_0}g_{J_0}(\overline{y})]\overline{p}.$$

If, condition (b) or (d) of Theorem 3.2. is satisfied for $\overline{x}$ and ($\overline{y}$, $\overline{u}$, $\overline{v}$, $\overline{p}$), then $\overline{x} = \overline{y}$.

Proof. We assume $\overline{x} \neq \overline{y}$ and exhibit a contradiction. Since $\overline{x}$ and ($\overline{y}$, $\overline{u}$, $\overline{v}$, $\overline{p}$), are feasible for (MOP) and (GMWD) respectively, then $\overline{v} \geqq 0$, $g(\overline{x}) \leqq 0$ and

$$- \overline{v}_{J_k}g_{J_k}(\overline{y}) + \tfrac{1}{2}\overline{p}\nabla^2\overline{v}_{J_k}g_{J_k}(\overline{y})\overline{p} \leqq 0, \quad k = 1, 2, \dots , \nu.$$

Using hypothesis, we have

$$\overline{u}f(\overline{x}) + \overline{v}_{J_0}g_{J_0}(\overline{x}) \leqq \overline{u}f(\overline{y}) + \overline{v}_{J_0}g_{J_0}(\overline{y}) -\tfrac{1}{2}\overline{p}\nabla^2[\overline{u}f(\overline{y}) + \overline{v}_{J_0}g_{J_0}(\overline{y})]\overline{p}.$$

Using condition (b) of Theorem 3.2 , we get

$$F(\overline{x}\ ,\ \overline{y}\ ;\ \Delta(\overline{u}f(\cdot) + \overline{v}_{J_0}g_{J_0}(\cdot),\ \overline{x}\ ,\ \overline{y},\ \alpha^0,\ \overline{p})) < \text{-}\ \rho^0 d^2(\overline{x}\ ,\ \overline{y})$$
$$F(\overline{x}\ ,\ \overline{y}\ ;\ \Delta(\overline{v}_{J_k}g_{J_k}(\cdot),\ \overline{x}\ ,\ \overline{y},\ \alpha^k,\ \overline{p})) \leqq \text{-}\rho^k d^2(\overline{x}\ ,\ \overline{y}),\ \ k = 1,\ 2,\ \dots\ ,\ \nu.$$

Since $\alpha^0(\overline{x}\ ,\ \overline{y}) > 0,\ \ \alpha^k(\overline{x}\ ,\ \overline{y}) > 0, k = 1, 2, \dots , \nu$, we obtain

$$F(\overline{x}\ ,\ \overline{y}\ ;\ \Delta(\overline{u}f(\cdot) + \overline{v}_{J_0}g_{J_0}(\cdot),\ \overline{x}\ ,\ \overline{y},\ 1,\ \overline{p})) < \text{-}\ \alpha^0(\overline{x}\ ,\ \overline{y})^{-1}\rho^0 d^2(\overline{x}\ ,\ \overline{y})$$
$$F(\overline{x}\ ,\ \overline{y}\ ;\ \Delta(\overline{v}_{J_k}g_{J_k}(\cdot),\ \overline{x}\ ,\ \overline{y},\ 1,\ \overline{p})) \leqq \text{-}\alpha^k(\overline{x}\ ,\ \overline{y})^{-1}\rho^k d^2(\overline{x}\ ,\ \overline{y}), k = 1, 2, \dots , \nu.$$

Using condition (d) of Theorem 3.2 , we get

$$F(\overline{x}\ ,\ \overline{y}\ ;\ \Delta(\overline{u}f(\cdot) + \overline{v}_{J_0}g_{J_0}(\cdot),\ \overline{x}\ ,\ \overline{y},\ 1,\ \overline{p})) \leqq \text{-}\ \alpha^0(\overline{x}\ ,\ \overline{y})^{-1}\rho^0 d^2(\overline{x}\ ,\ \overline{y})$$
$$F(\overline{x}\ ,\ \overline{y}\ ;\ \Delta(\overline{v}_{J_k}g_{J_k}(\cdot),\ \overline{x}\ ,\ \overline{y},\ 1,\ \overline{p})) < \text{-}\alpha^k(\overline{x}\ ,\ \overline{y})^{-1}\rho^k d^2(\overline{x}\ ,\ \overline{y}), k = 1, 2, \dots , \nu.$$

By q-sublinearity of F, both systems gives:

$$0 = F(\overline{x}\ ,\ \overline{y}\ ;\ 0) = F(\overline{x}\ ,\ \overline{y}\ ;\ \overline{u}\nabla f(\overline{y}) + \overline{u}\nabla^2 f(\overline{y})\overline{p} + \overline{v}\nabla g(\overline{y}) + \overline{v}\nabla^2 g(\overline{y})\overline{p}) =$$
$$= F(\overline{x}\ ,\ \overline{y}\ ;\ \Delta(\overline{u}f(\cdot) + \overline{v}_{J_0}g_{J_0}(\cdot),\ \overline{x}\ ,\ \overline{y}\ ,\ 1,\ \overline{p}) + \sum_{k=1}^{\nu}\Delta(\overline{v}_{J_k}g_{J_k}(\cdot),\ \overline{x}\ ,\ \overline{y}\ ,\ 1,\ \overline{p})) \leq$$
$$\leq \max\{F(\overline{x}\ ,\ \overline{y}\ ;\ \Delta(\overline{u}f(\cdot) + \overline{v}_{J_0}g_{J_0}(\cdot),\ \overline{x}\ ,\ \overline{y}\ ,\ 1,\ \overline{p})),$$
$$\max_{k=\overline{1,\nu}}\ [F(\overline{x}\ ,\ \overline{y}\ ;\ \Delta(\overline{v}_{J_k}g_{J_k}(\cdot),\ \overline{x}\ ,\ \overline{y}\ ,1,\ \overline{p}))]\} \leqq$$
$$\leqq \max\{\text{-}\ \alpha^0(\overline{x}\ ,\ \overline{y}\ )^{-1}\rho^0 d^2(\overline{x}\ ,\ \overline{y}\ ),\ \max_{k=\overline{1,\nu}}\ [\text{-}\alpha^k(\overline{x}\ ,\ \overline{y}\ )^{-1}\rho^k d^2(\overline{x}\ ,\ \overline{y}\ )]\} =$$
$$= \text{-}\ \min\{\ \alpha^0(\overline{x}\ ,\ \overline{y}\ )^{-1}\rho^0 d^2(\overline{x}\ ,\ \overline{y}\ ),\ \min_{k=\overline{1,\nu}}\ [\alpha^k(\overline{x}\ ,\ \overline{y}\ )^{-1}\rho^k d^2(\overline{x}\ ,\ \overline{y}\ )]\} \leqq 0,$$

contradicting.

Hence, in both cases we have contradicting, so $\overline{x} = \overline{y}$. $\square$

COROLLARY 3.7. *Let assumptions of Theorem 3.6 be verified, and let $\overline{x}$ be (weak) efficient for (MOP) and ($\overline{y}$, $\overline{u}$, $\overline{v}$, $\overline{p}$) be (weak) efficient for (GMWD). Then $\overline{x} = \overline{y}$, i.e. $\overline{y}$ is (weak) efficient for (MOP).*

Proof. The proof follows from Theorem 3.6. $\square$

THEOREM 3.8. *Let $\overline{x}$ be feasible solution for (MOP) and ($\overline{y}$, $\overline{u}$, $\overline{v}$, $\overline{p}$) be feasible solution for (GMWD) such that*

$$f(\overline{x}) \leqq f(\overline{y}) +\ \overline{v}_{J_0}g_{J_0}(\overline{y})e -\tfrac{1}{2}\overline{p}\nabla^2 [f(\overline{y}) +\ \overline{v}_{J_0}g_{J_0}(\overline{y})e]\overline{p}.$$

For each feasible x for (MOP) and (y, u, v, p) for (GMWD), if weak duality (any of Theorem 3.1 or 3.2) holds:
    (a) *at $\overline{y}$, then $\overline{x}$ is efficient for (MOP).*
    (b) *at y, then ($\overline{y}$, $\overline{u}$, $\overline{v}$, $\overline{p}$) is efficient for (GMWD).*

Proof. (a) Suppose that $\overline{x}$ is not an efficient solution for (MOP). Then, there exist a feasible x for (MOP) such that $f(x) \leq f(\overline{x})$. Using hypothesis, we have

$$f(x) \le f(\overline{y}) + \overline{v}_{J_0}g_{J_0}(\overline{y})e - \tfrac{1}{2}\overline{p}\nabla^2[f(\overline{y}) + \overline{v}_{J_0}g_{J_0}(\overline{y})e]\overline{p},$$

which contradicts the weak duality for feasible solutions x for (MOP) and ($\overline{y}$, $\overline{u}$, $\overline{v}$, $\overline{p}$) for (GMWD), so $\overline{x}$ is efficient for (MOP).

(b) Let us assume on the contrary that ($\overline{y}$, $\overline{u}$, $\overline{v}$, $\overline{p}$) is not an efficient solution for (GMWD). Then, there exist a feasible (y, u, v, p) for (GMWD) such that

$$f(\overline{y}) + \overline{v}_{J_0}g_{J_0}(\overline{y})e - \tfrac{1}{2}\overline{p}\nabla^2[f(\overline{y}) + \overline{v}_{J_0}g_{J_0}(\overline{y})e]\overline{p} \le$$
$$\le f(y) + v_{J_0}g_{J_0}(y)e - \tfrac{1}{2}p\nabla^2[f(y) + v_{J_0}g_{J_0}(y)e]p.$$

Using hypothesis, we obtain $f(\overline{x}) \le f(y) + v_{J_0}g_{J_0}(y)e - \tfrac{1}{2}p\nabla^2[f(y)+v_{J_0}g_{J_0}(y)e]p$, which contradicts the weak duality for feasible solutions $\overline{x}$ for (MOP) and (y, u, v, p) for (GMWD). Thus, ($\overline{y}$, $\overline{u}$, $\overline{v}$, $\overline{p}$) is efficient for (GMWD).

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