BALANCING AND LUCAS-BALANCING SUMS BY MATRIX METHODS

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The balancing number n and the balancer r are solution of a simple Diophantine equation $1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$. It is well known that if n is balancing number, then $8n^2+1$ is a perfect square and its positive square root is called a Lucas-balancing number. There is another way to generate balancing numbers and their related number sequences through matrices. The matrix representation indeed, gives many known and new formulas for these numbers. In this paper, two special types of 2×2 matrices $S=\begin{pmatrix}3 & 8\\ 1 & 3\end{pmatrix}$ and $T=\begin{pmatrix}0 & 8\\ 1 & 0\end{pmatrix}$ are introduced to derive some balancing and Lucas-balancing sums. Also, these matrices are used to establish some new identities for balancing and Lucas-balancing numbers.

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1. INTRODUCTION

The concept of balancing numbers was originally introduced by Behera *et al.* [1], in connection with the Diophantine equation

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r).$$

The balancing numbers B_n are the terms of the sequence {0, 1, 6, 35, 204, ...} beginning with the values $B_0=0$ and $B_1=1$ are the roots of the auxiliary equation (1.1) [1]. The sequence of balancing numbers $\{B_n\}$ satisfy the recurrence relation

$$B_{n+1}=6B_n-B_{n-1},\quad n\geq 1, \tag{1.1}$$

with $B_0=0, B_1=1$. The closed form (popularly known as Binet's formula) for balancing numbers is the expression $B_n=\frac{\lambda_1^n-\lambda_2^n}{\lambda_1-\lambda_2}$, where $\lambda_1=3+\sqrt{8}$, $\lambda_2=3-\sqrt{8}$ [1].

It is well known that if x is a balancing number, then $8x^2+1$ is a perfect square, and the positive square root of $8x^2+1$ is called a Lucas-balancing number which is denoted by C_n [8, 9]. Observe that $C_1 = 3,\ C_2 = 17$ and the Lucas-balancing numbers C_n satisfy the recurrence relation

$$C_{n+1} = 6C_n - C_{n-1}, \quad n \geq 2, \tag{1.2}$$

identical with that for the balancing numbers. The Binet's formula for the Lucas-balancing numbers is given by $C_n = \frac{\lambda_1^n + \lambda_2^n}{2}$.

With the help (1.1) and (1.2), we can extend the balancing and Lucas-balancing sequences backward to get

$$B_{-n} = 6B_{-n+1} - B_{-n+2} = -B_n, \tag{1.3}$$

$$C_{-n} = 6C_{-n+1} - C_{-n+2} = C_n. \tag{1.4}$$

In [4], K. Liptai searched for those balancing numbers which are Fibonacci numbers too. He proved that the only Fibonacci number in the sequence of balancing numbers is 1. In a similar manner, in [5], he proved that there are no Lucas numbers in the sequence of balancing numbers. In [9], Panda has shown that, the Lucas-balancing numbers are associated with balancing numbers in the way Lucas numbers are associated with Fibonacci numbers. Panda *et al.* [8], have proved that the Lucas-balancing numbers are nothing but the even ordered terms of the associated Pell sequence. Also they have shown that the n^{th} balancing numbers are product of n^{th} Pell numbers and n^{th} associated Pell numbers. Recently, Keskin and Karaatly [3], obtained some interesting properties of balancing numbers and square triangular numbers. In a subsequent paper, Liptai, *et al.* [6] added another interesting result to the theory of balancing numbers by generalizing these numbers. A. Berczes, *et al.* [2] and P. Olajos [7] surveyed some interesting properties and results on generalized balancing numbers.

Ray [10, 11], obtained some product formulas for both balancing and negatively subscripted balancing and Lucas-balancing numbers. Some curious congruence properties for balancing numbers and their applications are also studied in [12, 14]. In [13], Ray introduced Q-balancing matrix and using matrix algebra he obtained several interesting results on sequence of balancing numbers and its related numbers sequences. The matrix representation, indeed gives many known and new identities for these numbers.

This paper presents an interesting and valuable investigation about some special types of relations between matrices and balancing, Lucas-balancing numbers. It helps to obtain new balancing and Lucas-balancing identities by different methods. So, this paper contributes to balancing and Lucas-balancing numbers literature and encourage many researchers to investigate the properties of such number sequences.

2. SOME USEFUL RESULTS

THEOREM 2.1. *If B_n is the n^{th} balancing number and X be a square matrix with $X^2 = 6X - I$, where I is the identity matrix same order as A, then for all integers n*

$$X^n = B_n X - B_{n-1} I. \tag{2.1}$$

Proof. The proof proceeds by induction on n. Clearly the result holds for $n = 0, n = 1$. Assuming that the result is true for $n = k$, that is, $X^k = B_k X - B_{k-1} I$. Now, by the recurrence relation (1.1), we obtain

$$\begin{aligned} X^{k+1} &= X^k X \\ &= [B_k X - B_{k-1} I] X \\ &= B_k[6X - I] - B_{k-1} I X \\ &= [6B_k - B_{k-1}] X - B_k I \\ &= B_{k+1} X - B_k I. \end{aligned}$$

Thus, the result holds for all natural number n. Now to finish the proof, we need to show that, for all natural number n, $X^{-n} = B_{-n} X - B_{-n-1} I$. Let $Y = 6I - X = X^{-1}$ and since $X^2 = 6X - I$, we get

$$\begin{aligned} Y^2 &= 36I - 12X + X^2 \\ &= 6(6I - X) - I \\ &= 6Y - I. \end{aligned}$$

Continuing in this manner, we get $Y^n = B_n Y - B_{n-1} I$. It follows that for all natural number n,

$$\begin{aligned} X^{-n} &= (6B_n X - B_{n-1}) I - B_n X \\ &= B_{n+1} I - B_n X \\ &= B_{-n} X - B_{-n-1} I, \end{aligned}$$

which completes the proof. $\square$

The following corollary which is already shown in [10] is an immediate consequence of Theorem 2.1.

COROLLARY 2.2. *Let Q_B be the balancing Q-matrix defined by $Q_B = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}$, then for all integers n,* $Q_B^n = \begin{pmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{pmatrix}$.

We now introduce two special types of 2×2 matrices $S = \begin{pmatrix} 3 & 8 \\ 1 & 3 \end{pmatrix}$ and $T = \frac{1}{2}\left[S - S^{-1}\right] = \begin{pmatrix} 0 & 8 \\ 1 & 0 \end{pmatrix}$ which will be useful to establish some identities of the balancing and Lucas-balancing numbers. Also in the final Section of this article, we derive some balancing and Lucas-balancing sums with the help of the matrices S and T.

Notice that, for any numbers a, b, c, d, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then by usual matrix multiplication, we can write $TA = \begin{pmatrix} 8c & 8d \\ a & b \end{pmatrix}$.

The following corollary is also directly follows from Theorem 2.1.

COROLLARY 2.3. *If sequence* $S = \begin{pmatrix} 3 & 8 \\ 1 & 3 \end{pmatrix}$, *then for all integers* n, $S^n = \begin{pmatrix} C_n & 8B_n \\ B_n & C_n \end{pmatrix}$.

3. SOME KNOWN IDENTITIES OF BALANCING AND LUCAS-BALANCING NUMBERS *VIA* MATRICES

In this Section, we present some well known identities for balancing and Lucas-balancing numbers through the matrix S, which is defined in the Section 2 of this article.

LEMMA 3.1. *For every integer* n, $C_n^2 - 8B_n^2 = 1$.

Proof. Since the determinant, $\det S = 1$, it follows that, $\det S^n = 1$. So, For all integers n, $\det S^n = C_n^2 - 8B_n^2 = 1$. □

LEMMA 3.2. *For all integers* n *and* m, $B_{n+m} = B_nC_m + C_nB_m$ *and* $C_{n+m} = C_nC_m + 8B_nB_m$.

Proof. By virtue of Corollary 2.3, we find

$$\begin{aligned} \begin{pmatrix} C_{n+m} & 8B_{n+m} \\ B_{n+m} & C_{n+m} \end{pmatrix} &= S^{n+m} \\ &= S^n S^m \\ &= \begin{pmatrix} C_n & 8B_n \\ B_n & C_n \end{pmatrix} \begin{pmatrix} C_m & 8B_m \\ B_m & C_m \end{pmatrix} \\ &= \begin{pmatrix} C_nC_m + 8B_nB_m & 8(C_nB_m + B_nC_m) \\ C_nB_m + B_nC_m & C_nC_m + 8B_nB_m \end{pmatrix}. \end{aligned}$$

Thus, by comparing the corresponding entries of the matrices from both the sides, we get the desired results. □

LEMMA 3.3. *For all integers* n *and* m, $B_{n-m} = B_nC_m - C_nB_m$ *and* $C_{n-m} = C_nC_m - 8B_nB_m$.

Proof. Using Corollary 2.3, we obtain

$$\begin{pmatrix} C_{n-m} & 8B_{n-m} \\ B_{n-m} & C_{n-m} \end{pmatrix} = S^{n-m} = S^n S^{-m}$$

$$= \begin{pmatrix} C_n & 8B_n \\ B_n & C_n \end{pmatrix} \begin{pmatrix} C_m & -8B_m \\ -B_m & C_m \end{pmatrix}$$

$$= \begin{pmatrix} C_nC_m - 8B_nB_m & 8(B_nC_m - C_nB_m) \\ B_nC_m - C_nB_m & C_nC_m - 8B_nB_m \end{pmatrix}.$$

which follows the result. □

LEMMA 3.4. *For every integers n and m, $C_{n+m} + C_{n-m} = 2C_nC_m$ and $B_{n+m} + B_{n-m} = 2B_nC_m$.*

Proof. By Corollary 2.3, we get

$$\begin{pmatrix} C_{n+m} + C_{n-m} & 8(B_{n+m} + B_{n-m}) \\ B_{n+m} + B_{n-m} & C_{n+m} + C_{n-m} \end{pmatrix} = S^{n+m} + S^{n-m}$$

$$= S^nS^m + S^nS^{-m}$$

$$= \begin{pmatrix} C_n & 8B_n \\ B_n & C_n \end{pmatrix} \begin{pmatrix} C_m & 8B_m \\ B_m & C_m \end{pmatrix} + \begin{pmatrix} C_n & 8B_n \\ B_n & C_n \end{pmatrix} \begin{pmatrix} C_m & -8B_m \\ -B_m & C_m \end{pmatrix}$$

$$= \begin{pmatrix} 2C_nC_m & 16B_nC_m \\ 2B_nC_m & 2C_nC_m \end{pmatrix}.$$

The required results are obtained by comparing the corresponding entries from both the sides of the above matrices. □

4. BALANCING AND LUCAS-BALANCING SUMS *VIA* MATRICES

In this Section, we present some balancing and Lucas-balancing sums using the matrices S and T which are defined in Section 2.

THEOREM 4.1. *For every natural number n and for all integers m, k with $m \neq 0$,*

$$\sum_{r=0}^{n} C_{mr+k} = \frac{(C_k - C_{mn+m+k}) + (C_{mn+k} - C_{k-m})}{2(1 - C_m)}$$

and

$$\sum_{r=0}^{n} B_{mr+k} = \frac{(B_k - B_{mn+m+k}) + (B_{mn+k} - B_{k-m})}{2(1 - C_m)}.$$

Proof. Let I denotes the identity matrix same order as S and T. Then, it is clear that

$$I - (S^m)^{n+1} = (I - S^m)\sum_{r=0}^{n}(S^m)^r.$$

Further, by Lemma 3.1 and as $m \neq 0$, we have

$$\det(I - S^m) = (1 - C_m)^2 - 8B_m^2 = 2(1 - C_m) \neq 0.$$

Therefore by Corollary 2.3, it follows that

$$(I - S^m)^{-1}\left[I - (S^m)^{n+1}\right] S^k = \sum_{r=0}^{n} S^{mr+k} = \begin{pmatrix} \sum_{r=0}^{n} C_{mr+k} & 8\sum_{r=0}^{n} B_{mr+k} \\ \sum_{r=0}^{n} B_{mr+k} & \sum_{r=0}^{n} C_{mr+k} \end{pmatrix}. \tag{4.1}$$

But the inverse of the matrix $(I - S^m)$ is given by

$$(I - S^m)^{-1} = \frac{1}{2(1 - C_m)}[(1 - C_m)I + B_m T],$$

where $T = \begin{pmatrix} 0 & 8 \\ 1 & 0 \end{pmatrix}$. Therefore by virtue of Corollary 2.3, we get

$$\begin{aligned} &(I - S^m)^{-1}(S^k - S^{mn+m+k}) \\ &= \frac{1}{2(1 - C_m)}\left[(1 - C_m)\begin{pmatrix} C_k - C_{mn+m+k} & 8(B_k - B_{mn+m+k}) \\ B_k - B_{mn+m+k} & C_k - C_{mn+m+k} \end{pmatrix}\right. \\ &\qquad \left. + B_m \begin{pmatrix} 8(B_k - B_{mn+m+k}) & 8(C_k - C_{mn+m+k}) \\ C_k - C_{mn+m+k} & 8(B_k - B_{mn+m+k}) \end{pmatrix}\right]. \end{aligned} \tag{4.2}$$

Comparing (4.1) and (4.2) and by Lemma 3.2, we obtain

$$\sum_{r=0}^{n} C_{mr+k} = \frac{1}{2(1 - C_m)}\left[(1 - C_m)(C_k - C_{mn+m+k}) + 8B_m(B_k - B_{mn+m+k})\right]$$

$$= \frac{(C_k - C_{mn+m+k}) + (C_{mn+k} - C_{k-m})}{2(1 - C_m)}$$

and

$$\sum_{r=0}^{n} B_{mr+k} = \frac{1}{2(1 - C_m)}\left[(1 - C_m)(B_k - B_{mn+m+k}) + B_m(C_k - C_{mn+m+k})\right]$$

$$= \frac{(B_k - B_{mn+m+k}) + (B_{mn+k} - B_{k-m})}{2(1 - C_m)}. \quad \square$$

THEOREM 4.2. *Let n is a natural number and m, k be integers. Then, if n is even,*

$$\sum_{r=0}^{n}(-1)^r C_{mr+k} = \frac{(C_k + C_{mn+m+k}) + (C_{mn+k} + C_{k-m})}{2(1+C_m)}$$

and

$$\sum_{r=0}^{n}(-1)^r B_{mr+k} = \frac{(B_k + B_{mn+m+k}) + (B_{mn+k} + B_{k-m})}{2(1+C_m)}.$$

Again, if n is odd,

$$\sum_{r=0}^{n}(-1)^r C_{mr+k} = \frac{(C_k - C_{mn+m+k}) - (C_{mn+k} - C_{k-m})}{2(1+C_m)}$$

and

$$\sum_{r=0}^{n}(-1)^r B_{mr+k} = \frac{(B_k - B_{mn+m+k}) - (B_{mn+k} - B_{k-m})}{2(1+C_m)}.$$

Proof. If n is an even natural number, then it is obvious that

$$I + (S^m)^{n+1} = (I + S^m)\sum_{r=0}^{n}(-1)^r (S^m)^r.$$

By Lemma 3.1 and for all integers m, we obtain

$$\det(I + S^m) = (1 + C_m)^2 - 8B_m^2 = 2(1 + C_m) \neq 0.$$

Therefore, it follows that

$$(I+S^m)^{-1}(I+(S^m)^{n+1})S^k = \sum_{r=0}^{n}(-1)^r S^{mr+k} = \begin{pmatrix} \sum_{r=0}^{n}(-1)^r C_{mr+k} & 8\sum_{r=0}^{n}(-1)^r B_{mr+k} \\ \sum_{r=0}^{n}(-1)^r B_{mr+k} & \sum_{r=0}^{n}(-1)^r C_{mr+k} \end{pmatrix}. \tag{4.3}$$

The inverse of the matrix $I + S^m$ will be

$$(I + S^m)^{-1} = \frac{1}{2(1 + C_m)}\left[(1 + C_m)I - B_m T\right].$$

Therefore, by Corollary 2.3,

$$(I+S^m)^{-1}(S^k+S^{mn+m+k}) \tag{4.4}$$
$$= \frac{1}{2(1+C_m)}\left[(1+C_m)\begin{pmatrix} C_k+C_{mn+m+k} & 8(B_k+B_{mn+m+k}) \\ B_k+B_{mn+m+k} & C_k+C_{mn+m+k}\end{pmatrix}\right.$$
$$\left. -B_m\begin{pmatrix} 8(B_k+B_{mn+m+k}) & 8(C_k+C_{mn+m+k}) \\ C_k+C_{mn+m+k} & 8(B_k+B_{mn+m+k})\end{pmatrix}\right].$$

Again by comparing (4.3) and (4.4) and by Lemma 3.3, we get

$$\sum_{r=0}^{n}(-1)^r C_{mr+k} = \frac{1}{2(1+C_m)}\left[(1+C_m)(C_k+C_{mn+m+k})-8B_m(B_k+B_{mn+m+k})\right]$$
$$= \frac{(C_k+C_{mn+m+k})+(C_{mn+k}+C_{k-m})}{2(1+C_m)}$$

and

$$\sum_{r=0}^{n}(-1)^r B_{mr+k} = \frac{1}{2(1+C_m)}\left[(1+C_m)(B_k+B_{mn+m+k})-B_m(C_k+C_{mn+m+k})\right]$$
$$= \frac{(B_k+B_{mn+m+k})+(B_{mn+k}+B_{k-m})}{2(1+C_m)}.$$

Assuming if n is an odd natural number, then

$$\sum_{r=0}^{n}(-1)^r C_{mr+k} = \sum_{r=0}^{n-1}(-1)^r C_{mr+k} - C_{mr+k}.$$

Further, as $(n-1)$ is even, we have

$$\sum_{r=0}^{n}(-1)^r C_{mr+k} = \frac{C_k+C_{mn+k}+C_{mn+k-m}+C_{k-m}}{2(1+C_m)} - C_{mn+k}$$
$$= \frac{(C_k-C_{mn+m+k})-(C_{mn+k}-C_{k-m})}{2(1+C_m)}.$$

The other identity can be shown similarly. □

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