ASYMPTOTIC STABILITY FOR A CLASS OF MARKOV SEMIGROUPS

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Let $U \subset K$ be an open and dense subset of a compact metric space such that $\partial U \neq \emptyset$. Let $\Phi$ be a Markov operator with the strong Feller property acting on the space of bounded Borel measurable functions on $U$. Suppose that for each $x \in \partial U$ there exists a barrier $h \in C(K)$ at $x$ such that $\Phi(h) \geq h$. Suppose moreover that every real-valued $g \in C(K)$ with $\Phi(g) \geq g$ and which attains its global maximum at a point inside $U$ is constant. Then for each $f \in C(K)$ there exists the uniform limit $F = \lim_{n \to \infty} \Phi^n(f)$. Moreover $F \in C(K)$ agrees with $f$ on $\partial U$ and $\Phi(F) = F$.

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1. INTRODUCTION

Let $E$ be a locally compact Hausdorff space and let $M_b(E)$ be the space of all complex-valued, bounded and Borel measurable functions on $E$. Let also $C_b(E)$ be the set of all continuous functions in $M_b(E)$. A linear map

$$\Phi : M_b(E) \to M_b(E)$$

is called a Markov operator if for each $x \in E$ there exists a probability Borel measure $\mu_x$ on $E$ such that

$$\Phi(f)(x) = \int f \, d\mu_x \quad \forall f \in M_b(E).$$

A Markov operator $\Phi$ is positive, in the sense that $\Phi(f) \geq 0$ for every $f \in M_b(E)$ with $f \geq 0$ on $E$. A Markov operator $\Phi$ is said to have the strong Feller property if $\Phi(f) \in C_b(E)$ for every $f \in M_b(E)$.

In [1] J. Arazy and M. Engliš studied the iterates of Markov operators acting on $M_b(U)$ where $U$ is a bounded domain in $\mathbb{C}^d$. One of their main results is the following (see Theorem 1.4 in [1]):

**Theorem 1.1.** Let $U \subset \mathbb{C}^d$ be a bounded domain and let $H^\infty(U)$ be the algebra of all bounded analytic function on $U$. Let $\partial_p(U) \subset \partial U$ be the set of all peak points for $H^\infty(U) \cap C(\overline{U})$. Let

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be a Markov operator with the strong Feller property such that \( \Phi(h) = h \) for every \( h \in H^\infty(U) \).

Then for every continuous function \( F \) on the closure \( \bar{U} \) the sequence \( \{ \Phi^n(F) \} \) converges uniformly on every subset of \( U \) whose closure is contained in \( U \cup \partial_p(U) \) to a function \( G \in C_b(U) \). Moreover, \( \Phi(G) = G \) and

\[
\lim_{y \to x} G(y) = F(x)
\]

for every \( x \in \partial_p(U) \).

The authors provide a large class of such operators, including Berezin transforms on strictly pseudoconvex domains and convolution operators on bounded symmetric domains.

In this paper we prove a result concerning the iterates of a class of Markov operators with the strong Feller property associated with dense open subsets of compact metric spaces. Under suitable boundary conditions and assuming a maximum principle for the interior points, we obtain a uniform convergence similar to that of Theorem 1.1. Theorem 2.1 below can be used to give an alternate proof for the main result in [5]. Indeed, it may be easily seen that the Berezin-type transform defined there satisfies all the conditions appearing in Theorem 2.1. The methods used in [1] strongly motivated and inspired this research.

2. THE MAIN RESULT

The main result of this paper is the following:

**Theorem 2.1.** Let \( K \) be a compact metric space and let \( U \subset K \) be a dense open subset such that \( \partial U \neq \emptyset \). Let \( C(K) \) be the space of all continuous complex-valued functions on \( K \) and let \( C(\partial U) \) be the corresponding space for \( \partial U \).

Let \( \Phi : M_b(U) \to M_b(U) \) be a Markov operator with the strong Feller property. Suppose that \( \Phi \) satisfies the following conditions:

(A) For each point \( x \in \partial U \) there exists \( h \in C(K) \) such that \( h(x) = 0 \), \( h(y) < 0 \) for all \( y \in K \setminus \{ x \} \) and \( \Phi(h_U) \geq h_U \) on \( U \), where \( h_U \) is the restriction of \( h \) to \( U \);

(B) If \( g \in C(K) \) is a real valued function with \( \Phi(g_U) \geq g_U \) and if there exists \( z \in U \) such that \( g(z) = \max\{ g(x) : x \in K \} \) then \( g \) is constant on \( K \).

Then, for each \( f \in C(\partial U) \) there exists a unique function \( G \in C(K) \) such that \( \Phi(G_U) = G_U \) and \( G(x) = f(x) \) for all \( x \in \partial U \). Moreover, if \( F \in C(K) \) is an arbitrary continuous extension of \( f \) to \( K \) then the sequence \( \{ \Phi^n(F_U) \}_n \) converges uniformly on \( U \) to \( G_U \).
Proof. Let us assume that $\partial U$ contains at least two points.

(1) First of all, it can be proved, exactly as in Proposition 1.3 from [1], that the boundary condition (A) implies the following. For each $f \in C(K)$ and for each $x \in \partial U$

$$\lim_{y \to x} \sup_{n \geq 1} |(\Phi^n(f))(y) - f(x)| = 0.$$ 

The idea is to use a barrier at $x$ in order to show that, for any neighborhood $W$ of $x$,

$$\lim_{y \to x} \Phi^n(\chi_W)(y) = 1$$

uniformly for all $n \geq 1$, where $\chi_W$ is the characteristic function of $W$. We refer to [1] for details.

In particular, this shows that for each $f \in C(K)$ the function $\Phi(f_U)$ extends continuously up to $\partial U$ and this extension agrees with $f$ on $\partial U$. By abuse of notation, we shall continue to denote this extension by $\Phi(f)$.

(2) We show that if $f \in C(K)$ is a real-valued function such that $\Phi(f) \geq f$ then the sequence $\{\Phi^n(f)\}$ converges uniformly on $K$. Indeed, this sequence is monotone increasing and if $g$ is its pointwise limit then $g$ is Borel measurable and $\Phi(g_U) = g_U$. Since $\Phi$ has the strong Feller property, it follows that $g$ is continuous on $U$. Moreover (1) implies that

$$\lim_{y \to x} g(y) = f(x)$$

for every $x \in \partial U$. Since $g$ agrees with $f$ on $\partial U$ and $f \in C(K)$ we see that $g \in C(K)$ and Dini’s theorem shows that the convergence is uniform on $K$.

(3) Let

$$\mathcal{T}(\Phi) = \{h \in C(K) : \Phi(h) = h\}$$

and let $C^{*}(\mathcal{T}(\Phi))$ be the norm closed subalgebra of $C(K)$ generated by $\mathcal{T}(\Phi)$. Since $\mathcal{T}(\Phi)$ is a selfadjoint subspace of $C(K)$ it follows that $C^{*}(\mathcal{T}(\Phi))$ is a commutative $C^{*}$-algebra.

Let $C(\Phi)$ be the set of all $f \in C(K)$ for which the sequence $\{\Phi^n(f)\}$ is uniformly convergent on $K$ and denote $\pi(f)$ its limit. It is clear that $\pi(f) \in \mathcal{T}(\Phi)$ for every $f \in C(\Phi)$. Let also

$$C(\Phi)_0 = \{f \in C(\Phi) : \pi(f) = 0\}.$$ 

We will show that $C^{*}(\mathcal{T}(\Phi)) \subset C(\Phi)$. It suffices to prove that for any finite set of $k$ functions from $\mathcal{T}(\Phi)$ their product belongs to $C(\Phi)$.

Let $k = 2$. Let $h \in \mathcal{T}(\Phi)$. Since $\Phi(\|f\|^2) \geq |\Phi(f)|^2$ for every $f \in C(K)$ we see that $\Phi(\|h\|^2) \geq |h|^2$. In this case (2) shows that $|h|^2 \in C(\Phi)$ therefore $g = \pi(|h|^2) - |h|^2 \in C(\Phi)_0$. Since $g \geq 0$ we see that $gf \in C(\Phi)_0$ for every
$f \in C(K)$. Indeed, it is easy to see that if $F \in C(\Phi)_0$ is non-negative on $K$ then $Ff \in C(\Phi)_0$ for every $f \in C(K)$.

If $h_1, h_2 \in \mathcal{T}(\Phi)$ then

$$h_1 h_2 = (1/4) \sum_{m=0}^{3} i^m |g_m|^2$$

where $i = \sqrt{-1}$ and $g_m = (h_1 + i^m \bar{h}_2)$. Since $g_m \in \mathcal{T}(\Phi)$ we see that $h_1 h_2 \in C(\Phi)$.

Let $k \geq 3$ and assume that every product of at most $k-1$ elements from $\mathcal{T}(\Phi)$ belongs to $C(\Phi)$. Let $h_1, \ldots, h_k$ in $\mathcal{T}(\Phi)$ and let $g = h_1 \cdots h_k$. Then

$$g = (h_1 h_2 - \pi(h_1 h_2)) \cdot h_3 \cdots h_k + \pi(h_1 h_2) \cdot h_3 \cdots h_k.$$

By what we have already proved the first summand belongs to $C(\Phi)_0$ and the second belongs, by the induction hypothesis, to $C(\Phi)$. This shows that $C^*(\mathcal{T}(\Phi)) \subset C(\Phi)$.

(4) Consider the map

$$\rho : C^*(\mathcal{T}(\Phi)) \to C(\partial U)$$

that takes any $f \in C^*(\mathcal{T}(\Phi))$ into its restriction to $\partial U$. It turns out that $\rho$ is onto. To see this, we first observe that the boundary condition (A) together with (2) implies that for each $x \in \partial U$ there exists $g \in \mathcal{T}(\Phi)$ such that $g(x) = 0$ and $g(y) < 0$ for every $y \in \partial U \setminus \{x\}$. This shows that the range of $\rho$ separates the points of $\partial U$ and the Stone-Weierstrass theorem shows that this image is norm-dense in $C(\partial U)$. On the other hand, it is well-known (see Theorem I.5.7 in [6]) that the image of a *-homomorphism between two $C^*$-algebras is norm-closed. The conclusion is that $\rho$ is onto. Since $C^*(\mathcal{T}(\Phi)) \subset C(\Phi)$, and since for each $f \in C(\Phi)$ the function $\pi(f) \in \mathcal{T}(\Phi)$ and agrees with $f$ on $\partial U$ we see that for each $f \in C(\partial U)$ there exists a unique function $\theta(f) \in \mathcal{T}(\Phi)$ which agrees with $f$ on $\partial U$. Uniqueness follows easily from (B). The map $f \mapsto \theta(f)$ is obviously linear and unit preserving.

(5) Let $L = \{g \in C(K) : g = \pi(|h|^2) - |h|^2 \mbox{ for some } h \in \mathcal{T}(\Phi)\}$. We will show that $L \neq \{0\}$. Assume, on the contrary, that $L = \{0\}$ which means that $\pi(|h|^2) = |h|^2$ for every $h \in \mathcal{T}(\Phi)$. In this case, if $h_1$ and $h_2$ are functions in $\mathcal{T}(\Phi)$ then $\pi(h_1 h_2) - h_1 h_2$ can be written as a linear combination of elements from $L$ (see step (3)). It then follows that $\pi(h_1 h_2) = h_1 h_2$. It follows that $\mathcal{T}(\Phi)$ is closed under multiplication hence it equals $C^*(\mathcal{T}(\Phi))$. This shows that the map $\theta : C(\partial U) \to C(K)$ defined in (4) is multiplicative on $C(\partial U)$. It follows that there exists a continuous map $\gamma : K \to \partial U$ such that

$$\theta(f) = f \circ \gamma \quad \forall f \in C(\partial U).$$
Let \( z \in U \) and let \( x = \gamma(z) \). Let \( h \in C(\partial U) \) be a nonconstant real-valued function such that
\[
h(x) = \sup\{h(y); y \in \partial U\}.
\]
Then
\[
h(x) = (h \circ \gamma)(z) = (\theta(h))(z).
\]

However, since \( \theta(h) \in \mathcal{T}(\Phi) \) and is nonconstant, it attains its global maximum only on \( \partial U \). We get a contradiction. The conclusion is that \( L \neq \{0\} \).

(6) Let \( h \in \mathcal{T}(\Phi) \) such that the function \( g = \pi(|h|^2) - |h|^2 \) is not identically zero on \( U \). As pointed out in (3), \( g \geq 0 \) on \( K \) and \( g \in C(\Phi)_0 \). Moreover, one can see that \( \Phi(g) \leq g \). We will show that \( \Phi(g) = g \). Suppose, on the contrary, that there exists \( z \in U \) such that \( g(z) = 0 \). Since \( \Phi(g) \leq g \) and since \( g \geq 0 \) it follows from (B) that \( g = 0 \) on \( K \). This contradiction shows that \( g > 0 \) on \( U \). Since \( g = 0 \) on \( \partial U \) this shows that the closed ideal \( I(g) \) of \( C(K) \) generated by \( g \) equals
\[
Z(\partial U) = \{ f \in C(K) : f = 0 \text{ on } \partial U \}.
\]

Since, as we pointed out in (3), \( I(g) \subset C(\Phi)_0 \) this shows that \( f - \theta(f_{\partial U}) \in C(\Phi)_0 \) for every \( f \in C(K) \), where \( f_{\partial U} \) is the restriction of \( f \) to \( \partial U \). This means that the sequence \( \{\Phi^n(f)\} \) converges uniformly on \( K \) to \( \theta(f_{\partial U}) \) for every \( f \in C(K) \). This completes the proof for the case when \( \partial U \) contains at least two points.

(7) Suppose now that \( \partial U \) reduces to a singleton \( \{x_0\} \). The first two steps hold true in this case as well. Moreover, the boundary condition (A) shows that there exists \( g \in C(K) \) such that \( g(x_0) = 0 \), \( g(y) < 0 \) on \( U \) and \( \Phi(g) \geq g \). It then follows from (2) that \( g \in C(\Phi) \) and, since \( g = 0 \) on \( \partial U \), we see that \( \pi(g) = 0 \) hence \( g \in C(\Phi)_0 \). It then follows, exactly as in (6), that \( I(g) = Z(\partial U) \). This shows that \( f - f(x_0) \in C(\Phi)_0 \) for every \( f \in C(K) \) therefore
\[
\lim_{n \to \infty} \Phi^n(f) = f(x_0)
\]
uniformly on \( K \), for all \( f \in C(K) \). This completes the proof of this theorem. \( \square \)

Condition (B) (the maximum principle) holds true, for instance, in the case when every point \( y \in U \) is accessible for \( \Phi \). This means that for each \( x \in U \) and each open neighborhood \( \omega \) of \( y \) there exists \( k \geq 1 \) such that \( \Phi^k(\chi_\omega)(x) > 0 \), where \( \chi_\omega = 1 \) on \( \omega \) and 0 elsewhere.

3. CONNECTIONS WITH THE CLASSICAL DIRICHLET PROBLEM

Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain (open connected set). Let \( \delta : \Omega \to \mathbb{R}_+ \) be a strictly positive, continuous function on \( \Omega \) such that \( \delta(x) \leq \text{dist}(x, \partial \Omega) \)
for all $x \in \Omega$. For each $x \in \Omega$ let $B(x)$ denote the open ball of radius $\delta(x)$ centered at $x$. Let $\lambda$ be the Lebesgue measure on $\mathbb{R}^d$.

For each $x \in \Omega$ define a probability measure $\mu_x$ on $\Omega$ as follows:

$$\mu_x(\omega) = (\lambda(B(x)))^{-1}\lambda(\omega \cap B(x))$$

for every Borel subset $\omega \subset \Omega$. It is easy to see that the map $x \mapsto \mu_x(\omega)$ is continuous on $\Omega$ for any fixed Borel subset $\omega \subset \Omega$. It then follows that for any $f \in M_b(\Omega)$ the function

$$\Phi(f)(x) = \int_{\Omega} f(y) d\mu_x(y) \quad x \in \Omega$$

is continuous on $\Omega$ therefore we obtain in this way a Markov operator

$$\Phi : M_b(\Omega) \to M_b(\Omega)$$

with the strong Feller property. This operator can be expressed as follows:

$$\Phi(f)(x) = (\lambda(B(x)))^{-1} \int_{B(x)} f(y) d\lambda(y) \quad x \in \Omega$$

for all $f \in M_b(\Omega)$. Due to the connectedness of $\Omega$, $\Phi$ satisfies the maximum principle from Theorem 2.1. Such operators are usually called averaging operators.

If $h$ is a real valued bounded, subharmonic function on $\Omega$ then $\Phi(h) \geq h$ on $\Omega$. In particular, every bounded harmonic function on $\Omega$ is a fixed point for $\Phi$. The converse is also true but much more difficult to prove (see [4] and [2]).

Assume now that every point $x \in \partial \Omega$ is regular for the classical Dirichlet problem on $\Omega$ (see [3] for basics of potential theory). In this case, the Markov operator $\Phi$ defined above satisfies all the assumptions of Theorem 2.1. It then follows that for each function $F \in C(\Omega)$ the sequence $\{\Phi^n(F)\}_n$ converges uniformly on $\Omega$ and its limit $G$ is the solution of the classical Dirichlet problem with boundary value $f = F|_{\partial \Omega}$. The harmonicity of $G$ follows directly from the maximum principle, therefore there is no need to make appeal to the deep results mentioned above.

It might be interesting to find other examples of elliptic operators to which Theorem 2.1 could be applied in a similar manner. The results of this paper may be linked to the field of discrete potential theory and one of the best references here remains the classical monograph of J.L. Doob. [3].

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