DIFFERENTIAL SANDWICH THEOREMS OF P-VALENT FUNCTIONS

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In this paper, we obtain sandwich results involving Hadamard product for certain p-valent functions associated with Noor integral operator in the open unit disk.

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1. INTRODUCTION

Let Σ_p denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in N = 1, 2, 3, ...), \tag{1}$$

which are analytic in the punctured open unit disk $\mathbb{U} = \{z : |z| < 1\}$.

For functions $f \in \Sigma_p$ given by (1) and $g \in \Sigma_p$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k.$$

We define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k. \tag{2}$$

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. We say that the function $g(z)$ is subordinate to $f(z)$, if there exists a function $w(z)$ analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$, and such that $g(z) = f(w(z))$. In such a case, we write $g(z) \prec f(z)$. If the function f is univalent in $\mathbb{U}$, then $g(z) \prec f(z)$ if and only if $g(0) = f(0)$ and $g(\mathbb{U}) \subset f(\mathbb{U})$.

Let $H(\mathbb{U})$ denote the class of analytic functions in $\mathbb{U}$ and let $H(a, n)$ denote the subclass of functions $f \in H(\mathbb{U})$ of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots (n = 1, 2, 3, ...).$$

Denote by Q, the set of all functions $f(z)$ that are analytic and injective on $\overline{U}\backslash E(f)$, where $E(f) = \{\xi \in \partial U : \lim_{z\to\xi} f(z) = \infty\}$, and such that $f'(\xi) \neq 0$ for $\xi \in \partial U\backslash E(f)$.

Let $\psi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, let $h(z)$ be univalent in $\mathbb{U}$ and $q(z) \in Q$. Miller and Mocanu [1] considered the problem of determining conditions on admissible function ψ such that

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \tag{3}$$

implies $p(z) \prec q(z)$, for all functions $p(z) \in H(a,n)$ that satisfy the differential subordination (3). Moreover, they found conditions so that $q(z)$ is the smallest function with this property, called the best dominant of the subordination (3).

Let $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, let $h(z) \in H$ and $q(z) \in H(a,n)$. Recently, Miller and Mocanu [2] studied the dual problem and determined conditions on φ such that

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \tag{4}$$

implies $q(z) \prec p(z)$, for all functions $p(z) \in Q$ that satisfy the above superordination. They also found conditions so that the function $q(z)$ is the largest function with this property, called the best subordinant of the superordination (4).

Liu and Noor [3] introduced an integral operator $\mathcal{N}_{n,p}f(z) : \Sigma_p \longrightarrow \Sigma_p$ as follows:

Let $f_{n,p}(z) = \frac{z^p}{(1-z)^{n+p}} (n > -p)$, and let $f_{n,p}^{(+)}(z)$ be defined such that

$$f_{n,p}(z) * f_{n,p}^{(+)}(z) = \frac{z^p}{(1-z)^{1+p}}, \tag{5}$$

then

$$\mathcal{N}_{n,p}f(z) = f_{n,p}^{(+)}(z) * f_{n,p}(z) = (\frac{z^p}{(1-z)^{n+p}})^{(+)} * f_{n,p}(z). \tag{6}$$

If $f(z)$ is given by (1), then from (5) and (6), we deduce that

$$\mathcal{N}_{n,p}f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{(p+1)(p+2)\cdots k}{(n+p)(n+p+1)\cdots(n+k-1)} a_k z^k. \tag{7}$$

It follows from (7) that

$$z(\mathcal{N}_{n+1,p}f(z))' = (n+p)\mathcal{N}_{n,p}f(z) - n\mathcal{N}_{n+1,p}f(z). \tag{8}$$

We also note that $\mathcal{N}_{0,p}f(z) = \frac{zf'(z)}{p}$ and $\mathcal{N}_{1,p}f(z) = f(z)$. The operator $\mathcal{N}_{n,p}f(z)$ defined by (6) is called as the Noor integral operator of $(n+p-1)th$ order of $f(z)$ [1]. For $p = 1$, the operator $\mathcal{N}_{n,1}f(z)$ was introduced by Noor

[4] and Noor [5]. Several classes of analytic functions, defined by using the operator $\mathcal{N}_{n,1}f(z)$, have been studied by many authors [6–8].

In this paper, we will derive several subordination results, superordination results and sandwich results involving the operator $\mathcal{N}_{n,p}f(z)$ and some of its special operators.

In order to prove our main results, we need the following lemmas.

LEMMA 1 (see [9]). *Let $q(z)$ be univalent in $\mathbb{U}$, $\gamma \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and suppose that*

$$Re\{1+\frac{zq''(z)}{q'(z)}\} > max\{0, -Re\frac{1}{\gamma}\}. \tag{9}$$

If $p(z)$ is analytic in $\mathbb{U}$, with $p(0) = q(0)$ and

$$p(z) + \gamma z p'(z) \prec q(z) + \gamma z q'(z), \tag{10}$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant.

LEMMA 2 (see [10]). *Let $q(z)$ be convex in $\mathbb{U}$, $q(0) = a$ and $\gamma \in \mathbb{C}$, $Re\gamma > 0$. If $p \in H(a, 1)$ and $p(z) + \gamma z p'(z)$ is univalent in $\mathbb{U}$, and*

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z), \tag{11}$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

2. MAIN RESULTS

We shall assume in the reminder of this paper that $p, n \in \mathbb{N}$ and $z \in \mathbb{U}$.

THEOREM 1. *Let $q(z)$ be univalent in $\mathbb{U}$ with $q(0) = 1$, $\alpha \in \mathbb{C}^*$, and suppose that*

$$Re\{1+\frac{zq''(z)}{q'(z)}\} > max\{0, -Re\frac{n+p}{\alpha}\}. \tag{12}$$

If $f(z) \in \Sigma_p$ satisfies the subordination

$$\mathcal{R}(\alpha, n, p) \prec q(z) + \frac{\alpha}{n+p} z q'(z), \tag{13}$$

where $\mathcal{R}(\alpha, n, p)$ is given by

$$\mathcal{R}(\alpha, n, p) = (1-\alpha)\frac{\mathcal{N}_{n+1,p}f(z)}{z^p} + \alpha\frac{\mathcal{N}_{n,p}f(z)}{z^p}, \tag{14}$$

then

$$\frac{\mathcal{N}_{n+1,p}f(z)}{z^p} \prec q(z).$$

and $q(z)$ is the best dominant.

Proof. Let

$$p(z)=\frac{\mathcal{N}_{n+1,p}f(z)}{z^p}, \tag{15}$$

differentiating (15) with respect to z and using the identity (8) in the resulting equation, we have

$$\frac{zp'(z)}{p(z)}=(n+p)\{\frac{\mathcal{N}_{n,p}f(z)}{\mathcal{N}_{n+1,p}f(z)}-1\},$$

that is,

$$\frac{1}{(n+p)}zp'(z)=\frac{\mathcal{N}_{n,p}f(z)}{z^p}-\frac{\mathcal{N}_{n+1,p}f(z)}{z^p}.$$

Therefore, we have

$$\mathcal{R}(\alpha,n,p)=(1-\alpha)\frac{\mathcal{N}_{n+1,p}f(z)}{z^p}+\alpha\frac{\mathcal{N}_{n,p}f(z)}{z^p}=p(z)+\frac{\alpha}{(n+p)}zp'(z).$$

By (13), we obtain

$$p(z)+\frac{\alpha}{(n+p)}zp'(z)\prec q(z)+\frac{\alpha}{n+p}zq'(z).$$

By Lemma 1, $\frac{\mathcal{N}_{n+1,p}f(z)}{z^p}\prec q(z)$, and the proof of Theorem 1 is completed. □

Taking the convex function $q(z)=\frac{1+Az}{1+Bz}$ in Theorem 1, we have the following corollary.

COROLLARY 1. *Let $A,B,\alpha\in\mathbb{C}$, $A\neq B$, $|B|<1$, $Re\alpha>0$. If $f(z)\in\Sigma_p$ satisfies the subordination*

$$\mathcal{R}(\alpha,n,p)\prec\frac{1+Az}{1+Bz}+\frac{\alpha}{n+p}\frac{(A-B)z}{(1+Bz)^2},$$

where $\mathcal{R}(\alpha,n,p)$ is given by (14), then

$$\frac{\mathcal{N}_{n+1,p}f(z)}{z^p}\prec\frac{1+Az}{1+Bz},$$

and the function $\frac{1+Az}{1+Bz}$ is the best dominant.

Taking $n=0$ in Theorem 1, we obtain the following result.

COROLLARY 2. *Let $q(z)$ be univalent in $\mathbb{U}$ with $q(0)=1$, $\alpha\in\mathbb{C}^*$, and suppose that (12) holds. If $f(z)\in\Sigma_p$ satisfies the subordination*

$$\mathcal{R}(\alpha,p)\prec q(z)+\frac{\alpha}{p}zq'(z), \tag{16}$$

where $\mathcal{R}(\alpha, p)$ *is given by*

$$\mathcal{R}(\alpha, p) = (1-\alpha)\frac{f(z)}{z^p} + \frac{\alpha}{p}\frac{zf'(z)}{z^p}, \tag{17}$$

then

$$\frac{f(z)}{z^p} \prec q(z).$$

Taking $p = 1$ in Theorem 1, we have the following result.

COROLLARY 3. *Let* $q(z)$ *be univalent in* $\mathbb{U}$ *with* $q(0) = 1$, $\alpha \in \mathbb{C}^*$, *and suppose that (12) holds. If* $f(z) \in \Sigma_p$ *satisfies the subordination*

$$\mathcal{R}(\alpha, n) \prec q(z) + \frac{\alpha}{n+1}zq'(z), \tag{18}$$

where $\mathcal{R}(\alpha, n)$ *is given by*

$$\mathcal{R}(\alpha, n) = (1-\alpha)\frac{\mathcal{N}_{n+1}f(z)}{z} + \alpha\frac{\mathcal{N}_n f(z)}{z}, \tag{19}$$

then

$$\frac{\mathcal{N}_{n+1}f(z)}{z} \prec q(z).$$

THEOREM 2. *Let* $q(z)$ *be convex in* $\mathbb{U}$, $q(0) = 1$ *and* $\alpha \in \mathbb{C}$, $Re\alpha > 0$. *If* $f(z) \in \Sigma_p$ *such that* $\frac{\mathcal{N}_{n+1,p}f(z)}{z^p} \in H(q(0), 1) \bigcap Q$, *and* $\mathcal{R}(\alpha, n, p)$ *is univalent in* $\mathbb{U}$ *and satisfies the superordination*

$$q(z) + \frac{\alpha}{n+p}zq'(z) \prec \mathcal{R}(\alpha, n, p), \tag{20}$$

where $\mathcal{R}(\alpha, n, p)$ *is given by (14), then*

$$q(z) \prec \frac{\mathcal{N}_{n+1,p}f(z)}{z^p},$$

and $q(z)$ *is the best subordinant.*

Proof. Let $p(z)$ be given by (15) and proceeding as in the proof of Theorem 1, the subordination (20) becomes

$$q(z) + \frac{\alpha}{n+p}zq'(z) \prec p(z) + \frac{\alpha}{(n+p)}zp'(z).$$

The proof follows by an application of Lemma 2. □

Taking $n = 0$ in Theorem 2, we obtain the following result.

COROLLARY 4. *Let $q(z)$ be convex in $\mathbb{U}$, $q(0) = 1$ and $\alpha \in \mathbb{C}$, $Re\alpha > 0$. If $f(z) \in \Sigma_p$ such that $\frac{f(z)}{z^p} \in H(q(0), 1) \bigcap Q$, and $\mathcal{R}(\alpha, p)$ is univalent in $\mathbb{U}$ and satisfies the superordination*

$$q(z) + \frac{\alpha}{p} zq'(z) \prec \mathcal{R}(\alpha, p), \tag{21}$$

where $\mathcal{R}(\alpha, p)$ is given by(17), then

$$q(z) \prec \frac{f(z)}{z^p}.$$

Taking $p = 1$ in Theorem 2, we have the following result.

COROLLARY 5. *Let $q(z)$ be convex in $\mathbb{U}$, $q(0) = 1$ and $\alpha \in \mathbb{C}$, $Re\alpha > 0$. If $f(z) \in \Sigma_p$ such that $\frac{\mathcal{N}_{n+1}f(z)}{z} \in H(q(0), 1) \bigcap Q$, and $\mathcal{R}(\alpha, n)$ is univalent in $\mathbb{U}$ and satisfies the superordination*

$$\mathcal{R}(\alpha, n) \prec q(z) + \frac{\alpha}{n+1} zq'(z), \tag{22}$$

where $\mathcal{R}(\alpha, n)$ is given by (19), then

$$q(z) \prec \frac{\mathcal{N}_{n+1}f(z)}{z}.$$

Combining Theorems 1 and 2, we have the following sandwich theorem.

THEOREM 3. *Let q_1 and $q_2(z)$ be convex in $\mathbb{U}$, $q_1(0) = q_2(0) = 1$ and $q_2(z)$ satisfies (12), and $\alpha \in \mathbb{C}$, $Re\alpha > 0$. If $f(z) \in \Sigma_p$ such that $\frac{\mathcal{N}_{n+1,p}f(z)}{z} \in H(q(0), 1) \bigcap Q$, and $\mathcal{R}(\alpha, n, p)$ is univalent in $\mathbb{U}$ and satisfies*

$$q_1(z) + \frac{\alpha}{n+p} zq_1'(z) \prec \mathcal{R}(\alpha, n, p) \prec q_2(z) + \frac{\alpha}{n+p} zq_2'(z), \tag{23}$$

where $\mathcal{R}(\alpha, n, p)$ is given by (14), then

$$q_1(z) \prec \frac{\mathcal{N}_{n+1,p}f(z)}{z^p} \prec q_2(z).$$

and $q_1(z)$, $q_2(z)$ are the best subordinant and the best dominant, respectively.

Remark. Combining Corollaries 2, 4 and Corollaries 3, 5, we obtain the corresponding sandwich results for the operators $\mathcal{N}_p$ and $\mathcal{N}_{n+1}$

3. OPEN PROBLEM

If $f(z)$ is meromorphically multivalent functions, a new operator can be defined. New results which is about differential Sandwich theorems of the new operator can be obtained.

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