NUMERICAL METHOD FOR A SINGULARLY PERTURBED
DIFFERENTIAL SYSTEM

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A second-order accurate difference scheme is developed to solve Boussinesq system. For the time integration, a Crank-Nicolson type scheme is used. The error estimates for the numerical solution are obtained. Numerical results are also presented.

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Key words: Boussinesq system, difference scheme, error estimates.

1. INTRODUCTION

We shall be concerned with developing difference schemes for approximating the solution \{u(x, t), v(x, t)\}, arising in the long water waves theory.

\[
L_1[u, v] := \frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial t \partial x^2} + \alpha u \frac{\partial u}{\partial x} + a_0(x, t)u + a_1(x, t)v + a_2(x, t) \frac{\partial v}{\partial x} = f_1(x, t),
\]

\((x, t) \in Q,\)

\[
L_2[u, v] := \frac{\partial v}{\partial t} - \frac{\partial^3 v}{\partial t \partial x^2} + \beta \frac{\partial}{\partial x}(uv) + b_0(x, t)v + b_1(x, t)u + b_2(x, t) \frac{\partial u}{\partial x} = f_2(x, t),
\]

\((x, t) \in Q,\)

\[
u(x, 0) = \varphi(x), v(x, 0) = \psi(x), x \in \Omega,
\]

\(u(0, t) = v(0, t) = u(l, t) = v(l, t) = 0, t \in (0, T],\)

where \(Q = \Omega \cup (0, T], \Omega = (0, l)\) and \(\overline{\Omega}\) the closure of \(\Omega\). \(a_k(x, t), b_k(x, t)(k = 0, 1, 2), f_k(x, t)(k = 1, 2), \varphi(x), \psi(x)\) are given sufficiently smooth function such as

\[
\frac{\partial^s a_k}{\partial x^s}, \frac{\partial^s b_k}{\partial x^s}, \frac{\partial^s a_k}{\partial t^s}, \frac{\partial^s b_k}{\partial t^s} \in C(\overline{Q})(k, s = 0, 1, 2)
\]

\[
\frac{\partial^s f_k}{\partial x^s}, \frac{\partial^s f_k}{\partial t^s} \in C(\overline{Q})(k = 1, 2; \ s = 0, 1, 2)
\]

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and \( \alpha, \beta \) are given constants.

The system (1.1)–(1.2) provides an approximate model for long gravity waves of small amplitude in one dimension \([11, 5]\). Here, \( u \) denotes horizontal velocity at the level of fluid, and \( v \) is the height of the surface above the undisturbed level.

The difference methods having first-order in time direction accuracy was investigated in \([1, 2]\). In \([2]\) two-dimensional case and in \([1]\) singularly perturbed case was considered. Note also, that many studies have been devoted to the so-called Sobolev or pseudo-parabolic equation \([3, 4, 6–10, 12]\) (see, also references cited in them).

In this paper, we present a difference method with exponential fitting factors to problem (1.1)–(1.4). For the time variable we use Crank-Nicolson type discretization. The fully discrete scheme is shown to be second-order both accurate in space and time variables.

### 2. DIFFERENCE APPROXIMATION AND CONVERGENCE

Let a set of mesh nodes that discretises \( Q \) be given by

\[
\omega_{h\tau} = \omega_h \times \omega_\tau
\]

with

\[
\omega_h = \{x_i = ih, i = 1, 2, \ldots, N - 1, h = l/N\}, \\
\omega_\tau = \{t_j = j\tau, i = 1, 2, \ldots, N_0, \tau = T/N_0\}
\]

and

\[
\omega_{h^+} = \omega_h \cup \{x_N = l\}, \quad \bar{\omega}_h = \omega_h \cup \{x_0 = 0, x_N = l\}, \\
\bar{\omega}_\tau = \omega_\tau \cup \{t_0 = 0\}, \quad \bar{\omega}_{h\tau} = \bar{\omega}_h \times \bar{\omega}_\tau.
\]

Define the following finite differences for any mesh function \( g_i = g(x_i) \) given on \( \bar{\omega}_h \) by

\[
g_{\bar{x},i} = \frac{g_{i+1} - g_i}{h}, \quad g_{x,i} = \frac{g_{i+1} - 2g_i + g_{i-1}}{2h}, \quad g_{\bar{x}x,i} = \frac{g_{i+1} - 2g_i + g_{i-1}}{h^2}.
\]

Introduce the inner products for the mesh functions \( v_i \) and \( w_i \) defined on \( \bar{\omega}_h \) as follows

\[
(v, w)_0 \equiv (v, w)_{\omega_h} := \sum_{i=1}^{N-1} hv_i w_i,
\]

\[
(v, w) \equiv (v, w)^\dagger_{\omega_h} := \sum_{i=1}^{N} hv_i w_i.
\]
For any mesh function \( v_i \), vanishing for \( i = 0 \) and \( i = N \) we introduce the norms
\[
\| v \|_0^2 \equiv \| v \|_{0, \omega_h}^2 := (v, v)_0, \quad \| v \|_1^2 := (v_x, v_x),
\]
\[
\| v \|_1^2 := \| v \|_0^2 + \| v \|_1^2.
\]

Given a function \( g \equiv g_j^i \equiv g(x_i, t_j) \) defined on \( \omega_{h \tau} \), we shall also use the notation
\[
g_j^i = \frac{g_j^i - g_j^{i-1}}{\tau}, \quad g_j^{i+1} = \frac{g_j^{i+1} - g_j^i}{\tau}.
\]

On the mesh \( \omega_{h \tau} \), we approximate (1.1)–(1.4) by the following difference problem of Crank-Nicolson type
\[
l_1[y_1, y_2] := y_1\bar{t} - \theta y_1\bar{t}xx + \alpha S^{(0.5)}(y_1) + a_0^{(0.5)}y_1^{(0.5)} + a_1^{(0.5)}y_2^{(0.5)} + a_2^{(0.5)}y_2^0 = f_1^{(0.5)},
\]
(2.1)
\[
l_2[y_1, y_2] := y_2\bar{t} - \theta y_2\bar{t}xx + \beta(y_1y_2)_x + b_0^{(0.5)}y_2^{(0.5)} + b_1^{(0.5)}y_1^{(0.5)} + b_2^{(0.5)}y_1^0 = f_2^{(0.5)},
\]
(2.2)
\[
y_1(x, 0) = \varphi(x), \quad y_2(x, 0) = \psi(x), \quad x \in \bar{\omega}_h,
\]
(2.3)
\[
y_1 = y_2 = 0, \quad \text{for } x = 0, \ell \text{ and } t \in \bar{\omega}_\tau,
\]
(2.4)
where
\[
g^{(0.5)} \equiv g^{(0.5)}_j = \frac{1}{2} \left( g_j^i + g_j^{i-1} \right),
\]
\[
S(y_j^{i, i}) = \frac{1}{3} \left\{ \left( y_1^2 \right)_0 + \left( y_1^i y_j^{i, 0} \right)_{1x, i} \right\}, \quad \theta = \frac{h^2}{4 \sinh^2(h/2)}.
\]

Our method of approximating the convective term in equation (1.1) has the property that \( (S(y_1), y_1)_{\omega_h} = 0 \), i.e., it preserves the analogous principle for the differential case.

We note that assumption (1.5) implies the existence of a unique solution \( \{ u, v \} \) to problem (1.1)–(1.4), belonging to \( C^3(\bar{Q}) \). Then, by the similar manner to that in [1–4], it can be shown that
\[
\ell_k[u, v] + R_k = f_k, \quad k = 1, 2; \quad (x, t) \in \omega_{h \tau}
\]
with truncation errors \( R_k \), such that
\[
\| R_k \|_0 = O(h^2 + \tau^2), \quad k = 1, 2; \quad t \in \omega_{\tau}.
\]

Further, from the identity
\[
\tau \sum_{p=1}^{j} (\ell_1[y_p^1, y_p^2] \,-\, \ell_1[u^p, v^p], \, z_1^p)_{0} + \tau \sum_{p=1}^{j} (\ell_2[y_p^1, y_p^2] \,-\, \ell_2[u^p, v^p], \, z_2^p)_{0} = \tau \sum_{p=1}^{j} 2 \sum_{k=1}^{2} (R_k, z_k^p)_{0},
\]

using summation by parts and appropriate difference embedding inequalities, for the errors \( z_1 = y_1 - u \), \( z_2 = y_2 - v \) by the similar technique described in \([1, 2]\), we have

\[
\delta_j \leq C \left\{ \delta_* + \tau \sum_{p=1}^{j} (\delta_j + \delta_{j-1} + \delta_j^2 + \delta_{j-1}^2) \right\}
\]

with

\[
\delta_j = \sum_{k=1}^{2} \left\{ \theta \| z_k^j \|_1^2 + \| z_k \|_0^2 \right\} \quad (\delta_0 = 0),
\]

\[
\delta_* = \tau \sum_{w_r} \sum_{k=1}^{2} \| R_k \|_0^2
\]

and constant \( C \), that is independent of the mesh parameters. From (2.5), by an application of the difference analogue of Bernoulli inequality (see, also \([1, 2]\)), for sufficiently small \( \tau \), we obtain

\[
\delta_j \leq C \delta_*, \; j = 1, 2, ..., N_0.
\]

We summarize this result in the following theorem.

**Theorem 2.1.** Under the conditions on \( a_k, b_k, f_k \) indicated in (1.5), the solution of (2.1)-(2.4) converges to the exact solution in the mesh norm \( \| \cdot \|_1 \) with the rate \( O(h^2 + \tau^2) \):

\[
\| y_1 - u \|_1 + \| y_2 - v \|_1 \leq C(h^2 + \tau^2).
\]

### 3. Numerical Results

In this section, we present numerical results obtained by applying the numerical method (2.1)-(2.4) to a particular problem

\[
\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial t \partial x^2} + 3u \frac{\partial u}{\partial x} + (e^{-t} + x^2)u = f(x, t), \; (x, t) \in [0, 1] \times [0, 1],
\]

\[
u(x, 0) = \sin(\pi x) - 0.5x^2(1-x),
\]

\[
u(0, t) = u(1, t) = 0,
\]

where \( f \) is such that the exact solution is given by

\[
u(x, t) = -\frac{1}{2}x^2(1-x) + e^{-t} \sin \pi x.
\]
In the computations, the quasilinearization technique to the difference equation was used (see, [3]). The step sizes \( h \) and \( \tau \) were both taken to be 0.025. The results at selected points are presented in Table 1.

<table>
<thead>
<tr>
<th>((x, t))</th>
<th>Exact Solution</th>
<th>Approximate Solution</th>
<th>Pointwise Errors</th>
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</thead>
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<td>(0.05, 0.05)</td>
<td>0.14761757</td>
<td>0.14761772</td>
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<tr>
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<tr>
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</table>

The obtained results show an excellent degree of accuracy for the above described method.

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REFERENCES


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