In this paper, we introduce the concepts of commutative \((\varepsilon, \varepsilon \lor q_k))-\text{fuzzy ideals}
and commutative \((\bar{\varepsilon}, \bar{\varepsilon} \lor \bar{q}_k))-\text{fuzzy ideals of BCH-algebras and investigate some}
of their properties. The concept of a commutative fuzzy ideal with thresholds,
commutative fuzzyfying ideal and commutative t-implication-based fuzzy ideal
are introduced and studied.

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Key words: BCH-algebra, commutative \((\varepsilon, \varepsilon \lor q_k))-\text{fuzzy ideal, commutative}
\((\bar{\varepsilon}, \bar{\varepsilon} \lor \bar{q}_k))-\text{fuzzy ideal, commutative fuzzy ideal with threshold;}
commutative t-implication-based fuzzy ideal.

1. INTRODUCTION

The theory of BCH-algebras was first introduced by Hu and Li in [13]
and gave examples of proper BCH-algebras [14]. The theories were further
enriched by many authors (see [1, 6–8, 11, 25, 28–30]). The concept of a fuzzy
set, which was introduced by Zadeh in his definitive paper [36] of 1965, was
applied by many researchers to generalize some of the basic concepts of al-
gebras. The fuzzy algebraic structures play a vital role in Mathematics with
wide applications in many other branches such as theoretical physics, computer
sciences, control engineering, information sciences, coding theory, topological
spaces, logic, set theory, real analysis, measure theory etc. Chang applied it to
the topological spaces in [5]. Das and Rosenfeld applied it to the fundamental
theory of fuzzy groups in [9, 27]. In [15], Hong et al. applied the concept to
BCH-algebras and studied fuzzy dot subalgebras of BCH-algebras. Jun, gives
characterizations of BCI/BCH-algebras in [17]. In 2001, Jun et al. discussed on
imaginable T-fuzzy subalgebras and imaginable T-fuzzy closed ideals of BCH-
gebras [18]. Kim [22] studied intuitionistic \((T, S))-\text{normed fuzzy closed ideals}
of BCH-algebras. In [20], Jun et al. discussed N-structures applied to closed
ideals in BCH-algebras. Jun and Park investigated filters of BCH-algebras
based on bipolar-valued fuzzy sets in [19]. In [10], Dudek and Rousseau, give
the idea of set-theoretic relations and BCH-algebras with trivial structure. In [21], Kazanci et al. studied soft set and soft BCH-algebras. Yin studied fuzzy dot ideals and fuzzy dot H-ideals of BCH-algebras in [32]. In [31], Saeid et al. discussed fuzzy n-fold ideals in BCH-algebras.

In 1971, Rosenfeld’s laid the foundation of fuzzy groups in [27]. Murali defined the concept of belongingness of a fuzzy point to a fuzzy set under a natural equivalence on a fuzzy set in [24]. In [26], the idea of fuzzy point and its belongingness to and quasi-coincidence with a fuzzy set were used to define ($\alpha, \beta$)-fuzzy subgroups, where $\alpha, \beta \in \{\in, q, \in \lor q, \in \land q\}$ with $\alpha \neq \in \land q$. This concept was further discussed by Yuan et al. [35]. In particular, ($\in, \in \lor q$)-fuzzy subgroup is an important and useful generalization of the Rosenfeld’s fuzzy subgroups. The ($\in \lor q$)-level subsets was discussed in [2]. In [3], Bhakat introduced the concept of ($\in, \in \lor q$)-fuzzy normal, quasi-normal and maximal subgroups. In [4], Bhakat and Das studied ($\in, \in \lor q$)-fuzzy subgroups.

In this paper, we show that a fuzzy set $\mu$ of a BCH-algebra $X$ is a commutative fuzzy ideal of $X$ if and only if $\mu_t \neq \phi$ is a commutative ideal of $X$. We also prove that a fuzzy set $\mu$ of a BCH-algebra $X$ is a commutative ($\in, \in \lor q_k$)-fuzzy ideal of $X$ if and only if it satisfies (K) and (L), where

(K) $\mu(0) \geq \mu(x) \land \frac{1-k}{2}$,

(L) $\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \geq \mu((x \ast y) \ast z) \land \mu(z) \frac{1-k}{2}$, for all $x, y, z \in X$.

We further show that a fuzzy set $\mu$ of a BCH-algebra $X$, the following are equivalent:

(U) $\mu$ is a commutative ($\in, \in \lor q_k$)-fuzzy ideal of $X$.

(V) $[\mu]^k_t \neq \phi \Rightarrow [\mu] \triangleright X$, for all $t \in (0, 1]$.

We call $[\mu]^k_t$ a commutative ($\in \lor q_k$)-level ideal of $\mu$.

We prove that if $\{\mu_i | i \in \Lambda\}$ be a family of commutative ($\in, \in \lor q_k$)-fuzzy ideals of a BCH-algebra $X$, then $\mu = \cap_{i \in \Lambda} \mu_i$ is a commutative ($\in, \in \lor q_k$)-fuzzy ideal of $X$.

In Section 2, we recall some ideal and define commutative ideal of BCH-algebra; in Section 3, we review some fuzzy logic concepts and define commutative fuzzy ideal and discuss some of their level ideal; in Section 4, we define ($\in, \in \lor q_k$)-fuzzy subalgebra, ($\in, \in \lor q_k$)-fuzzy ideal and commutative ($\in, \in \lor q_k$)-fuzzy ideal and investigate some of their properties; in Section 5, we define the concepts of ($\in, \bar{\in} \lor \bar{q}_k$)-fuzzy ideal and commutative ($\in, \bar{\in} \lor \bar{q}_k$)-fuzzy ideal of BCH-algebras and investigate some of their properties; in Section 6, we define the concepts of fuzzy ideal with thresholds and commutative fuzzy ideal with thresholds. Finally, in Section 7, we consider the concepts of fuzzifying ideal, commutative fuzzifying ideal, t-implication-based fuzzy ideal and commutative t-implication-based fuzzy ideal of BCH-algebras. In particular, we
discuss the implication operators by Lukasiewicz system of continuous-valued logic.

2. CRISP SETS – LEVEL 0

Throughout this paper X always means a BCH-algebra without any specification. We also include some basic results that are necessary for this paper.

By a BCH-algebra [1], we mean an algebra $(X, \ast, 0)$ of type $(2, 0)$ satisfying the axioms:

(BCH-I) $x \ast x = 0$
(BCH-II) $x \ast y = 0$ and $y \ast x = 0$ imply $x = y$
(BCH-III) $(x \ast y) \ast z = (x \ast z) \ast y$

for all $x, y, z \in X$. A BCH-algebra $X$ is said to be a BCI-algebra if it satisfies the identity:

(BCI-I) $((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0$

for all $x, y, z \in X$.

BCC-algebras (introduced by Komori [23]) are generalizations of BCK-algebras, weak BCC-algebras are generalizations of BCI-algebras. By many mathematicians, especially from China and Korea, weak BCC-algebras are called BZ-algebras ([12, 37, 38]). A weak BCC-algebra satisfying the identity

$0 \ast x = 0$

is called a BCC-algebra. A weak BCC-algebra satisfying the identity

$(x \ast y) \ast z = (x \ast z) \ast y$

is called a BCI-algebra. A weak BCC-algebra which is neither a BCI-algebra or a BCC-algebra is called proper. A BCC-algebra with the condition

$(x \ast (x \ast y)) \ast y = 0$

is called a BCK-algebra. BCK-algebras and BCI-algebras are two important classes of logical algebras introduced by Imai and Iseki [16] in 1966. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Since then, a great deal of literature has been produced on the theory of BCK/BCI-algebras. In [13], Hu and Li introduced a wide class of abstract algebras called BCH-algebras based upon BCK/BCI-algebras, and subsequently gave examples of proper BCH-algebras [14]. For the general development of the BCK/BCI/BCH-algebras the subalgebras play a central role. It is known that every BCI-algebra is a BCH-algebra but not conversely. A BCH-algebra $X$ is called proper if it is not a BCI-algebra. It is known that proper BCH-algebras exist. In any BCH / BCI-algebra $X$ we can define a partial order $\leq$ by putting $x \leq y$ if and only if $x \ast y = 0$. 
**Proposition 2.1 ([31]).** In any BCH-algebra X, the following are true:

1. \( x \ast (x \ast y) \leq y \)
2. \( 0 \ast (x \ast y) = (0 \ast x) \ast (0 \ast y) \)
3. \( x \ast 0 = x \)
4. \( x \leq 0 \) implies \( x = 0 \)

for all \( x, y \in X \).

**Definition 2.2 ([15]).** A nonempty subset \( S \) of a BCH-algebra \( X \) is called a subalgebra of \( X \) if it satisfies

\[ x \ast y \in S, \text{ for all } x, y \in S. \]

**Definition 2.3 ([29]).** A subset \( I \) of a BCH-algebra \( X \) is called an ideal of \( X \) if it satisfies (I1) and (I2), where

1. \( 0 \in I \),
2. \( x \ast y \in I \) and \( y \in I \) imply \( x \in I \),

for all \( x, y \in X \).

**Definition 2.4.** A subset \( I \) of a BCH-algebra \( X \) is called a commutative ideal of \( X \), denoted by \( I \cong X \), if it satisfies (I1) and (I3), where

1. \( 0 \in I \),
2. \( (x \ast y) \ast z \in I \) and \( z \in I \) imply \( x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \in I \),

for all \( x, y, z \in X \).

**Theorem 2.5.** A commutative ideal of a BCH-algebra must be an ideal, but the converse does not hold.

**Proof.** Suppose \( I \) is a commutative ideal of \( X \) and for all \( x, y, z \in X \), we have

\[(x \ast y) \ast z \in I \text{ and } z \in I \text{ imply } x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \in I.\]

Put \( y = 0 \) in above we get
\[(x \ast 0) \ast z \in I \text{ and } z \in I \text{ imply } x \ast ((0 \ast (0 \ast x)) \ast (0 \ast (0 \ast x))) \in I \text{ (by Proposition 2.1(3))}.\]

This means that I satisfy (I2). Combining (I1) implies that I is an ideal. The last part is shown by the example.

**Example 2.6.** Suppose \( X = \{0, 1, 2, 3, 4\} \), the operation is given by the table

```markdown
<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>1</td>
<td>1</td>
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<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>
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Then \((X, *, 0)\) is a BCH-algebra. It is easy to verify that \(\{0, 2\}\) is an ideal but \(\{0, 1, 3\}\) is not commutative ideal.

**Theorem 2.7.** Let \(X\) be a BCH-algebra. Then an ideal \(I\) of \(X\) is a commutative ideal if and only if the condition

\[
(\text{for all } x, y \in X) \ (x * y) \in I \Rightarrow x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in I
\]

is satisfied.

**Proof.** Straightforward. \(\square\)

3. **LEVEL 1 OF FUZZIFICATION**

We now review some fuzzy logic concepts.

**Definition 3.1 ([36]).** A fuzzy set \(\mu\) of a universe \(X\) is a function from \(X\) into the unit closed interval \([0, 1]\), that is \(\mu : X \to [0, 1]\).

**Definition 3.2 ([9]).** For a fuzzy set \(\mu\) of a BCH-algebra \(X\) and \(t \in (0, 1]\), the crisp set

\[
\mu_t = \{x \in X \mid \mu(x) \geq t\}
\]

is called the level subset of \(\mu\).

Recall that \([0, 1], \land = \min, \lor = \max, 0, 1\) is a complete lattice (chain).

**Definition 3.3 ([22]).** Let \(X\) be a BCH-algebra. A fuzzy set \(\mu\) in \(X\) is said to be a fuzzy subalgebra of \(X\) if it satisfies, for all \(x, y \in X\),

\[
\mu(x * y) \geq \mu(x) \land \mu(y).
\]

**Theorem 3.4.** Let \(\mu\) be a fuzzy set of a BCH-algebra \(X\). Then \(\mu\) is a fuzzy subalgebra of \(X\) if and only if \(\mu_t = \{x \in X \mid \mu(x) \geq t\}\) is a subalgebra of \(X\) for all \(t \in (0, 1]\), for our convenience, the empty set \(\phi\) is regarded as a subalgebra of \(X\).

**Proof.** Straightforward. \(\square\)
Definition 3.5 ([31]). A fuzzy set \( \mu \) of a BCH-algebra \( X \) is called a fuzzy ideal of \( X \) if it satisfies (F1) and (F2), where

(F1) \( \mu(0) \geq \mu(x) \),
(F2) \( \mu(x) \geq \mu(x \ast y) \land \mu(y) \),

for all \( x, y \in X \).

Theorem 3.6. A fuzzy set \( \mu \) of a BCH-algebra \( X \) is a fuzzy ideal if and only if \( \mu_t \neq \phi \) is an ideal of \( X \).

Proof. The proof of the following theorem is obvious. \( \square \)

Definition 3.7. A fuzzy set \( \mu \) of a BCH-algebra \( X \) is called a commutative fuzzy ideal of \( X \) if it satisfies (F1) and (F3), where

(F1) \( \mu(0) \geq \mu(x) \),
(F3) \( \mu((x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq \mu((x \ast y) \ast z) \land \mu(z) \),

for all \( x, y, z \in X \).

Theorem 3.8. A fuzzy set \( \mu \) of a BCH-algebra \( X \) is a commutative fuzzy ideal of \( X \) if and only if \( \mu_t \neq \phi \) is a commutative ideal of \( X \).

Proof. Let \( \mu \) be a commutative fuzzy ideal of \( X \) and \( \mu_t \neq \phi \) for \( t \in (0, 1] \). Since \( \mu(0) \geq \mu(x) \geq t \) for all \( x \in \mu_t \), we get \( 0 \in \mu_t \). If \( (x \ast y) \ast z \in \mu_t \) and \( z \in \mu_t \), then

\[ \mu((x \ast y) \ast z) \geq t \text{ and } \mu(z) \geq t. \]

It follows from (F3) that

\[ \mu((x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq \mu((x \ast y) \ast z) \land \mu(z) \]
\[ \geq t \land t \geq t. \]

Hence, \( x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \in \mu_t \). This shows that \( \mu_t \) is a commutative ideal of \( X \) by (I3).

Conversely, suppose that for each \( t \in (0, 1] \), \( \mu_t \) is either empty or a commutative ideal of \( X \). For any \( x \in X \), setting \( 0 \neq \mu(x) = t \), then \( x \in \mu_t \). Since \( \mu_t(\neq \phi) \) is a commutative ideal of \( X \), we have \( 0 \in \mu_t \) and hence, \( \mu(0) \geq t = \mu(x) \). If \( \mu(x) = 0 \) then obviously \( \mu(0) \geq 0 = \mu(x) \). Thus, \( \mu(0) \geq \mu(x) \) for all \( x \in X \). Now we prove that \( \mu \) satisfies (F3). If not, then there exist \( x_1, y_1, z_1 \in X \) such that

\[ \mu(x_1 \ast ((y_1 \ast (y_1 \ast x_1)) \ast (0 \ast (0 \ast (x_1 \ast y_1)))))) < \mu((x_1 \ast y_1) \ast z_1) \land \mu(z_1). \]

Select \( t \in (0, 1] \) such that

\[ \mu(x_1 \ast ((y_1 \ast (y_1 \ast x_1)) \ast (0 \ast (0 \ast (x_1 \ast y_1)))))) < t \leq \mu((x_1 \ast y_1) \ast z_1) \land \mu(z_1). \]
Hence, \((x_1 \ast y_1) \ast z_1 \in \mu_t\) and \(z_1 \in \mu_t\), but \(x_1 \ast ((y_1 \ast (y_1 \ast x_1)) \ast (0 \ast (0 \ast (x_1 \ast y_1)))) \in \mu_t\), which is a contradiction. Therefore,

\[\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq \mu((x \ast y) \ast z) \land \mu(z).\]

Consequently, \(\mu\) is a commutative fuzzy ideal of \(X\). \(\Box\)

**Theorem 3.9.** Any commutative fuzzy ideal of a BCH-algebra is a fuzzy ideal.

**Proof.** Let \(\mu\) be a fuzzy commutative ideal of a BCH-algebra \(X\) and let \(x, z \in X\).

Then

\[\mu(x \ast z) \land \mu(z) = \mu(((x \ast 0) \ast z) \land \mu(z) \text{ (by Proposition 2.1(3))} \leq \mu(x \ast ((0 \ast (0 \ast x)) \ast (0 \ast (0 \ast (x \ast 0)))) (by F3) = \mu(x \ast ((0 \ast (0 \ast x)) \ast (0 \ast (0 \ast x)))) (by Proposition 2.1(3)) = \mu(x \ast 0) \text{ (BCH-I)} = \mu(x) \text{ (by Proposition 2.1(3))}\]

Hence, \(\mu\) is a fuzzy ideal of \(X\). \(\Box\)

**Remark 3.10.** The converse of the above theorem may not be true as shown in the following example.

**Example 3.11.** Let \(X = \{0, 1, 2, 3, 4\}\) in which the operation is defined by

| \* | 0 1 2 3 4 |
|---|---|---|---|---|
| 0 | 0 0 0 0 0 |
| 1 | 1 0 1 0 0 |
| 2 | 2 2 0 0 0 |
| 3 | 3 3 3 0 0 |
| 4 | 4 3 4 3 0 |

Then \((X, \ast, 0)\) is a BCH-algebra. Let \(t_0, t_1, t_2 \in [0, 1]\) be such that \(t_0 > t_1 > t_2\). We define a map \(\mu : X \rightarrow [0, 1]\) by \(\mu(0) = t_0, \mu(1) = t_1\) and \(\mu(2) = \mu(3) = \mu(4) = t_2\). By simple calculations show that \(\mu\) is a fuzzy ideal of \(X\).

But \(\mu\) is not a commutative fuzzy ideal of \(X\), because

Put \(x = 2, y = 3, z = 0\) in (F3) we get

\[\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq \mu((x \ast y) \ast z) \land \mu(z)\]

\[\mu(2 \ast ((3 \ast (3 \ast 2)) \ast (0 \ast (0 \ast (2 \ast 3)))))) \geq \mu((2 \ast 3) \ast 0) \land \mu(0)\]

\[\mu(2 \ast ((3 \ast 3) \ast (0 \ast (0 \ast 0)))) \geq \mu(0) \land \mu(0)\]

\[\mu(2 \ast (0 \ast (0 \ast 0))) \geq \mu(0) \land \mu(0)\]

\[\mu(2 \ast (0 \ast 0)) \geq \mu(0) \land \mu(0)\]
Theorem 3.12. A fuzzy ideal $\mu$ of a BCH-algebra $X$ is a commutative fuzzy ideal of $X$ if and only if it satisfies the condition

$$\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y))))) \geq \mu(x \ast y) \ (i)$$

for all $x, y \in X$.

Proof. Assume that $\mu$ is a commutative fuzzy ideal of $X$. By (F3), we have

$$\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y))))) \geq \mu((x \ast y) \ast z) \land \mu(z).$$

Put $z = 0$ in above, we get

$$\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y))))) \geq \mu((x \ast y) \ast 0) \land \mu(0)$$

$$\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y))))) \geq \mu(x \ast y) \land \mu(0) \ (\text{by Proposition 2.1(3)})$$

$$\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y))))) \geq \mu(x \ast y) \ (\text{by using condition (F1)})$$

Conversely, suppose $\mu$ satisfies (i). As $\mu$ is a fuzzy ideal, hence

$$\mu(x \ast y) \geq \mu((x \ast y) \ast z) \land \mu(z) \ (\text{ii})$$

for all $x, y, z \in X$. Combining (ii) and (i), then we obtain

$$\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y))))) \geq \mu((x \ast y) \ast z) \land \mu(z).$$

Hence, $\mu$ is a commutative fuzzy ideal of $X$.

4. LEVEL 2.1 OF FUZZIFICATION

In this section, we define $(\in, \in \lor q_k)$-fuzzy subalgebra, $(\in, \in \lor q_k)$-fuzzy ideal and commutative $(\in, \in \lor q_k)$-fuzzy ideal and investigate some of their properties.

Definition 4.1 ([31]). A fuzzy set $\mu$ of a BCH-algebra $X$ having the form

$$\mu(y) = \begin{cases} 
    t \in (0, 1], & \text{if } y = x \\
    0, & \text{if } y \neq x
\end{cases}$$

is said to be a fuzzy point with support $x$ and value $t$ and is denoted by $x_t$. 
For a fuzzy point \( x_t \) and a fuzzy set \( \mu \) in a set \( X \), Pu and Liu [26] gave meaning to the writing \( x_t \alpha \mu \), where \( \alpha \in \{\in, q, \in \lor q, \in \land q\} \).

A fuzzy point \( x_t \) is said to belong to (resp., quasi-coincident with) a fuzzy set \( \mu \), written as \( x_t \in \mu \) (resp. \( x_t q \mu \)) if \( \mu(x) \geq t \) (resp. \( \mu(x) + t > 1 \)).

To say that \( x_t \in \lor q \mu \) means that \( x_t \in \mu \) or \( x_t q \mu \).

Let \( k \) denote an arbitrary element of \([0, 1)\) unless otherwise specified. For a fuzzy point \( x_t \) and a fuzzy set \( \mu \) of a BCH-algebra \( X \), we say that

(A0) \( x_t q \mu \) if \( \mu(x) + t > 1 \), \( q = q_0 \).

(A) \( x_t \in q k \mu \) if \( \mu(x) + t + k > 1 \).

(B) \( x_t \lor q k \mu \) if \( x_t \in \mu \) or \( x_t q k \mu \).

(C) \( x_t \alpha \mu \) if \( x_t \alpha \mu \) does not hold for \( \alpha \in \{q, \lor q k\} \).

**Definition 4.2.** A fuzzy set \( \mu \) of a BCH-algebra \( X \) is called an \((\in, \lor q k)\)-fuzzy subalgebra of \( X \) if it satisfies

\[
x_{t_1} \in \mu, \; y_{t_2} \in \mu \Rightarrow (x \ast y)_{t_1 \land t_2} \in \lor q k \mu,
\]

for all \( x, y \in X \) and \( t_1, t_2 \in (0, 1] \).

**Theorem 4.3.** Let \( \mu \) be a fuzzy set of a BCH-algebra \( X \). Then the following are equivalent:

(D) \( \mu_t \neq \phi \Rightarrow \mu_t \triangleright X \), for all \( t \in (\frac{1-k}{2}, 1] \).

(E) \( \mu \) satisfies the following assertions:

(i) \( \mu(x) \leq \mu(0) \lor \frac{1-k}{2} \),

(ii) \( \mu((x \ast y) \ast z) \land \mu(z) \leq \mu((0 \ast (x \ast y))) \lor \frac{1-k}{2} \),

for all \( x, y, z \in X \).

**Proof.** Suppose that (D) is hold. If there is \( a \in X \) such that the condition (i) is not true, that is, there exist \( a \in X \) such that

\[
\mu(a) > \mu(0) \lor \frac{1-k}{2}
\]

then \( \mu(a) \in (\frac{1-k}{2}, 1] \) and \( a \in \mu(a) \). But \( \mu(0) < \mu(a) \) implies that \( 0 \notin \mu(a) \), a contradiction. Hence, (i) is hold. Suppose that (ii) is false, i.e.,

\[
w = \mu((a \ast b) \ast c) \land \mu(c) > \mu((0 \ast (a \ast b))) \lor \frac{1-k}{2}
\]

for some \( a, b, c \in X \). Then

\[
w \in (\frac{1-k}{2}, 1] \text{ and } (a \ast b) \ast c, c \in \mu_w.
\]

But

\[
a \ast ((b \ast (b \ast a)) \ast (0 \ast (a \ast b))) \notin \mu_w
\]
since $\mu((a \ast (b \ast (b \ast a)) \ast (0 \ast (0 \ast (a \ast b)))) < w$. This is a contradiction, and so (ii) holds.

Conversely, suppose that $\mu$ satisfies conditions (i) and (ii). Let $t \in (1-k/2, 1]$ be such that $\mu_t \neq \phi$. For any $x \in \mu_t$, we have

$$\mu(0) \lor \frac{1-k}{2} \geq \mu(x) \geq t > \frac{1-k}{2}$$

and so

$$\mu(0) = \mu(0) \lor \frac{1-k}{2} \geq t.$$  

Hence, $0 \in \mu_t$. Let $x, y, z \in X$ be such that $(x \ast y) \ast z \in \mu_t$ and $z \in \mu_t$.

Then

$$\mu((y \ast (0 \ast (x \ast y))) \lor \frac{1-k}{2} \geq \mu((x \ast y) \ast z) \lor \mu(z) \geq t \land t \geq t > \frac{1-k}{2}$$

and thus,

$$\mu((y \ast (0 \ast (x \ast y))) \lor \frac{1-k}{2} \geq \mu((x \ast y) \ast z) \lor \mu(z) \geq t \land t \geq t > \frac{1-k}{2}$$

i.e.,

$$x \ast ((y \ast (0 \ast (x \ast y))) \lor \frac{1-k}{2} \geq \mu((x \ast y) \ast z) \lor \mu(z) \geq t \land t \geq t > \frac{1-k}{2}$$

Therefore $\mu_t$ is a commutative ideal of $X$. □

Setting $k = 0$ in Theorem 4.3, then we have the following Corollary.

**Corollary 4.4.** Let $\mu$ be a fuzzy set of a BCH-algebra $X$. Then the following are equivalent:

- (F) $\mu_t \neq \phi \Rightarrow \mu_t \triangleright X$, for all $t \in (0.5, 1]$.
- (G) $\mu$ satisfies the following assertions:
  - (iii) $\mu(x) \leq \mu(0) \lor 0.5$.
  - (iv) $\mu((x \ast y) \ast z) \land \mu(z) \leq \mu((y \ast (0 \ast (x \ast y))) \lor 0.5)$, for all $x, y, z \in X$.

**Definition 4.5.** A fuzzy set $\mu$ of a BCH-algebra $X$ is called an $(\in, \in \lor q_k)$-fuzzy ideal of $X$ if it satisfies (H) and (I), where

- (H) $x_t \in \mu \Rightarrow 0_t \in \lor q_k \mu$,
- (I) $(x \ast y)_{t_1} \in \mu, y_{t_2} \in \mu \Rightarrow x_{t_1 \land t_2} \in \lor q_k \mu$,

for all $x, y \in X$ and $t, t_1, t_2 \in (0, 1]$.

**Example 4.6.** Let $X = \{0, 1, 2, 3, 4\}$ in which is defined by
Then \((X, *, 0)\) is a BCH-algebra. Define \(\mu : X \rightarrow [0, 1]\) by \(\mu(0) = 0.4, \mu(1) = \mu(3) = 0.7\) and \(\mu(2) = \mu(4) = 0.2\). By simple calculations show that \(\mu\) is an \((\in, \in \cap \mu_k)\)-fuzzy ideal of \(X\) for \(k = 0.4\) and \(\mu\) is not fuzzy ideal of \(X\).

**Definition 4.7.** A fuzzy set \(\mu\) of a BCH-algebra \(X\) is called a commutative \((\in, \in \cap \mu_k)\)-fuzzy ideal of \(X\) if it satisfies (H) and (J), where

\[
\begin{align*}
(H) \quad & x_t \in \mu \Rightarrow 0_t \in \mu_k, \\
(J) \quad & ((x*y)*z)_{t_1} \in \mu, z_{t_2} \in \mu \Rightarrow (x*((y*(y*x))*(0*(0*(x*y))))))_{t_1 \cap t_2} \in \mu_k, \\
\end{align*}
\]

for all \(x, y, z \in X\) and \(t, t_1, t_2 \in (0, 1]\).

**Theorem 4.8.** Every commutative \((\in, \in \cap \mu_k)\)-fuzzy ideal of a BCH-algebra \(X\) is an \((\in, \in \cap \mu_k)\)-fuzzy ideal.

**Proof.** Let \(\mu\) be a commutative \((\in, \in \cap \mu_k)\)-fuzzy ideal of \(X\). Then for all \(t_1, t_2 \in (0, 1]\) and for all \(x, y, z \in X\), we have

\[
((x*y)*z)_{t_1} \in \mu, z_{t_2} \in \mu \Rightarrow (x*((y*(y*x))*(0*(0*(x*y))))))_{t_1 \cap t_2} \in \mu_k.
\]

Putting \(y = 0\) in above, we get

\[
\begin{align*}
(x*0)*z)_{t_1} \in \mu, z_{t_2} \in \mu \Rightarrow (x*((0*(0*x))*(0*(0*(x*0))))))_{t_1 \cap t_2} \in \mu_k \\
(x*z)_{t_1} \in \mu, z_{t_2} \in \mu \Rightarrow (x*((0*(0*x))*(0*(0*x))))_{t_1 \cap t_2} \in \mu_k \quad \text{(by Proposition 2.1(3))} \\
(x*z)_{t_1} \in \mu, z_{t_2} \in \mu \Rightarrow (x*0)_{t_1 \cap t_2} \in \mu_k \quad \text{(BCH-I)} \\
(x*z)_{t_1} \in \mu, z_{t_2} \in \mu \Rightarrow x_{t_1 \cap t_2} \in \mu_k \quad \text{(by Proposition 2.1(3))}
\end{align*}
\]

This means that \(\mu\) satisfies the condition (I). This implies that \(\mu\) is an \((\in, \in \cap \mu_k)\)-fuzzy ideal of \(X\).

A commutative \((\in, \in \cap \mu_k)\)-fuzzy ideal of a BCH-algebra \(X\) with \(k = 0\) is called a commutative \((\in, \in \cap \mu)\)-fuzzy ideal of \(X\).

**Theorem 4.9.** Let \(X\) be a BCH-algebra. A fuzzy set \(\mu\) of \(X\) is a commutative \((\in, \in \cap \mu_k)\)-fuzzy ideal of \(X\) if and only if it satisfies (K) and (L), where

\[
\begin{align*}
(K) \quad & \mu(0) \geq \mu(x) \land \frac{1-k}{2}, \\
(L) \quad & \mu(x ((y*(y*x)))*(0*(0*(x*y)))) \geq \mu((x*y)*z) \land \mu(z) \land \frac{1-k}{2},
\end{align*}
\]

for all \(x, y, z \in X\).
Proof. Assume that $\mu$ is a commutative ($\in, \in \triangledown q_k$)-fuzzy ideal of $X$. Let $x \in X$ and suppose that $\mu(x) < \frac{1-k}{2}$. If $\mu(0) < \mu(x)$, then $\mu(0) < t \leq \mu(x)$ for some $t \in (0, \frac{1-k}{2})$. It follows that $x_t \in \mu$ but $0_t \notin \mu$. Since $\mu(0) + t < 2t < 1 - k$, we get $0_t \notin \mu$. Therefore $0_t \notin \mu$, which is a contradiction. Hence, $\mu(0) \geq \mu(x)$.

Now if $\mu(x) \geq \frac{1-k}{2}$, then $x_{\frac{1-k}{2}} \in \mu$ and so $0_{\frac{1-k}{2}} \in \triangledown q_k \mu$. This implies that

$$\mu(0) \geq \frac{1-k}{2}$$

or

$$\mu(0) + \frac{1-k}{2} > 1 - k.$$

Hence,

$$\mu(0) \geq \frac{1-k}{2}.$$

Otherwise,

$$\mu(0) + \frac{1-k}{2} < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k,$$

a contradiction. Consequently,

$$\mu(0) \geq \mu(x) \land \frac{1-k}{2}$$

for all $x \in X$. Let $x, y, z \in X$ and suppose that

$$\mu((x \ast y) \ast z) \land \mu(z) < \frac{1-k}{2}.$$

Now we have to show that

$$\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \ast (0 \ast (0 \ast (x \ast y)))) \geq \mu((x \ast y) \ast z) \land \mu(z).$$

If not, then

$$\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \ast (0 \ast (0 \ast (x \ast y)))) \ast (0 \ast (0 \ast (x \ast y)))) < t \leq \mu((x \ast y) \ast z) \land \mu(z)$$

for some $t \in (0, \frac{1-k}{2})$. It follows that

$$((x \ast y) \ast z)_t \in \mu$$

and

$$\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \ast (0 \ast (0 \ast (x \ast y)))) \ast (0 \ast (0 \ast (x \ast y)))) + t < 2t < 1 - k,$$

i.e.,

$$(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \ast (0 \ast (0 \ast (x \ast y)))) \ast (0 \ast (0 \ast (x \ast y)))) \ast \overline{q_k} \mu.$$
This is a contradiction. Thus,

\[ \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq \mu((x \ast y) \ast z) \land \mu(z) \]

whenever

\[ \mu((x \ast y) \ast z) \land \mu(z) < \frac{1-k}{2}. \]

If \( \mu((x \ast y) \ast z) \land \mu(z) \geq \frac{1-k}{2} \), then

\[ ((x \ast y) \ast z)_{\frac{1-k}{2}} \in \mu \text{ and } z^{1-k}_{\frac{1}{2}} \in \mu. \]

Since \( \mu \) is a commutative \( (\in, \in \lor q_k) \)-fuzzy ideal, it follows that

\[ (x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y))))))_{\frac{1-k}{2}} \in \mu. \]

So that

\[ \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq \frac{1-k}{2} \]

or

\[ \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) + \frac{1-k}{2} > 1 - k. \]

If \( \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) < \frac{1-k}{2} \), then

\[ \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) + \frac{1-k}{2} < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k \]

a contradiction. Therefore

\[ \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq \frac{1-k}{2}. \]

Consequently,

\[ \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq \mu((x \ast y) \ast z) \land \mu(z) \land \frac{1-k}{2} \]

for all \( x, y, z \in X \).

Conversely, suppose that (K) and (L) are hold. Let \( x \in X \) and \( t \in (0, 1] \) be such that \( x_t \in \mu \). Then \( \mu(x) \geq t \). Suppose \( \mu(0) < t \). If \( \mu(x) < \frac{1-k}{2} \), then

\[ \mu(0) \geq \mu(x) \land \frac{1-k}{2} = \mu(x) \geq t, \]

a contradiction. Hence, \( \mu(x) \geq \frac{1-k}{2} \). This implies that

\[ \mu(0) + t > 2\mu(0) \geq 2(\mu(x) \land \frac{1-k}{2}) = 1 - k. \]

Thus, \( 0_t \in \lor q_k \mu \). Let \( x, y, z \in X \) and \( t_1, t_2 \in (0, 1] \) be such that \( ((x \ast y) \ast z)_{t_1} \in \mu \) and \( z_{t_2} \in \mu \). Then
\[
\mu((x \ast y) \ast z) \geq t_1 \text{ and } \mu(z) \geq t_2.
\]

Suppose that
\[
\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) < t_1 \land t_2.
\]
If \(\mu((x \ast y) \ast z) \land \mu(z) < \frac{1-k}{2}\), then
\[
\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq \mu((x \ast y) \ast z) \land \mu(z) \land \frac{1-k}{2} = \mu((x \ast y) \ast z) \land \mu(z) \geq t_1 \land t_2.
\]
This is not possible, and so
\[
\mu((x \ast y) \ast z) \land \mu(z) \geq \frac{1-k}{2}.
\]
It follows that
\[
\mu((x \ast y) \ast z) \land \mu(z) \geq 1 - k.
\]
So that
\[
(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y))))))_{t_1 \land t_2} \in \ \lor_k \mu.
\]
Hence, \(\mu\) is a commutative \((\in, \in \lor_k)\)-fuzzy ideal of \(X\). \(\Box\)

Setting \(k = 0\) in Theorem 4.9, then we have the following Corollary.

**Corollary 4.10.** Let \(X\) be a BCH-algebra. A fuzzy set \(\mu\) of \(X\) is a commutative \((\in, \in \lor_k)\)-fuzzy ideal of \(X\) if and only if it satisfies (M) and (N), where

- (M) \(\mu(0) \geq \mu(x) \land 0.5,\)
- (N) \(\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq \mu((x \ast y) \ast z) \land \mu(z) \land 0.5,\)

for all \(x, y, z \in X\).

**Theorem 4.11.** A \((\in, \in \lor_k)\)-fuzzy ideal \(\mu\) of a BCH-algebra \(X\) is a commutative \((\in, \in \lor_k)\)-fuzzy ideal of \(X\) if and only if it satisfies the condition
\[
\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq \mu(x \ast y) \land \frac{1-k}{2} \ (iii)
\]
for all \(x, y \in X\).

**Proof.** Assume that \(\mu\) is a commutative \((\in, \in \lor_k)\)-fuzzy ideal of \(X\). By (L), we have
\[
\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq \mu((x \ast y) \ast z) \land \mu(z) \land \frac{1-k}{2}.
\]
Put \( z = 0 \) in above, we get
\[
\begin{align*}
\mu(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) & \geq \mu((x * y) * 0) \land \mu(0) \land \frac{1-k}{2} \\
\mu(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) & \geq \mu(x * y) \land \mu(0) \land \frac{1-k}{2} \quad \text{(by Proposition 2.1(3))} \\
\mu(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) & \geq \mu(x * y) \land \frac{1-k}{2} \quad \text{(by using condition (K))}
\end{align*}
\]

Conversely suppose \( \mu \) satisfies (iii). As \( \mu \) is a \((\varepsilon, \varepsilon \lor q_k)\)-fuzzy ideal, hence,
\[
\mu(x * y) \geq \mu((x * y) * z) \land \mu(z) \land \frac{1-k}{2} \quad \text{(iv)}
\]
for all \( x, y, z \in X \). Combining (iv) and (iii), then we obtain
\[
\begin{align*}
\mu(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) & \geq \mu((x * y) * z) \land \mu(z) \land \frac{1-k}{2} \land \frac{1-k}{2} \\
& \geq \mu((x * y) * z) \land \mu(z) \land \frac{1-k}{2}
\end{align*}
\]

Hence, \( \mu \) is a commutative \((\varepsilon, \varepsilon \lor q_k)\)-fuzzy ideal of \( X \). \( \square \)

Setting \( k = 0 \) in Theorem 4.11, then we have the following Corollary.

**Corollary 4.12.** A \((\varepsilon, \varepsilon \lor q)\)-fuzzy ideal \( \mu \) of a BCH-algebra \( X \) is a commutative \((\varepsilon, \varepsilon \lor q)\)-fuzzy ideal of \( X \) if and only if it satisfies the condition
\[
\mu(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) \geq \mu(x * y) \land 0.5
\]
for all \( x, y \in X \).

Clearly, every commutative fuzzy ideal is a commutative \((\varepsilon, \varepsilon \lor q_k)\)-fuzzy ideal.

Here we give a condition for a commutative \((\varepsilon, \varepsilon \lor q_k)\)-fuzzy ideal to be a commutative fuzzy ideal.

**Theorem 4.13.** Let \( \mu \) be a commutative \((\varepsilon, \varepsilon \lor q_k)\)-fuzzy ideal of a BCH-algebra \( X \). If \( \mu(0) < \frac{1-k}{2} \), then \( \mu \) is a commutative fuzzy ideal of \( X \).

**Proof.** Suppose that \( \mu(0) < \frac{1-k}{2} \). Then \( \mu(x) < \frac{1-k}{2} \) and so \( \mu(x) \leq \mu(0) < \frac{1-k}{2} \) for all \( x \in X \) by Theorem 4.9(K). It follows from Theorem 4.9(L) that
\[
\begin{align*}
\mu(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) & \geq \mu((x * y) * z) \land \mu(z) \land \frac{1-k}{2} \\
& = \mu((x * y) * z) \land \mu(z)
\end{align*}
\]
Hence, \( \mu \) is a commutative fuzzy ideal of \( X \). \( \square \)

**Corollary 4.14.** Let \( \mu \) be a commutative \((\varepsilon, \varepsilon \lor q)\)-fuzzy ideal of a BCH-algebra \( X \). If \( \mu(0) < 0.5 \), then \( \mu \) is a commutative fuzzy ideal of \( X \).

**Proof.** It follows from Theorem 4.13 by letting \( k = 0 \). \( \square \)

**Theorem 4.15.** Let \( X \) be a BCH-algebra. If \( 0 \leq k < r < 1 \), then every commutative \((\varepsilon, \varepsilon \lor q_k)\)-fuzzy ideal is a commutative \((\varepsilon, \varepsilon \lor q_r)\)-fuzzy ideal.

**Proof.** Straightforward. \( \square \)
Theorem 4.16. For a fuzzy set \( \mu \) of a BCH-algebra \( X \), the following are equivalent:

(O) \( \mu \) is a commutative \((\in, \in \vee q_k)\)-fuzzy ideal of \( X \).

(P) \( \mu_t \neq \phi \Rightarrow \mu_t \triangleright X \), for all \( t \in (0, \frac{1-k}{2}] \).

We say that \( \mu_t \) is an \((\in \vee q_k)\)-level ideal of \( \mu \) in \( X \).

Proof. Suppose that \( \mu \) is a commutative \((\in, \in \vee q_k)\)-fuzzy ideal of \( X \) and let \( t \in (0, \frac{1-k}{2}] \) be such that \( \mu_t \neq \phi \). By Theorem 4.9(K), we have

\[
\mu(0) \geq \mu(x) \wedge \frac{1-k}{2}
\]

for any \( x \in \mu_t \). It follows that

\[
\mu(0) \geq t \wedge \frac{1-k}{2} = t.
\]

So that \( 0 \in \mu_t \). Let \( x, y, z \in X \) be such that \( (x \ast y) \ast z \in \mu_t \) and \( z \in \mu_t \) for \( t \in (0, \frac{1-k}{2}] \). Then \( \mu((x \ast y) \ast z) \geq t \) and \( \mu(z) \geq t \). By Theorem 4.9(L) implies that

\[
\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq \mu((x \ast y) \ast z) \wedge \mu(z) \wedge \frac{1-k}{2}
\]

\[
\geq t \wedge t \wedge \frac{1-k}{2} \geq t \wedge \frac{1-k}{2} = t.
\]

Thus, \( (x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \in \mu_t \), and so \( \mu_t \) is a commutative ideal of \( X \). Conversely, let \( \mu \) be a fuzzy set of \( X \) such that \( \mu_t \) is non-empty and is a commutative ideal of \( X \) for all \( t \in (0, \frac{1-k}{2}] \). If there exists \( a \in X \) such that

\[
\mu(0) < \mu(a) \wedge \frac{1-k}{2},
\]

then

\[
\mu(0) < t_0 \leq \mu(a) \wedge \frac{1-k}{2}
\]

for some \( t_0 \in (0, \frac{1-k}{2}] \), and so \( 0 \notin \mu_{t_0} \). This is a contradiction. Therefore

\[
\mu(0) \geq \mu(x) \wedge \frac{1-k}{2}
\]

for all \( x \in X \). Suppose that there exist \( a, b, c \in X \) such that

\[
\mu(a \ast ((b \ast (b \ast a)) \ast (0 \ast (0 \ast (a \ast b)))))) < \mu((a \ast b) \ast c) \wedge \mu(c) \wedge \frac{1-k}{2}.
\]

Then

\[
\mu(a \ast ((b \ast (b \ast a)) \ast (0 \ast (0 \ast (a \ast b)))))) < t_w \leq \mu((a \ast b) \ast c) \wedge \mu(c) \wedge \frac{1-k}{2}.
\]

for some \( t_w \in (0, \frac{1-k}{2}] \). It follows that

\[
(a \ast b) \ast c \in \mu_{t_w} \text{ and } c \in \mu_{t_w}, \text{ but } a \ast ((b \ast (b \ast a)) \ast (0 \ast (0 \ast (a \ast b)))) \notin \mu_{t_w}.
\]
Which is not possible, and thus,
\[ \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq \mu((x \ast y) \ast z) \land \mu(z) \land \frac{1-k}{2} \]
for all \( x, y, z \in X \). By Theorem 4.9, we conclude that \( \mu \) is a commutative \((\in, \in \lor q_k)\)-fuzzy ideal of \( X \). \( \square \)

Letting \( k = 0 \) in Theorem 4.16 induces the following Corollary.

**Corollary 4.17.** For a fuzzy set \( \mu \) of a BCH-algebra \( X \), the following are equivalent:

(Q) \( \mu \) is a commutative \((\in, \in \lor q)\)-fuzzy ideal of \( X \).

(R) \( \mu_t \neq \phi \Rightarrow \mu_t \triangleright X \), for all \( t \in (0, 0.5] \).

For a fuzzy point \( x_t \) and a fuzzy set \( \mu \) of a BCH-algebra \( X \), we say that

(S) \( x_t q_k \mu \) if \( \mu(x) + t > 1 \),

(T) \( x_t q_k \mu \) if \( \mu(x) + t + k > 1 \).

Denote by \( Q(\mu; t) \) (resp. \( Q(\mu; t) \)) the set \( \{ x \in X | x_t q_k \mu \} \) (resp. \( \{ x \in X | x_t q_k \mu \} \)), and

\[ Q^k(\mu; t) = \{ x \in X | x_t q_k \mu \} \]
\[ [\mu]_t^k = \{ x \in X | x_t \in \lor q_k \mu \} \]
\[ Q^k(\mu; t) = \{ x \in X | x_t q_k \mu \} \]
\[ [\mu]_t^k = \{ x \in X | x_t \in \lor q_k \mu \} \]

Obviously,

\[ [\mu]_t^k = \mu_t \cup Q^k(\mu; t) \]

and

\[ [\mu]_t^k = \mu_t \cup Q^k(\mu; t). \]

**Theorem 4.18.** If \( \mu \) is a commutative \((\in, \in \lor q_k)\)-fuzzy ideal of a BCH-algebra \( X \), then \( Q^k(\mu; t) \neq \phi \Rightarrow Q^k(\mu; t) \triangleright X \), for all \( t \in (\frac{1-k}{2}, 1] \).

*Proof.* Assume \( t \in (\frac{1-k}{2}, 1] \) be such that \( Q^k(\mu; t) \neq \phi \). Then there exists \( x_0 \in Q^k(\mu; t) \), and so \( \mu(x_0) + t \geq 1 - k \). By Theorem 4.9(K), we have

\[ \mu(0) \geq \mu(x_0) \land \frac{1-k}{2} \geq (1 - k - t) \land \frac{1-k}{2} = 1 - k - t \]
i.e., \( 0_t q_k \mu \). Hence, \( 0 \in Q^k(\mu; t) \). Let \( x, y, z \in X \) be such that

\( (x \ast y) \ast z \in Q^k(\mu; t) \) and \( z \in Q^k(\mu; t) \).
Then

\[ \mu((x \ast y) \ast z) + t \geq 1 - k \text{ and } \mu(z) + t \geq 1 - k. \]

From Theorem 4.9(L) it follows that

\[ \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq \mu((x \ast y) \ast z) \land \mu(z) \land \frac{1-k}{2} \geq (1 - k - t) \land (1 - k - t) \land \frac{1-k}{2} = 1 - k - t. \]

So that

\[ (x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \in Q_k^{\mu}, \]

i.e.,

\[ (x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \in Q_k^{\mu}(\mu; t). \]

Hence, \( Q_k^{\mu}(\mu; t) \) is a commutative ideal of \( X \).

**Corollary 4.19.** If \( \mu \) is a commutative \((\in, \in \lor q_k)\)-fuzzy ideal of a BCH-algebra \( X \), then

\[ Q(\mu; t) \neq \phi \Rightarrow Q(\mu; t) \triangleright X, \]

for all \( t \in (0.5, 1] \).

**Corollary 4.20.** Let \( \mu \) be a commutative \((\in, \in \lor q_k)\)-fuzzy ideal of a BCH-algebra \( X \). If \( k < r < 1 \), then

\[ Q_r(\mu; t) \neq \phi \Rightarrow Q_r(\mu; t) \triangleright X, \]

for all \( t \in (\frac{1-r}{2}, 1] \).

*Proof.* It is straightforward by Theorems 4.15 and 4.18.

**Theorem 4.21.** For a fuzzy set \( \mu \) of a BCH-algebra \( X \), the following are equivalent:

(U) \( \mu \) is a commutative \((\in, \in \lor q_k)\)-fuzzy ideal of \( X \).

(V) \([\mu]^k_t \neq \phi \Rightarrow [\mu]^k_t \triangleright X\), for all \( t \in (0, 1] \).

We call \([\mu]^k_t \) a commutative \((\in \lor q_k)\)-level ideal of \( \mu \).

*Proof.* Suppose that \( \mu \) is a commutative \((\in, \in \lor q_k)\)-fuzzy ideal of \( X \) and let \( t \in (0, 1] \) such that \([\mu]^k_t \neq \phi \). Then there exists \( a \in [\mu]^k_t \neq \phi \), and so \( a \in \mu_t \) or \( a \in Q^k(\mu; t) \), i.e.,

\[ \mu(x) \geq t \text{ or } \mu(x) + t \geq 1 - k. \]

By Theorem 4.9(K), we get
\[ \mu(0) \geq \mu(a) \land \frac{1-k}{2} \quad (1) \]

Here we consider two cases:
Case 1: \( \mu(a) \leq \frac{1-k}{2} \) and
Case 2: \( \mu(a) > \frac{1-k}{2} \).

Case 1: We have \( \mu(0) \geq \mu(a) \) by (1). Thus, if \( \mu(a) \geq t \), then \( \mu(0) \geq t \) and so

\[ 0 \in \mu_t \subseteq [\mu]_t^k. \]

If \( \mu(a) + t \geq 1 - k \), then

\[ \mu(0) + t \geq \mu(a) + t = 1 - k. \]

This implies that \( 0_t {\underline{q}}_k \mu \), i.e.,

\[ 0 \in Q^k(\mu; t) \subseteq [\mu]_t^k. \]

Combining the Case 2 and (1) induces

\[ \mu(0) \geq \frac{1-k}{2}. \]

If \( t \leq \frac{1-k}{2} \), then \( \mu(0) \geq t \) and hence, \( 0 \in \mu_t \subseteq [\mu]_t^k. \)

Case 2: If \( t > \frac{1-k}{2} \), then

\[ \mu(0) + t > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k. \]

This implies that

\[ 0 \in Q^k(\mu; t) \subseteq Q^k(\mu; t) \subseteq [\mu]_t^k. \]

Therefore \( [\mu]_t^k \) satisfies the condition (I1). Let \( x, y, z \in X \) be such that

\( (x * y) * z \in [\mu]_t^k \) and \( z \in [\mu]_t^k. \)

Then

\( (x * y) * z \in \mu_t \) or \( ((x * y) * z)_t {\underline{q}}_k \mu, \)

and

\( z \in \mu_t \) or \( z_t {\underline{q}}_k \mu, \)

that is,

\[ \mu((x * y) * z) \geq t \) or \( \mu((x * y) * z) + t \geq 1 - k, \]
and

\[ \mu(z) \geq t \text{ or } \mu(z) + t \geq 1 - k. \]

We consider the following cases:

(a) \( \mu((x \ast y) \ast z) \geq t \) and \( \mu(z) \geq t \),

(b) \( \mu((x \ast y) \ast z) \geq t \) and \( \mu(z) + t \geq 1 - k \),

(c) \( \mu((x \ast y) \ast z) + t \geq 1 - k \) and \( \mu(z) + t \geq 1 - k \).

Since \( \mu \) is a commutative \((\varepsilon, \varepsilon \lor q_k)\)-fuzzy ideal of \( X \), we have

\[ \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \geq \mu((x \ast y) \ast z) \land \mu(z) \land \frac{1-k}{2} \quad (2) \]

by Theorem 4.9(L). By using (a) and (2), we get

\[ \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \geq \mu((x \ast y) \ast z) \land \mu(z) \land \frac{1-k}{2} \]

\[ \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \geq t \land t \land \frac{1-k}{2} \]

\[ \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \geq t \land \frac{1-k}{2}. \]

If \( t \leq \frac{1-k}{2} \), then \( \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \geq t \), i.e.,

\[ (x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \in \mu_t \subseteq [\mu]_t. \]

If \( t > \frac{1-k}{2} \), then \( \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \geq \frac{1-k}{2} \) and so

\[ \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \geq \frac{1-k}{2} + \frac{1-k}{2} = 1 - k. \]

Hence,

\[ (x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \in Q^k(\mu; t) \subseteq Q^k(\mu; t) \subseteq [\mu]_t. \]

Case (b) and (2) imply that

\[ \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \geq \mu((x \ast y) \ast z) \land \mu(z) \land \frac{1-k}{2} \]

\[ \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \geq t \land (1 - k - t) \land \frac{1-k}{2}. \]

If \( t \leq \frac{1-k}{2} \), then

\[ \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \geq t \land (1 - k - t) = t \]

and so

\[ (x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \in \mu_t \subseteq [\mu]_t. \]

If \( t > \frac{1-k}{2} \), then

\[ \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \geq (1 - k - t) \land \frac{1-k}{2} = 1 - k - t \]

and thus,
\[(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \in Q^k(\mu; t) \subseteq [\mu]^k_t.\]

Similarly, we have \((x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \in [\mu]^k_t\) from the
case (c) and (2).

Finally, case (d) and (2) induces
\[
\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq \mu((x \ast y) \ast z) \wedge \mu(z) \wedge \frac{1-k}{2}
\]
\[
\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq (1 - k - t) \wedge (1 - k - t) \wedge \frac{1-k}{2}
\]
\[
\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq (1 - k - t) \wedge \frac{1-k}{2}.
\]

If \(t \leq \frac{1-k}{2}\), then
\[
\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq \frac{1-k}{2} \geq t.
\]

Hence,
\[
(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \in \mu_t \subseteq [\mu]^k_t.
\]

If \(t > \frac{1-k}{2}\), then
\[
\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq 1 - k - t.
\]

This implies that
\[
(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \in Q^k(\mu; t) \subseteq [\mu]^k_t.
\]

Consequently, \([\mu]^k_t\) is a commutative ideal of \(X\). Conversely, assume that
\((V)\) is hold. If there exists \(a \in X\) such that
\[
\mu(0) < \mu(a) \wedge \frac{1-k}{2},
\]
then
\[
\mu(0) < t_0 \leq \mu(a) \wedge \frac{1-k}{2}
\]
for some \(t_0 \in (0, \frac{1-k}{2}]\). It follows that \(a \in \mu_{t_0} \subseteq [\mu]^k_{t_0}\) but \(0 \notin \mu_{t_0}\). Also, we have
\[
\mu(0) + t_0 < 2t_0 \leq 1 - k,
\]
and so \(0_{t_0 \overline{Q}^k_m}\), i.e., \(0 \notin Q^k(\mu; t)\). Therefore \(0 \notin [\mu]^k_{t_0}\), a contradiction. Hence,
\[
\mu(0) \geq \mu(x) \wedge \frac{1-k}{2}
\]
for all \(x \in X\). Suppose that there exist \(a, b, c \in X\) such that
\[
\mu(a \ast ((b \ast (b \ast a)) \ast (0 \ast (0 \ast (a \ast b)))))) < \mu((a \ast b) \ast c) \wedge \mu(c) \wedge \frac{1-k}{2}.
\]

Then
\[
\mu(a \ast ((b \ast (b \ast a)) \ast (0 \ast (0 \ast (a \ast b)))))) < t_w \leq \mu((a \ast b) \ast c) \wedge \mu(c) \wedge \frac{1-k}{2}
\]
for some $t_w \in (0, \frac{1-k}{2}]$. It follows that $(a*b)*c, c \in \mu_{w_0} \subseteq [\mu]_{t_w}^k$ from (I3) that

$$(a * ((b * (b * a)) * (0 * (0 * (a * b)))))) \in [\mu]_{t_w}^k.$$ 

Thus,

$$\mu(a * ((b * (b * a)) * (0 * (0 * (a * b)))))) \geq t_w$$

or

$$\mu(a * ((b * (b * a)) * (0 * (0 * (a * b)))))) + t_w \geq 1 - k,$$

a contradiction. Therefore

$$\mu((y * (y * x)) * (0 * (0 * (x * y)))) \geq \mu((x * y) * z) \land \mu(z) \land \frac{1-k}{2},$$

for all $x, y, z \in X$. From Theorem 4.9, we conclude that $\mu$ is a commutative $(\in, \lor q_k)$-fuzzy ideal of $X$. □

**Corollary 4.22.** For a fuzzy set $\mu$ of a BCH-algebra $X$, the following are equivalent:

(W) $\mu$ is a commutative $(\in, \lor q_k)$-fuzzy ideal of $X$,

(Y) $[\mu]_t \neq \phi \Rightarrow [\mu]_t > X$, for all $t \in (0, 1]$,

where $[\mu]_t = \{ x \in X | x_t \in \lor q_k \} = \mu_t \cup Q(\mu; t)$.

A fuzzy set $\mu$ of a BCH-algebra $X$ is said to be proper if $\text{Im}(\mu)$ has at least two elements. Two fuzzy sets are said to be equivalent if they have the same family of level subsets. Otherwise, they are said to be non-equivalent.

**Theorem 4.23.** Let $\mu$ be a commutative $(\in, \lor q_k)$-fuzzy ideal of a BCH-algebra $X$ such that

$$\# \{ \mu(x) | \mu(x) < \frac{1-k}{2} \} \geq 2,$$

where $\#$ mean cardinal.

Then there exist two proper non-equivalent commutative $(\in, \lor q_k)$-fuzzy ideals of $X$ such that $\mu$ can be expressed as the union of them.

**Proof.** Suppose that

$$\{ \mu(x) | \mu(x) < \frac{1-k}{2} \} = \{ t_1, t_2, ..., t_n \},$$

where $t_1 > t_2 > ... > t_n$ and $n \geq 2$. Then the chain of commutative $\lor q_k$-level ideal of $\mu$ is

$$[\mu]_{\frac{1-k}{2}}^k \subseteq [\mu]_{t_1}^k \subseteq [\mu]_{t_2}^k \subseteq ... \subseteq [\mu]_{t_n}^k = X.$$
Let \( \tau \) and \( \sigma \) be fuzzy sets of \( X \) defined by

\[
\tau(x) = \begin{cases} 
  t_1 & \text{if } x \in [\mu]_{t_1}^k, \\
  t_2 & \text{if } x \in [\mu]_{t_2}^k \setminus [\mu]_{t_1}^k, \\
  \ldots \\
  t_n & \text{if } x \in [\mu]_{t_n}^k \setminus [\mu]_{t_{n-1}}^k, 
\end{cases}
\]

and

\[
\sigma(x) = \begin{cases} 
  \mu(x) & \text{if } x \in [\mu]_{1-k}^k, \\
  k & \text{if } x \in [\mu]_{t_2}^k \setminus [\mu]_{1-k}^k, \\
  t_3 & \text{if } x \in [\mu]_{t_3}^k \setminus [\mu]_{t_2}^k, \\
  \ldots \\
  t_n & \text{if } x \in [\mu]_{t_n}^k \setminus [\mu]_{t_{n-1}}^k, 
\end{cases}
\]

respectively, where \( t_3 < k < t_2 \). Then \( \tau \) and \( \sigma \) are commutative \((\in, \in \lor q_k)\)-fuzzy ideals of \( X \), and \( \tau, \sigma \leq \mu \). The chains of commutative \( \in \lor q_k \)-level ideals of \( \tau \) and \( \sigma \) are, respectively, given by

\[
[\mu]_{t_1}^k \subseteq [\mu]_{t_2}^k \subseteq \ldots \subseteq [\mu]_{t_n}^k
\]

and

\[
[\mu]_{1-k}^k \subseteq [\mu]_{t_2}^k \subseteq \ldots \subseteq [\mu]_{t_n}^k.
\]

Therefore \( \tau \) and \( \sigma \) are non-equivalent and clearly \( \mu = \tau \cup \sigma \). \( \Box \)

**Theorem 4.24.** Let \( \{ \mu_i | i \in \Lambda \} \) be a family of commutative \((\in, \in \lor q_k)\)-fuzzy ideals of a BCH-algebra \( X \). Then \( \mu = \cap_{i \in \Lambda} \mu_i \) is a commutative \((\in, \in \lor q_k)\)-fuzzy ideal of \( X \).

**Proof.** Let \( x \in X \) and \( t \in (0, 1] \) be such that \( x_t \in \mu \). Suppose that \( 0 \in \mu \). Then

\[
\mu(0) < t \text{ and } \mu(x) + t \leq 1 - k.
\]

This implies that

\[
\mu(0) < \frac{1-k}{2}. \quad (3)
\]

Let

\[
\Omega_1 = \{ i \in \Lambda | \mu_i(0) \geq t \}
\]

and
\[
\Omega_2 = \{i \in \Lambda | 0_t q_k \mu_i \text{ and } \mu_i(0) < t\}.
\]

Then

\[
\Lambda = \Omega_1 \cup \Omega_2
\]

and

\[
\Omega_1 \cap \Omega_2 = \emptyset.
\]

If \(\Omega_2 = \emptyset\), then \(\mu_i(0) \geq t\) for all \(i \in \Lambda\), and so \(\mu(0) \geq t\) is a contradiction. Hence, \(\Omega_2 \neq \emptyset\), and so \(\mu_i(0) + t > 1 - k\) and \(\mu_i(0) < t\) for every \(i \in \Omega_2\). It follows that \(t > \frac{1-k}{2}\) so that

\[
\mu_i(x) \geq \mu(x) \geq t > \frac{1-k}{2}
\]

for all \(i \in \Lambda\). Now, assume that

\[
t_0 = \mu_i(x) < \frac{1-k}{2}
\]

for some \(i \in \Lambda\). Let \(t'_0 \in (0, \frac{1-k}{2})\) be such that \(t_0 < t'_0\). Then

\[
\mu_i(x) > \frac{1-k}{2} > t'_0,
\]

i.e., \(x_{t'_0} \in \mu_i\). But \(\mu_i(0) = t_0 < t'_0\) and \(\mu_i(0) + t'_0 < 1 - k\), that is, \(x_{t'_0} \in \overline{\vee q_k \mu_i}\). Which is a contradiction, and so \(\mu_i(x) \geq \frac{1-k}{2}\) for all \(i \in \Lambda\). Thus, \(\mu(x) \geq \frac{1-k}{2}\). This is not possible. Therefore \(0_t \in \vee q_k \mu\). Let \(x, y, z \in X\) and \(t_1, t_2 \in (0, 1]\) be such that \(((x * y) * z)_{t_1} \in \mu\) and \(z_{t_2} \in \mu\). Suppose that

\[
(x * (((y * (y * x)) * (0 * (0 * (x * y)))))_{t_1} \wedge t_2) \in \overline{\vee q_k \mu}.
\]

Then

\[
\mu(x * (((y * (y * x)) * (0 * (0 * (x * y))))) < t_1 \wedge t_2
\]

and

\[
\mu(x * (((y * (y * x)) * (0 * (0 * (x * y))))) + t_1 \wedge t_2 \leq 1 - k.
\]

It follows that

\[
\mu(x * (((y * (y * x)) * (0 * (0 * (x * y)))))) < \frac{1-k}{2}.
\]

Let

\[
\Omega_3 = \{i \in \Lambda | \mu_i(x * (((y * (y * x)) * (0 * (0 * (x * y)))))) \geq t_1 \wedge t_2\}
\]

and
\[ \Omega_4 = \{ i \in \Lambda | (x * ((y * (y * x)) * (0 * (0 * (x * y))))))_{t_1 \wedge t_2} q_k \mu_i \text{ and } \\
\mu_i(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) < t_1 \wedge t_2 \}. \]

Then

\[ \Lambda = \Omega_3 \cup \Omega_4 \]

and

\[ \Omega_3 \cap \Omega_4 = \phi. \]

If \( \Omega_4 = \phi \), then

\[ \mu_i(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) \geq t_1 \wedge t_2 \]

for all \( i \in \Lambda \), and so

\[ \mu(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) \geq t_1 \wedge t_2. \]

This is a contradiction. Hence, \( \Omega_4 \neq \phi \), and thus,

\[ (x * ((y * (y * x)) * (0 * (0 * (x * y))))))_{t_1 \wedge t_2} q_k \mu_i, \]

i.e.,

\[ \mu_i(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) + t_1 \wedge t_2 > 1 - k, \]

and

\[ \mu_i(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) < t_1 \wedge t_2. \]

It follows that

\[ t_1 \wedge t_2 > \frac{1-k}{2}. \]

So that

\[ \mu_i((x * y) * z) \geq \mu((x * y) * z) \geq t_1 \geq t_1 \wedge t_2 > \frac{1-k}{2} \]

for all \( i \in \Lambda \). Similarly, we have

\[ \mu_i(z) > \frac{1-k}{2} \]

for all \( i \in \Lambda \). Now, suppose that

\[ t = \mu_i(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) < \frac{1-k}{2} \]

for some \( i \in \Lambda \). Let \( t' \in (0, \frac{1-k}{2}) \) be such that \( t < t' \). Then
\[ \mu_i((x * y) * z) > \frac{1-k}{2} > t' \]

and

\[ \mu_i(z) > \frac{1-k}{2} > t', \]

that is,

\[ ((x * y) * z)_{t'} \in \mu_i \text{ and } z_{t'} \in \mu_i. \]

But

\[ \mu_i(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) = t < t' \]

and

\[ \mu_i(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) + t' < 1 - k, \]

that is,

\[ (x * ((y * (y * x)) * (0 * (0 * (x * y))))))_{t' \in \sqrt{q_k} \mu_i}. \]

Which is a contradiction, and hence,

\[ \mu_i(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) \geq \frac{1-k}{2} \]

for all \( i \in \Lambda \). Therefore

\[ \mu(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) \geq \frac{1-k}{2}. \]

This does not hold. Consequently,

\[ (x * ((y * (y * x)) * (0 * (0 * (x * y))))))_{t_1 \land t_2 \in \sqrt{q_k} \mu}. \]

Hence, \( \mu \) is a commutative \((\in, \in \lor q_k)\)-fuzzy ideal of \( X \). \( \square \)

Setting \( k = 0 \) in Theorem 4.24, we have the following Corollary.

**Corollary 4.25.** Let \( \{\mu_i | i \in \Lambda\} \) be a family of commutative \((\in, \in \lor q)\)-fuzzy ideals of a BCH-algebra \( X \). Then \( \mu = \cap_{i \in \Lambda} \mu_i \) is a commutative \((\in, \in \lor q)\)-fuzzy ideal of \( X \).

Consider the number \( t \in (\frac{1-k}{2}, 1] \) for which \( \mu_t \) is a commutative ideal of \( X \), we consider a new kind of a commutative fuzzy ideal as follows.
5. LEVEL 2.2 OF FUZZIFICATION

In this section, we introduce the concepts of \((\bar{\varepsilon}, \bar{\varepsilon} \lor q_k)\)-fuzzy ideal and commutative \((\bar{\varepsilon}, \bar{\varepsilon} \lor q_k)\)-fuzzy ideal in BCH-algebras and investigate some of their properties.

**Definition 5.1.** A fuzzy set \(\mu\) of a BCH-algebra \(X\) is called an \((\bar{\varepsilon}, \bar{\varepsilon} \lor q_k)\)-fuzzy ideal of \(X\) if it satisfies \((Z)\) and \((A1)\), where
\[
\begin{align*}
\text{(Z)} & \quad 0_t \bar{\varepsilon} \mu \Rightarrow x_t \bar{\varepsilon} \lor q_k \mu, \\
\text{(A1)} & \quad x_t \land r \bar{\varepsilon} \mu \Rightarrow (x \ast y)_t \bar{\varepsilon} \lor q_k \mu \text{ or } y_r \bar{\varepsilon} \lor q_k \mu, 
\end{align*}
\]
for all \(x, y \in X\) and \(t, r \in (0, 1]\).

**Definition 5.2.** A fuzzy set \(\mu\) of a BCH-algebra \(X\) is called a commutative \((\bar{\varepsilon}, \bar{\varepsilon} \lor q_k)\)-fuzzy ideal of \(X\) if it satisfies \((Z)\) and \((B1)\), where
\[
\begin{align*}
\text{(Z)} & \quad 0_t \bar{\varepsilon} \mu \Rightarrow x_t \bar{\varepsilon} \lor q_k \mu, \\
\text{(B1)} & \quad (x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y))))))_t \land r \bar{\varepsilon} \mu \Rightarrow ((x \ast y) \ast z)_t \bar{\varepsilon} \lor q_k \mu \text{ or } z_r \bar{\varepsilon} \lor q_k \mu, 
\end{align*}
\]
for all \(x, y, z \in X\) and \(t, r \in (0, 1]\).

**Proposition 5.3.** Every commutative \((\bar{\varepsilon}, \bar{\varepsilon} \lor q_k)\)-fuzzy ideal of a BCH-algebra \(X\) is an \((\bar{\varepsilon}, \bar{\varepsilon} \lor q_k)\)-fuzzy ideal.

**Proof.** Let \(\mu\) be a commutative \((\bar{\varepsilon}, \bar{\varepsilon} \lor q_k)\)-fuzzy ideal of \(X\). Then for all \(t, r \in (0, 1]\) and for all \(x, y, z \in X\), we have
\[
(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y))))))_t \land r \bar{\varepsilon} \mu \Rightarrow ((x \ast y) \ast z)_t \bar{\varepsilon} \lor q_k \mu \text{ or } z_r \bar{\varepsilon} \lor q_k \mu.
\]

Putting \(y = 0\) in above, we get
\[
(x \ast ((0 \ast (0 \ast x)) \ast (0 \ast (0 \ast (x \ast 0))))))_t \land r \bar{\varepsilon} \mu \Rightarrow ((x \ast 0) \ast z)_t \bar{\varepsilon} \lor q_k \mu \text{ or } z_r \bar{\varepsilon} \lor q_k \mu
\]
\[
(x \ast ((0 \ast (0 \ast x)) \ast (0 \ast (0 \ast x)))))_t \land r \bar{\varepsilon} \mu \Rightarrow (x \ast z)_t \bar{\varepsilon} \lor q_k \mu \text{ or } z_r \bar{\varepsilon} \lor q_k \mu \text{ (by Proposition 2.1(3))}
\]
\[
(x \ast 0)_t \land r \bar{\varepsilon} \mu \Rightarrow (x \ast z)_t \bar{\varepsilon} \lor q_k \mu \text{ or } z_r \bar{\varepsilon} \lor q_k \mu \text{ (BCH-I)}
\]
\[
x_t \land r \bar{\varepsilon} \mu \Rightarrow (x \ast z)_t \bar{\varepsilon} \lor q_k \mu \text{ or } z_r \bar{\varepsilon} \lor q_k \mu \text{ (by Proposition 2.1(3))}
\]

This means that \(\mu\) satisfies the condition \((A1)\). This implies that \(\mu\) is an \((\bar{\varepsilon}, \bar{\varepsilon} \lor q_k)\)-fuzzy ideal of \(X\). \(\Box\)

A commutative \((\bar{\varepsilon}, \bar{\varepsilon} \lor q_k)\)-fuzzy ideal of a BCH-algebra \(X\) with \(k = 0\) is called a commutative \((\bar{\varepsilon}, \bar{\varepsilon} \lor \bar{q})\)-fuzzy ideal of \(X\).

**Theorem 5.4.** A \((\bar{\varepsilon}, \bar{\varepsilon} \lor q_k)\)-fuzzy ideal of a BCH-algebra \(X\) is a commutative \((\bar{\varepsilon}, \bar{\varepsilon} \lor q_k)\)-fuzzy ideal of \(X\) if and only if it satisfies the condition
\[
\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \lor \frac{1-k}{2} \geq \mu(x \ast y)
\]
for all \( x, y \in X \).

**Proof.** The proof is similar to the proof of Theorem 4.11. \(\square\)

Setting \( k = 0 \) in Theorem 5.4, we have the following Corollary.

**Corollary 5.5.** A \((\bar{\xi}, \bar{\xi} \lor \bar{q})\)-fuzzy ideal of a BCH-algebra \( X \) is a commutative \((\bar{\xi}, \bar{\xi} \lor \bar{q})\)-fuzzy ideal of \( X \) if and only if it satisfies the condition
\[
\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \lor 0.5 \geq \mu(x \ast y)
\]
for all \( x, y \in X \).

**Corollary 5.6.** Every commutative fuzzy ideal is a commutative \((\bar{\xi}, \bar{\xi} \lor \bar{q}_k)\)-fuzzy ideal.

Let \( \mu \) be a commutative \((\bar{\xi}, \bar{\xi} \lor \bar{q}_k)\)-fuzzy ideal of a BCH-algebra \( X \). Assume that there exists \( a \in X \) such that
\[
\mu(a) > \mu(0) \lor \frac{1-k}{2}.
\]

Then
\[
\mu(a) \geq t > \mu(0) \lor \frac{1-k}{2}
\]
for some \( t \in (\frac{1-k}{2}, 1] \). It follows that \( 0_t \bar{\in} \mu, a_t \bar{\in} \mu \) and \( \mu(0) + t \geq 2t > 1 - k \), i.e., \( a_t q_k \mu \). Which is a contradiction, and so the following inequality holds.

(C1) \( \mu(a) \leq \mu(0) \lor \frac{1-k}{2} \) for all \( x \in X \).

Assume that
\[
\mu(a \ast ((b \ast (b \ast a)) \ast (0 \ast (0 \ast (a \ast b)))))) \lor \frac{1-k}{2} < \mu((a \ast b) \ast c) \land \mu(c)
\]
for some \( a, b, c \in X \). Then there exists \( t \in (\frac{1-k}{2}, 1] \) such that
\[
\mu(a \ast ((b \ast (b \ast a)) \ast (0 \ast (0 \ast (a \ast b)))))) \lor \frac{1-k}{2} < t \leq \mu((a \ast b) \ast c) \land \mu(c).
\]

Thus,
\[
(a \ast ((b \ast (b \ast a)) \ast (0 \ast (0 \ast (a \ast b))))))_t \bar{\in} \mu.
\]

From \( t \leq \mu((a \ast b) \ast c) \land \mu(c) \), we have
\[
((a \ast b) \ast c)_t \in \mu, c_t \in \mu, \mu((a \ast b) \ast c) + t \geq 2t > 1 - k,
\]
i.e.,
\[(a * b) * c)_{t} q_{k} \mu, \text{ and } \mu(c) + t \geq 2t > 1 - k,\]

i.e., \(c_{t} q_{k} \mu\). Which is not possible, and hence, we know that \(\mu\) satisfies the following assertion:

\[(D1) \ \mu(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) \lor \frac{1-k}{2} \geq \mu((x * y) * z) \land \mu(z), \text{ for all } x, y, z \in X.\]

Suppose \(\mu\) is a fuzzy set of a BCH-algebra \(X\) satisfying (C1) and (D1). Let \(t \in \left(\frac{1-k}{2}, 1\right]\) be such that \(\mu_{t} \neq \phi\). Then there exists \(a \in \mu_{t}\), and so

\[\frac{1-k}{2} > t \leq \mu(a) \leq \mu(0) \lor \frac{1-k}{2} = \mu(0)\]

by (C1). Hence, \(0 \in \mu_{t}\). Let \(x, y, z \in X\) be such that \((x * y) * z \in \mu_{t}\) and \(z \in \mu_{t}\). Then

\[\mu((x * y) * z) \geq t > \frac{1-k}{2} \text{ and } \mu(z) \geq t > \frac{1-k}{2}.\]

By using (D1), we get

\[\mu(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) \lor \frac{1-k}{2} \geq \mu((x * y) * z) \land \mu(z) \geq t \land t \geq t > \frac{1-k}{2}.\]

This implies that

\[\mu(x*((y*(y*x))*(0*(0*(x*y)))))) = \mu(x*((y*(y*x))*(0*(0*(x*y)))))) \lor \frac{1-k}{2} \geq t.\]

Thus, \((x * ((y * (y * x)) * (0 * (0 * (x * y)))))) \in \mu_{t}\). Consequently, \(\mu_{t} \supset X\). Thus, we conclude that if a fuzzy set \(\mu\) of \(X\) satisfies conditions (C1) and (D1), then the following assertion holds.

\[(E1) \ \mu_{t} \neq \phi \Rightarrow \mu_{t} \supset X, \text{ for all } t \in \left[\frac{1-k}{2}, 1\right].\]

Let \(\mu\) be a fuzzy set of a BCH-algebra \(X\) satisfying (E1). Let \(x \in X\) and \(t \in (0, 1]\) be such that \(x_{t} \in \lor q_{k} \mu\). Then \(x_{t} \in \mu\) and \(x_{t} q_{k} \mu\). Hence, \(x \in \mu_{t}\), i.e., \(\mu_{t} \neq \phi\), and so \(\mu_{t} \supset X\) by (E1). Thus, \(0 \in \mu_{t}\), and thus, \(\mu(0) \geq t\), i.e., \(0_{t} \in \mu\).

This shows that condition (Z) is held. Let \(x, y, z \in X\) and \(t_{1}, t_{2} \in (0, 1]\) be such that

\[((x * y) * z)_{t_{1}} \in \lor q_{k} \mu \text{ and } z_{t_{2}} \in \lor q_{k} \mu.\]

Then

\[((x * y) * z)_{t_{1}} \in \mu, \text{ } z_{t_{2}} \in \mu, \text{ } ((x * y) * z)_{t_{1}} q_{k} \mu \text{ and } z_{t_{2}} q_{k} \mu.\]

This implies that
\[(x*y)*z \in \mu_{t_1} \subseteq \mu_{t_1 \land t_2} \text{ and } z \in \mu_{t_1} \subseteq \mu_{t_1 \land t_2}.
\]

Since \(\mu_{t_1 \land t_2} \supset X\) by (E1), it follows from (I3) that
\[(x* ((y*(y*x))*(0*(0*(x*y)))))) \in \mu_{t_1 \land t_2},
\]
i.e.,
\[\mu ((x* ((y*(y*x))*(0*(0*(x*y)))))) \geq t_1 \land t_2.
\]
So that
\[(x* ((y*(y*x))*(0*(0*(x*y))))))_{t_1 \land t_2} \in \mu.
\]
Hence, (B1) holds, and so \(\mu\) is a commutative \((\bar{\epsilon}, \bar{\epsilon} \lor \bar{q}_k)\)-fuzzy ideal of \(X\).

So we have the following theorem.

**Theorem 5.7.** For a fuzzy set \(\mu\) of a BCH-algebra \(X\), the following are equivalent:

(G1) \(\mu\) is a commutative \((\bar{\epsilon}, \bar{\epsilon} \lor \bar{q}_k)\)-fuzzy ideal of \(X\).

(H1) \(\mu\) satisfies the condition (E1).

(J1) \(\mu\) satisfies conditions (C1) and (D1).

**Corollary 5.8.** For a fuzzy set \(\mu\) of a BCH-algebra \(X\), the following are equivalent:

(K1) \(\mu\) is a commutative \((\bar{\epsilon}, \bar{\epsilon} \lor \bar{q})\)-fuzzy ideal of \(X\).

(L1) \(\mu_t \neq \phi \Rightarrow \mu_t \supset X\), for all \(t \in (0.5, 1]\). (M1) \(\mu\) satisfies the following conditions:

(i) \(\mu(x) \leq \mu(0) \lor 0.5,
\)
(ii) \(\mu((x* ((y*(y*x))*(0*(0*(x*y))))) \lor 0.5 \geq \mu((x*y)*z) \land \mu(z),
\)
for all \(x, y, z \in X\).

For a fuzzy set \(\mu\) of a BCH-algebra \(X\), we consider the following set
\[\tau = \{t \in (0, 1]|\mu_t \neq \phi \Rightarrow \mu_t \supset X\}.
\]
Then
(a) If \(\tau = (0, 1]\), then \(\mu\) is a commutative fuzzy ideal of \(X\).
(b) If \(\tau = (0, 1-\frac{k}{2}]\), then \(\mu\) is a commutative \((\bar{\epsilon}, \bar{\epsilon} \lor \bar{q}_k)\)-fuzzy ideal of \(X\).
(c) If \(\tau = (1-\frac{k}{2}, 1]\), then \(\mu\) is a commutative \((\bar{\epsilon}, \bar{\epsilon} \lor \bar{q}_k)\)-fuzzy ideal of \(X\).

**6. LEVEL 3 OF FUZZIFICATION**

In this section, we introduce the concepts of fuzzy ideal with thresholds and commutative fuzzy ideal with thresholds of BCH-algebras and investigate some of their properties.
Definition 6.1. A fuzzy set $\mu$ of a BCH-algebra $X$ is called a fuzzy ideal with thresholds $\varepsilon$ and $\delta$ of $X$, $\varepsilon, \delta \in (0,1]$ with $\varepsilon < \delta$, if it satisfies (N1) and (O1), where

(N1) $\mu(0) \vee \varepsilon \geq \mu(x) \wedge \delta,$

(O1) $\mu(x) \vee \varepsilon \geq \mu(x \ast y) \wedge \mu(y) \wedge \delta,$

for all $x, y \in X$.

Example 6.2. Let $X = \{0, 1, 2, a, b\}$ be a BCH-algebra where the $\ast$- operation is defined by Table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
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<th>a</th>
<th>b</th>
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</table>

Let $\mu$ be a fuzzy set of the BCH-algebra $X$ defined by $\mu(0) = 0.4$, $\mu(1) = \mu(b) = 0.1$, $\mu(2) = 0.5$ and $\mu(a) = 0.3$. By simple calculations show that $\mu$ is a fuzzy ideal of $X$ with thresholds $\varepsilon = 0.3$ and $\delta = 0.4$.

Definition 6.3. A fuzzy set $\mu$ of a BCH-algebra $X$ is called a commutative fuzzy ideal with thresholds $\varepsilon$ and $\delta$ of $X$, $\varepsilon, \delta \in (0,1]$ with $\varepsilon < \delta$, if it satisfies (N1) and (P1), where

(N1) $\mu(0) \vee \varepsilon \geq \mu(x) \wedge \delta,$

(P1) $\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \vee \varepsilon \geq \mu((x \ast y) \ast z) \wedge \mu(z) \wedge \delta,$

for all $x, y, z \in X$.

Example 6.4. Let $X = \{0, 1, 2, 3, 4\}$ be a BCH-algebra where the $\ast$- operation is defined by Table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
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</table>

Let $\mu$ be a fuzzy set of the BCH-algebra $X$ defined by $\mu(0) = \mu(1) = 0.6$, $\mu(2) = 1$, $\mu(3) = 0$ and $\mu(4) = 0.2$. By simple calculations show that $\mu$ is a commutative fuzzy ideal of $X$ with thresholds $\varepsilon = 0.2$ and $\delta = 0.6$. 
Theorem 6.5. Every commutative fuzzy ideal with thresholds of a BCH-algebra $X$ is a fuzzy ideal with thresholds.

Proof. Let $\mu$ be a commutative fuzzy ideal with thresholds of $X$. Then for all $\varepsilon, \delta \in (0, 1]$ and for all $x, y, z \in X$, we have

$$\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \vee \varepsilon \geq \mu((x \ast y) \ast z) \wedge \mu(z) \wedge \delta$$

Putting $y = 0$ in above, we get

$$\mu(x \ast ((0 \ast (0 \ast x)) \ast (0 \ast (0 \ast (x \ast 0)))) \vee \varepsilon \geq \mu((x \ast 0) \ast z) \wedge \mu(z) \wedge \delta$$

$$\mu(x \ast (0 \ast (0 \ast x)) \ast (0 \ast (0 \ast x)) \vee \varepsilon \geq \mu(x \ast z) \wedge \mu(z) \wedge \delta$$

(by Proposition 2.1(3))

$$\mu(x \ast 0) \vee \varepsilon \geq \mu(x \ast z) \wedge \mu(z) \wedge \delta$$

(BCH-I)

$$\mu(x) \vee \varepsilon \geq \mu(x \ast z) \wedge \mu(z) \wedge \delta$$

(by Proposition 2.1 (3))

This means that $\mu$ satisfies the condition (M1). This implies that $\mu$ is fuzzy ideal with thresholds of $X$. □

Example 6.6. Let $X = \{0, 1, 2, 3, 4\}$ in which the operation $\ast$ is defined by

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</table>

Then $(X, \ast, 0)$ is a BCH-algebra. Let $s_0, s_1, s_2 \in [0, 1]$ be such that $s_0 > s_1 > s_2$. We define a map $\mu : X \to [0, 1]$ by $\mu(0) = s_0$, $\mu(1) = s_1$ and $\mu(2) = \mu(3) = \mu(4) = s_2$. Routine calculations give that $\mu$ is a fuzzy ideal of $X$ with thresholds $\varepsilon = s_2$ and $\delta = s_0$. But $\mu$ is not a commutative fuzzy ideal of $X$ with thresholds $\varepsilon = s_2$ and $\delta = s_0$, because

Put $x = 2, y = 3, z = 0$ in (P1) we get

$$\mu(2 \ast ((3 \ast (3 \ast 2)) \ast (0 \ast (0 \ast (2 \ast 3)))) \vee \varepsilon \geq \mu((2 \ast 3) \ast 0) \wedge \mu(0) \wedge \delta$$

$$\mu(2 \ast ((3 \ast 3) \ast (0 \ast (0 \ast 0))) \vee \varepsilon \geq \mu(0 \ast 0) \wedge \mu(0) \wedge \delta$$

$$\mu(2 \ast (0 \ast (0 \ast 0))) \vee \varepsilon \geq \mu(0) \wedge \mu(0) \wedge \delta$$

$$\mu(2 \ast (0 \ast 0)) \vee \varepsilon \geq \mu(0) \wedge \mu(0) \wedge \delta$$

$$\mu(2) \vee \varepsilon \geq \mu(0) \wedge \mu(0) \wedge \delta$$

$s_2 \vee \varepsilon \geq \mu(0) \wedge \mu(0) \wedge \delta$

$s_2 \vee s_2 \geq s_0 \wedge s_0 \wedge s_0$
\[ s_2 \geq s_0 \]
\[ s_2 \not\geq s_0. \]

**Theorem 6.7.** A fuzzy ideal \( \mu \) with thresholds \( \varepsilon \) and \( \delta \) of a BCH-algebra \( X \) is a commutative fuzzy ideal with thresholds of \( X \) if and only if it satisfies the condition
\[
\mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \lor \varepsilon \geq \mu(x \ast y) \land \delta
\]
for all \( x, y \in X \).

**Proof.** The proof is similar to the proof of Theorem 3.12, Theorem 4.11 and Theorem 5.4. \( \square \)

**Theorem 6.8.** Let \( \mu \) be a fuzzy set of a BCH-algebra \( X \), \( \varepsilon, \delta \in (0, 1] \) with \( \varepsilon < \delta \). Then \( \mu \) is a commutative fuzzy ideal with thresholds \( \varepsilon \) and \( \delta \) of \( X \) if and only if it satisfies
\[
\mu_t \neq \phi \Rightarrow \mu_t \triangleright X, \ (4)
\]
for all \( t \in (\varepsilon, \delta] \).

**Proof.** The proof is similar to the proof of Theorems 4.15 and 5.7. \( \square \)

**Theorem 6.9.** Let \( \mu \) be a fuzzy set of a BCH-algebra \( X \), \( \varepsilon, \delta \in (0, 1] \) with \( \varepsilon < \delta \). Then
\begin{enumerate}
  \item[(Q1)] \( \mu \) is a commutative fuzzy ideal of \( X \) if and only if \( \mu \) is a commutative fuzzy ideal of \( X \) with thresholds \( \varepsilon = 0 \) and \( \delta = 1 \).
  \item[(R1)] \( \mu \) is a commutative \((\bar{\varepsilon}, \bar{\varepsilon} \lor q_k)\)-fuzzy ideal of \( X \) if and only if \( \mu \) is a commutative fuzzy ideal of \( X \) with thresholds \( \varepsilon = 0 \) and \( \delta = \frac{1-k}{2} \).
  \item[(S1)] \( \mu \) is a commutative \((\bar{\varepsilon}, \bar{\varepsilon} \lor \bar{q}_k)\)-fuzzy ideal of \( X \) if and only if \( \mu \) is a commutative fuzzy ideal of \( X \) with thresholds \( \varepsilon = \frac{1-k}{2} \) and \( \delta = 1 \).
\end{enumerate}

**Proof.** Straightforward. \( \square \)

### 7. Level 4 of Fuzzification

In this section, we introduce the concepts of fuzzifying ideal, commutative fuzzifying ideal, \( t \)-implication-based fuzzy ideal and commutative \( t \)-implication-based fuzzy ideal of BCH-algebras and investigate some of their related properties.

Fuzzy propositional calculus is an extension of the Aristotelean propositional calculus. In fuzzy propositional calculus the truth set is taken \([0, 1]\) instead of \(\{0, 1\}\), which is the truth set in Aristotelean propositional calculus. In fuzzy logic some of the operators like \( \land, \lor, \neg, \rightarrow \) can be defined by using
truth tables. One can also use the extension principle to obtain the definitions of these operators.

In fuzzy logic the truth value of a fuzzy proposition $\mu$ is denoted by $[\mu]$: In the following we give fuzzy logic and its corresponding set theoretical notations, which we will use in the paper hereafter.

$$[x \in \mu] = \mu(x) \quad (5)$$
$$[\emptyset \land \psi] = [\emptyset] \land [\psi] \quad (6)$$
$$[\emptyset \rightarrow \psi] = 1 \land (1 - [\emptyset] + [\psi]) \quad (7)$$
$$[\forall x \theta(x)] = \land_{x \in U} [\emptyset(x)] \quad (8)$$
$$\models \emptyset \text{ if and only if } [\emptyset] = 1 \text{ for all valuations.} \quad (9)$$

The truth valuation rules given in (7) are those in the Lukasiewicz system of continuous-valued logic. Of course, various implication operators can be similarly defined. We consider in the following important implication operators:

(a) Gaines-Rescher implication operator ($I_{GR}$):

$$I_{GR}(x, y) = \begin{cases} 
1 & \text{if } x \leq y \\
0 & \text{otherwise},
\end{cases}$$

(b) Gödel implication operator ($I_G$):

$$I_G(x, y) = \begin{cases} 
1 & \text{if } x \leq y \\
y & \text{otherwise},
\end{cases}$$

(c) The contraposition of Gödel implication operator ($I_G$):

$$I_G(x, y) = \begin{cases} 
1 & \text{if } x \leq y \\
1 - x & \text{otherwise},
\end{cases}$$

where $x$ is the degree of truth (or degree of membership) of the premise and $y$ is the respective value for the consequence and $I$ is the resulting degree of truth for the implication. The quality of these implication operators could be evaluated either by empirically or by axiomatically methods. Ying introduced the concept of fuzzifying topology in [33]. Here we can expand his idea to BCH-algebras, and we define a fuzzifying ideal and commutative fuzzifying ideal as follows.

**Definition 7.1.** A fuzzy set $\mu$ of a BCH-algebra $X$ is called a fuzzifying ideal of $X$ if it satisfies (T1) and (U1), where

(T1) $\models [x \in \mu] \rightarrow [0 \in \mu],$

(U1) $\models [(x \ast y) \in \mu] \land [y \in \mu] \rightarrow [x \in \mu],$

for all $x, y \in X.$
Obviously, conditions (T1) and (U1) are equivalent to (F1) and (F2), respectively. Therefore a fuzzifying ideal is an ordinary fuzzy ideal.

**Example 7.2.** Let $X = \{0, a, b, c\}$ in which $*$ is defined in

\[
\begin{array}{|c|c|c|c|c|}
\hline
* & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
\hline
a & a & 0 & 0 & 0 \\
\hline
b & b & a & 0 & a \\
\hline
c & c & c & c & 0 \\
\hline
\end{array}
\]

Then $(X, *, 0)$ is a BCH-algebra. Let $s_0, s_1, s_2 \in [0, 1]$ be such that $s_0 > s_1 > s_2$. We define a map $\mu : X \rightarrow [0, 1]$ by $\mu(0) = s_0$, $\mu(a) = s_1$ and $\mu(b) = \mu(c) = s_2$. By simple calculations show that $\mu$ is a fuzzifying ideal of $X$.

**Definition 7.3.** A fuzzy set $\mu$ of a BCH-algebra $X$ is called a commutative fuzzifying ideal of $X$ if it satisfies (T1) and (V1), where

(T1) $\models [x \in \mu] \rightarrow [0 \in \mu],$

(V1) $\models [(x * y) * z \in \mu] \land [z \in \mu] \rightarrow [x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in \mu],$

for all $x, y, z \in X$.

Obviously, conditions (T1) and (V1) are equivalent to (F1) and (F3), respectively. Therefore a commutative fuzzifying ideal is an ordinary commutative fuzzy ideal.

**Example 7.4.** Consider the BCH-algebra $X = \{0, 1, 2, 3\}$ with the following Cayley table:

\[
\begin{array}{|c|c|c|c|c|}
\hline
* & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 0 & 0 & 0 \\
\hline
2 & 2 & 1 & 0 & 1 \\
\hline
3 & 3 & 1 & 1 & 0 \\
\hline
\end{array}
\]

Define a fuzzy set $\mu$ in BCH-algebra $X$ by $\mu(0) = 0.8$, $\mu(1) = \mu(2) = 0.2$, $\mu(3) = 0$. By simple calculations show that $\mu$ is a commutative fuzzifying ideal of $X$.

In [34], the concept of t-tautology is introduced, i.e.,

$\models_t \emptyset$ if and only if $[\emptyset] \geq t$ for all valuations. (10)

**Definition 7.5.** Let $\mu$ be a fuzzy set of a BCH-algebra $X$ and $t \in (0, 1]$. $\mu$ is called a t-implication-based fuzzy ideal of $X$ if it satisfies (W1) and (X1),
where

\[(W1) \ \models_t [x \in \mu] \rightarrow [0 \in \mu],\]
\[(X1) \ \models_t [(x \ast y) \in \mu] \land [y \in \mu] \rightarrow [x \in \mu],\]

for all \(x, y \in X\).

**Definition 7.6.** Let \(\mu\) be a fuzzy set of a BCH-algebra \(X\) and \(t \in (0, 1]\). \(\mu\) is called a commutative \(t\)-implication-based fuzzy ideal of \(X\) if it satisfies \((W1)\) and \((Y1)\), where

\[(X1) \ \models_t [x \in \mu] \rightarrow [0 \in \mu],\]
\[(Y1) \ \models_t [(x \ast y) \ast z \in \mu] \land [z \in \mu] \rightarrow [x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \in \mu],\]

for all \(x, y, z \in X\).

**Theorem 7.7.** Every commutative \(t\)-implication-based fuzzy ideal of a BCH-algebra \(X\) is a \(t\)-implication-based fuzzy ideal.

**Proof.** The proof is similar to the proof of Theorem 6.5. \(\square\)

**Theorem 7.8.** A \(t\)-implication-based fuzzy ideal of a BCH-algebra \(X\) is a commutative \(t\)-implication-based fuzzy ideal of \(X\) if and only if it satisfies the condition

\[\models_t [(x \ast y) \in \mu] \rightarrow [x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \in \mu] \text{ for all } x, y \in X.\]

**Proof.** The proof is similar to the proof of Theorem 6.5. \(\square\)

Let \(I\) be an implication operator. Clearly, \(\mu\) is a commutative \(t\)-implication-based fuzzy ideal of a BCH-algebra \(X\) if and only if it satisfies \((Z1)\) and \((A2)\), where

\[(Z1) \ I(\mu(x), \mu(0)) \geq t,\]
\[(A2) \ I(\mu((x \ast y) \ast z) \land \mu(z), \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq t,\]

for all \(x, y, z \in X\).

**Theorem 7.9.** For any fuzzy set \(\mu\) of a BCH-algebra \(X\), we have

\[(B2) \text{ If } I = I_{GR}, \text{ then } \mu\text{ is a commutative } 0.5\text{-implication-based fuzzy ideal of } X \text{ if and only if } \mu\text{ is a commutative fuzzy ideal of } X.\]
\[(C2) \text{ If } I = I_G, \text{ then } \mu\text{ is a commutative } \frac{1-k}{2}\text{-implication-based fuzzy ideal of } X \text{ if and only if } \mu\text{ is a commutative } (\varepsilon, \varepsilon \lor \bar{q}_k)\text{-fuzzy ideal of } X.\]
\[(D2) \text{ If } I = \overline{I}_G, \text{ then } \mu\text{ is a commutative } \frac{1-k}{2}\text{-implication-based fuzzy ideal of } X \text{ if and only if } \mu\text{ is a commutative } (\bar{\varepsilon}, \bar{\varepsilon} \lor \bar{q}_k)\text{-fuzzy ideal of } X.\]

**Proof.** (B2) Straightforward.

(C2): Suppose that \(\mu\) is a commutative \(\frac{1-k}{2}\)-implication-based fuzzy ideal of \(X\). Then

\[I_G(\mu(x), \mu(0)) \geq \frac{1-k}{2}\]
and

\[ I_G(\mu((x * y) * z) \land \mu(z), \mu(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \geq \frac{1-k}{2}. \]

It follows that

\[ \mu(0) \geq \mu(x) \]

or

\[ \mu(x) \geq \mu(0) \geq \frac{1-k}{2}, \]

and

\[ \mu(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \geq \mu((x * y) * z) \land \mu(z) \]

or

\[ \mu((x * y) * z) \land \mu(z) \geq \mu(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \geq \frac{1-k}{2}. \]

Hence,

\[ \mu(0) \lor 0 = \mu(0) \geq \mu(x) \land \frac{1-k}{2} \]

and

\[ \mu(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \lor 0 = \mu(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \geq \mu((x * y) * z) \land \mu(z) \land \frac{1-k}{2} \]

Therefore, \( \mu \) is a commutative fuzzy ideal of \( X \) with thresholds \( \varepsilon = 0 \) and \( \delta = \frac{1-k}{2} \), and hence, \( \mu \) is a commutative \((\varepsilon, \varepsilon \lor \eta_k)\)-fuzzy ideal of \( X \) by Theorem 6.9.

Conversely, assume that \( \mu \) is a commutative \((\varepsilon, \varepsilon \lor \eta_k)\)-fuzzy ideal of \( X \). Then

\[ \mu(0) = \mu(0) \lor 0 \geq \mu(x) \land \frac{1-k}{2} \]

and

\[ \mu(x * ((y * (y * x)) * (0 * (0 * (x * y))))) = \mu(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \lor 0 \geq \mu((x * y) * z) \land \mu(z) \land \frac{1-k}{2}. \]

Case 1: If \( \mu(x) \land \frac{1-k}{2} = \mu(x) \), then

\[ I_G(\mu(x), \mu(0)) = 1 \geq \frac{1-k}{2}. \]

If \( \mu(x) \land \frac{1-k}{2} = \frac{1-k}{2} \), then
\[ \mu(0) \geq \frac{1-k}{2} \]

and so

\[ I_G(\mu(x), \mu(0)) \geq \frac{1-k}{2}. \]

Case 2: If \( \mu((x \ast y) \ast z) \wedge \mu(z) \wedge \frac{1-k}{2} = \mu((x \ast y) \ast z) \wedge \mu(z), \) then

\[ \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y))))))) \geq \mu((x \ast y) \ast z) \wedge \mu(z) \]

and thus,

\[ I_G(\mu((x \ast y) \ast z) \wedge \mu(z), \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y))))))) = 1 \geq \frac{1-k}{2}. \]

Let

\[ \mu((x \ast y) \ast z) \wedge \mu(z) \wedge \frac{1-k}{2} = \frac{1-k}{2}. \]

Then \( \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y))))))) \geq \frac{1-k}{2}, \) and hence,

\[ I_G(\mu((x \ast y) \ast z) \wedge \mu(z), \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y))))))) \geq \frac{1-k}{2}. \]

Therefore \( \mu \) is a commutative \( \frac{1-k}{2} \)-implication-based fuzzy ideal of \( X. \)

(D2): Assume that \( \mu \) is a commutative \((\bar{c}, \bar{e} \lor \bar{q}_k)\)-fuzzy ideal \( X. \) Then \( \mu \) is a commutative fuzzy ideal of \( X \) with thresholds \( \varepsilon = \frac{1-k}{2} \) and \( \delta = 1 \) by Theorem 6.9. Thus,

\[ \mu(0) \lor \frac{1-k}{2} \geq \mu(x) \land 1 \]

and

\[ \mu(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y))))))) \lor \frac{1-k}{2} \geq \mu((x \ast y) \ast z) \wedge \mu(z) \land 1. \]

Case 1: If \( \mu(x) = 1, \) then

\[ \mu(0) \lor \frac{1-k}{2} = 1 \]

and thus,

\[ \bar{I}_G(\mu(x), \mu(0)) = 1 \geq \frac{1-k}{2}. \]

If \( \mu(x) < 1, \) then

\[ \mu(0) \lor \frac{1-k}{2} \geq \mu(x). \]

Thus, if \( \mu(0) \lor \frac{1-k}{2} = \mu(0), \) then

\[ \mu(0) \geq \mu(x) \]
and so
\[ I_G(\mu(x), \mu(0)) = 1 \geq \frac{1-k}{2}. \]

If \( \mu(0) \vee \frac{1-k}{2} = \frac{1-k}{2} \), then
\[ \mu(x) \leq \frac{1-k}{2}. \]

This implies that
\[ I_G(\mu(x), \mu(0)) = 1 \geq \frac{1-k}{2} \]
when \( \mu(0) \geq \mu(x) \)
and
\[ I_G(\mu(x), \mu(0)) = 1 - \mu(x) \geq \frac{1-k}{2} \]
when \( \mu(0) < \mu(x) \).

Case 2: If \( \mu((x * y) * z) \wedge \mu(z) \wedge 1 = 1 \), then
\[ I_G(\mu((x * y) * z) \wedge \mu(z), \mu(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) = 1 \geq \frac{1-k}{2}. \]

Suppose that
\[ \mu((x * y) * z) \wedge \mu(z) \wedge 1 = \mu((x * y) * z) \wedge \mu(z). \]

Then
\[ \mu(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) \vee \frac{1-k}{2} \geq \mu((x * y) * z) \wedge \mu(z). \]

If
\[ \mu(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) \vee \frac{1-k}{2} = \mu(x * ((y * (y * x)) * (0 * (0 * (x * y))))), \]
then
\[ \mu(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) \geq \mu((x * y) * z) \wedge \mu(z) \]
and so
\[ I_G(\mu((x * y) * z) \wedge \mu(z), \mu(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) = 1 \geq \frac{1-k}{2}. \]

If \( \mu(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) \vee \frac{1-k}{2} = \frac{1-k}{2} \), then
\[ \mu(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) \leq \frac{1-k}{2} \]
and
Hence,
\[
\overline{I}_G(\mu((x \ast y) \ast z) \wedge \mu(z), \mu((x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y))))))) = 1 \geq \frac{1-k}{2}
\]
when \(\mu((x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) \geq \mu((x \ast y) \ast z) \wedge \mu(z)\) and
\[
\overline{I}_G(\mu((x \ast y) \ast z) \wedge \mu(z), \mu((x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y))))))) = 1 - \mu((x \ast y) \ast z) \wedge \mu(z) \geq \frac{1-k}{2}
\]
when \(\mu((x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))))) < \mu((x \ast y) \ast z) \wedge \mu(z)\).
Consequently, \(\mu\) is a commutative \(1-k\)-implication-based fuzzy ideal of \(X\).

Conversely, assume that \(\mu\) is a commutative \(1-k\)-implication-based fuzzy ideal of \(X\). Then
\[
\overline{I}_G(\mu(x), \mu(0)) \geq \frac{1-k}{2}
\]
and
\[
\overline{I}_G(\mu((x \ast y) \ast z) \wedge \mu(z), \mu((x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y))))))) = 1 \geq \frac{1-k}{2}
\]
for all \(x, y, z \in X\). It follows that
\[
\mu(x) \leq \mu(0) \text{ or } 1 - \mu(x) \geq \frac{1-k}{2},
\]
i.e., \(\mu(x) \leq \frac{1-k}{2}\),
and
\[
\mu((x \ast y) \ast z) \wedge \mu(z) \leq \mu((x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y))))))
\]
or
\[
\mu((x \ast y) \ast z) \wedge \mu(z) \leq \frac{1-k}{2}.
\]
Thus,
\[
\mu(0) \vee \frac{1-k}{2} \geq \mu(x) = \mu(x) \wedge 1
\]
and
\[
\mu((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \vee \frac{1-k}{2} \geq \mu((x \ast y) \ast z) \wedge \mu(z)
\]
\[
= \mu((x \ast y) \ast z) \wedge \mu(z) \wedge 1.
\]

Hence, \(\mu\) is a commutative \((\bar{\varepsilon}, \bar{\varepsilon} \vee \bar{q}_k)\)-fuzzy ideal of \(X\) by Theorem 6.9 (S1). \(\square\)
8. CONCLUSION

In the study of fuzzy algebraic system, we see that the commutative fuzzy ideals with special properties always play a fundamental role.

In this paper, we define commutative \((\in, \in \lor q_k)\)-fuzzy ideals and commutative \((\bar{\in}, \bar{\in} \lor \bar{q}_k)\)-fuzzy ideals in BCH-algebras and give several characterizations of commutative fuzzy ideals in BCH-algebras in terms of these notions. The concept of a commutative fuzzy ideal with thresholds, commutative fuzzy-fying ideal and commutative t-implication-based fuzzy ideal are introduced and studied.

We believe that the research along this direction can be continued, and in fact, some results in this paper have already constituted a foundation for further investigation concerning the further development of fuzzy BCH-algebras and their applications in other branches of algebra. In the future study of fuzzy BCH-algebras, perhaps the following topics are worth to be considered:

1. To characterize other classes of BCH-algebras by using this notion;
2. To apply this notion to some other algebraic structures;
3. To consider these results to some possible applications in computer sciences and information systems in the future.

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REFERENCES


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