NUMERICAL METHOD FOR A SINGULARLY PERTURBED DIFFERENTIAL SYSTEM

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A second-order accurate difference scheme is developed to solve Boussinesq system. For the time integration, a Crank-Nicolson type scheme is used. The error estimates for the numerical solution are obtained. Numerical results are also presented.

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1. INTRODUCTION

We shall be concerned with developing difference schemes for approximating the solution $\{u(x,t),v(x,t)\}$, arising in the long water waves theory.

$$L_1[u,v] := \frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial t \partial x^2} + \alpha u \frac{\partial u}{\partial x} + a_0(x,t)u + a_1(x,t)v + a_2(x,t)\frac{\partial v}{\partial x} = f_1(x,t),$$

$$(x,t) \in Q, \tag{1.1}$$

$$L_2[u,v] := \frac{\partial v}{\partial t} - \frac{\partial^3 v}{\partial t \partial x^2} + \beta \frac{\partial}{\partial x}(uv) + b_0(x,t)v + b_1(x,t)u + b_2(x,t)\frac{\partial u}{\partial x} = f_2(x,t),$$

$$(x,t) \in Q, \tag{1.2}$$

$$u(x,0) = \varphi(x), v(x,0) = \psi(x), x \in \Omega, \tag{1.3}$$

$$u(0,t) = v(0,t) = u(l,t) = v(l,t) = 0, t \in (0,T], \tag{1.4}$$

where $Q = \Omega \cup (0,T], \Omega = (0,l)$ and $\overline{\Omega}$ the closure of Ω. $a_k(x,t), b_k(x,t)(k = 0,1,2), f_k(x,t)(k = 1,2), \varphi(x), \psi(x)$ are given sufficiently smooth function such as

$$\frac{\partial^s a_k}{\partial x^s}, \frac{\partial^s b_k}{\partial x^s}, \frac{\partial^s a_k}{\partial t^s}, \frac{\partial^s b_k}{\partial t^s} \in C(\overline{Q})(k,s = 0,1,2)$$

$$\frac{\partial^s f_k}{\partial x^s}, \frac{\partial^s f_k}{\partial t^s}, \in C(\overline{Q})(k = 1,2;\ s = 0,1,2) \tag{1.5}$$

and α, β are given constants.

The system (1.1)–(1.2) provides an approximate model for long gravity waves of small amplitude in one dimension [11, 5]. Here, u denotes horizontal velocity at the level of fluid, and v is the height of the surface above the undisturbed level.

The difference methods having first-order in time direction accuracy was investigated in [1, 2]. In [2] two-dimentional case and in [1] singularly perturbed case was considered. Note also, that many studies have been devoted to the so-called Sobolev or pseudo-parabolic equation [3, 4, 6–10, 12] (see, also references cited in them).

In this paper, we present a difference method with exponential fitting factors to problem (1.1)–(1.4). For the time variable we use Crank-Nicolson type discretization. The fully discrete scheme is shown to be second-order both accurate in space and time variables.

2. DIFFERENCE APPROXIMATION AND CONVERGENCE

Let a set of mesh nodes that discretises Q be given by

$$\omega_{h\tau} = \omega_h \times \omega_\tau$$

with

$$\begin{aligned} \omega_h &= \{x_i = ih, i = 1, 2, ..., N-1, h = l/N\}, \\ \omega_\tau &= \{t_j = j\tau, i = 1, 2, ..., N_0, \tau = T/N_0\} \end{aligned}$$

and

$$\begin{aligned} \omega_h^+ &= \omega_h \cup \{x_N = l\},\ \bar{\omega}_h = \omega_h \cup \{x_0 = 0,\ x_N = l\}, \\ \bar{\omega}_\tau &= \omega_\tau \cup \{t_0 = 0\},\ \bar{\omega}_{h\tau} = \bar{\omega}_h \times \bar{\omega}_\tau. \end{aligned}$$

Define the following finite differences for any mesh function $g_i = g(x_i)$ given on $\bar{\omega}_h$ by

$$g_{\bar{x},i} = \frac{g_i - g_{i-1}}{h},\ g_{x,i} = \frac{g_{i+1} - g_i}{h},\ g_{\overset{0}{x},i} = \frac{g_{i+1} - g_{i-1}}{2h},$$
$$g_{\bar{x}x,i} = \frac{g_{i+1} - 2g_i + g_{i-1}}{h^2}.$$

Introduce the inner products for the mesh functions v_i and w_i defined on $\bar{\omega}_h$ as follows

$$\begin{aligned} (v, w)_0 \equiv (v, w)_{\omega_h} &:= \sum_{i=1}^{N-1} h v_i w_i, \\ (v, w] \equiv (v, w)_{\omega_h^+} &:= \sum_{i=1}^{N} h v_i w_i. \end{aligned}$$

For any mesh function v_i, vanishing for $i = 0$ and $i = N$ we introduce the norms

$$\|v\|_0^2 \equiv \|v\|_{0,\omega_h}^2 := (v,v)_0, \quad \|v\|_{\overset{0}{1}}^2 := (v_{\bar{x}}, v_{\bar{x}}],$$
$$\|v\|_1^2 := \|v\|_0^2 + \|v\|_{\overset{0}{1}}^2,$$

Given a function $g \equiv g_i^j \equiv g(x_i, t_j)$ defined on $\bar{\omega}_{h\tau}$, we shall also use the notation

$$g_{\bar{t},i}^j = \frac{g_i^j - g_i^{j-1}}{\tau}, \quad g_{t,i}^j = \frac{g_i^{j+1} - g_i^j}{\tau}.$$

On the mesh $\omega_{h\tau}$, we approximate (1.1)–(1.4) by the following difference problem of Crank-Nicolson type

$$l_1[y_1, y_2] := y_{1\bar{t}} - \theta y_{1\bar{t}\bar{x}x} + \alpha S^{(0.5)}(y_1) + a_0^{(0.5)} y_1^{(0.5)} + a_1^{(0.5)} y_2^{(0.5)} + a_2^{(0.5)} y_{2\overset{0}{x}} = f_1^{(0.5)},$$

(2.1) $$(x,t) \in \omega_{h\tau},$$

$$l_2[y_1, y_2] := y_{2\bar{t}} - \theta y_{2\bar{t}\bar{x}x} + \beta (y_1 y_2)_{\overset{0}{x}} + b_0^{(0.5)} y_2^{(0.5)} + b_1^{(0.5)} y_1^{(0.5)} + b_2^{(0.5)} y_{1\overset{0}{x}} = f_2^{(0.5)},$$

(2.2) $$(x,t) \in \omega_{h\tau},$$

(2.3) $$y_1(x,0) = \varphi(x), \; y_2(x,0) = \psi(x), x \in \bar{\omega}_h,$$

(2.4) $$y_1 = y_2 = 0, \text{ for } x = 0, \ell \text{ and } t \in \bar{\omega}_\tau,$$

where

$$g^{(0.5)} \equiv g_i^{(0.5)j} = \frac{1}{2}\left(g_i^j + g_i^{j-1}\right),$$
$$S(y_{1,i}^j) = \frac{1}{3}\left\{ (y_1^2)_{\overset{0}{x},i}^j + y_{1,i}^j y_{1\overset{0}{x},i}^j \right\}, \quad \theta = \frac{h^2}{4 sinh^2(h/2)}.$$

Our method of approximating the convective term in equation (1.1) has the property that $(S(y_1), y_1)_{\omega_h} = 0$, *i.e.*, it preserves the analogous principle for the differential case.

We note that assumption (1.5) implies the existence of a unique solution $\{u, v\}$ to problem (1.1)–(1.4), belonging to $C^3(\bar{Q})$. Then, by the similar manner to that in [1–4], it can be shown that

$$\ell_k[u,v] + R_k = f_k, \; k = 1,2; \; (x,t) \in \omega_{h\tau}$$

with truncation errors R_k, such that

$$\|R_k\|_0 = O(h^2 + \tau^2), \; k = 1,2; \; t \in \omega_\tau.$$

Further, from the identity

$$\tau\sum_{p=1}^{j}\left(\ell_1[y_1^p,y_2^p]-\ell_1[u^p,v^p],\ z_1^p\right)_0+\tau\sum_{p=1}^{j}\left(\ell_2[y_1^p,y_2^p]-\ell_2[u^p,v^p],\ z_2^p\right)_0$$
$$=\tau\sum_{p=1}^{j}\sum_{k=1}^{2}\left(R_k,z_k^p\right)_0,$$

using summation by parts and appropriate difference embedding inequalities, for the errors $z_1=y_1-u$, $z_2=y_2-v$ by the similar technique described in [1, 2], we have

$$\delta_j\le C\left\{\delta_*+\tau\sum_{p=1}^{j}\left(\delta_j+\delta_{j-1}+\delta_j^2+\delta_{j-1}^2\right)\right\} \tag{2.5}$$

with

$$\delta_j=\sum_{k=1}^{2}\left\{\theta\left\|z_k^j\right\|^2_{\overset{0}{1}}+\|z_k\|_0^2\right\}\quad(\delta_0=0),$$
$$\delta_*=\tau\sum_{w_\tau}\sum_{k=1}^{2}\|R_k\|_0^2$$

and constant C, that is independent of the mesh parameters. From (2.5), by an application of the difference analogue of Bernoulli inequality (see, also [1, 2]), for sufficiently small τ, we obtain

$$\delta_j\le C\delta_*,\ j=1,2,...,N_0.$$

We summarize this result in the following theorem.

THEOREM 2.1. *Under the conditions on a_k, b_k, f_k indicated in (1.5), the solution of (2.1)–(2.4) converges to the exact solution in the mesh norm $\|.\|_1$ with the rate $O(h^2+\tau^2)$:*

$$\|y_1-u\|_1+\|y_2-v\|_1\le C(h^2+\tau^2).$$

3. NUMERICAL RESULTS

In this section, we present numerical results obtained by applying the numerical method (2.1)–(2.4) to a particular problem

$$\frac{\partial u}{\partial t}-\frac{\partial^3 u}{\partial t\partial^2 x}+3u\frac{\partial u}{\partial x}+(e^{-t}+x^2)u=f(x,t),(x,t)\in[0,1]\times[0,1],$$
$$u(x,0)=\sin(\pi x)-0.5x^2(1-x),$$
$$u(0,t)=u(1,t)=0,$$

where f is such that the exact solution is given by

$$u(x,t)=-\frac{1}{2}x^2(1-x)+e^{-t}\sin\pi x.$$

In the computations, the quasilinearization technique to the difference equation was used (see, [3]). The step sizes h and τ were both taken to be 0.025. The results at selected points are presented in Table 1.

Table 1

(x,t)	Exact Solution	Approximate Solution	Pointwise Errors
$(0.05, 0.05)$	0.14761757	0.14761772	0.00000015
$(0.15, 0.15)$	0.38119074	0.38119121	0.00000047
$(0.25, 0.25)$	0.52725781	0.52725907	0.00000126
$(0.35, 0.35)$	0.58806919	0.58807119	0.00000200
$(0.45, 0.45)$	0.57409039	0.57409267	0.00000228
$(0.55, 0.55)$	0.50178410	0.50178641	0.00000231
$(0.65, 0.65)$	0.39120869	0.39121092	0.00000223
$(0.75, 0.75)$	0.26370109	0.26370237	0.00000128
$(0.85, 0.85)$	0.13985482	0.13985580	0.00000098
$(0.95, 0.95)$	0.03793713	0.03793732	0.00000019

The obtained results show an excellent degree of accuracy for the above described method.

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