RATIONAL GROMOV-WITTEN INVARIANTS OF HIGHER CODIMENSIONAL SUBVARIETIES

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Let X be a smooth projective variety. It was shown by A. Gathmann that in case X is a very ample hypersurface of some other smooth projective variety Y, the genus-0 (so-called "restricted" and "unrestricted" with a sufficiently low number of marked points) Gromov-Witten invariants of X can be computed in terms of genus-0 Gromov-Witten invariants of Y. The purpose of this article is to generalize this result. More precisely, we will try to answer the following questions: *"When can we compute the rational invariants of X from those of the projective space that contains X?"* or, if this is not possible, *"When can we compute the invariants of X from those of a bigger variety Z that contains X?"*. In the first section we prove our main theorem that allows the computation of Gromov-Witten invariants of nef hypersurfaces. We will then try to compute the invariants of an s-codimensional subvariety X of a given variety Z, in case we can find a sequence of varieties $Y_1, \ldots, Y_{s-1}$, such that $X \hookrightarrow Y_1 \hookrightarrow Y_2 \cdots \hookrightarrow Y_s := Z$ and each two consequent varieties in the above row respect the hypothesis of our main theorem. In particular, we develop an algorithm for the computation of the Gromov-Witten invariants of complete intersections. One example is the number of lines and conics on a degree-9 three-fold in $\mathbb{P}^5$ that is a complete intersection of two cubic hypersurfaces, numbers that were first predicted by A. Libgober and J. Teitelbaum ([12]) in 1993.

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1. INTRODUCTION

In case not otherwise stated, by a variety we will always mean a complex, smooth, projective variety. Let X be such a variety. The first ingredients we need are the moduli spaces of absolute and relative stable maps of a certain given homology class $\beta \in H_2(X)$ to X. As these spaces are known to have in general the "wrong dimension" we will also need to give valid definitions of their virtual fundamental classes. If for the absolute maps there is a well known construction, for the moduli space of relative maps we will use a recent construction of Li in order to define its virtual fundamental class. Let us first briefly recall the definitions we already mentioned.

Definition 1.1. We denote the set of all n-pointed stable maps to X of class $\beta \in H_2(X)$ by $M_n(X,\beta)$. By Theorem 1.1.14 of [4] $M_n(X,\beta)$ is a proper and separated Deligne-Mumford stack, that we will call the **moduli space of stable (absolute) maps** of class β to X. Moreover, by [1], [2] there exists a **virtual fundamental class** $[M_n(X,\beta)]^{\text{virt}} \in H_*(M_n(X,\beta))$ of expected complex dimension

$$\dim[M_n(X,\beta)]^{\text{virt}} = -K_X \cdot \beta + (\dim X - 3) + n.$$

Remark 1.2. Let X be a hypersurface of Y and $\beta \in H_2(X)$. Then it is easily seen that $M_n(X,\beta) \subset M_n(Y, i_*\beta)$ and $[M_n(X,\beta)]^{\text{virt}}$ is a class in $H_*(M_n(Y,i_*\beta))$.

Notation 1.3. Related to these spaces for any for $i = 1,\dots,n$ we consider the maps

(i) $ev_i : M_n(X,\beta) \to X$, the evaluation at the ith marked point, and

(ii) $\pi_i : M_n(X,\beta) \to M_{n-1}(X,\beta)$, the forgetful map that forgets the ith marked point.

It obviously makes sense to consider compositions of such forgetful maps. We denote by $\pi_{(n,m)}$ the composition of forgetful maps that forget the last $n-m$ marked points.

Definition 1.4. Let $\alpha = (\alpha_1,\dots,\alpha_n)$ a tuple of non-negative integers. Then we define $M_\alpha^X(Y,\beta)$ to be the locus in $M_n(Y,\beta)$ of all stable maps $(C, x_1,\dots,x_n, f)$ such that

- $f(x_i) \in X$ for all i with $\alpha_i > 0$,
- $f^*X - \sum_i \alpha_i x_i \in A_0(f^{-1}(X))$ is effective.

We will call any $(C, x_1,\dots,x_n, f) \in M_\alpha^X(Y,\beta)$ a **stable relative map** to X.

Construction 1.5. The cycle class $f^*X \in A_0(f^{-1}(X))$ is well defined by [3], Chapter 6, as the refined intersection product $X \cdot C$ in $X \times_Y C = f^{-1}(X)$.

Remark 1.6. For degree reasons, the space $M_\alpha^X(Y,\beta)$ is empty if $\sum_i \alpha_i > X \cdot \beta$.

As already stated, we need to endow this space (so far we have just a set-theoretic definition) with additional structures. For this we use the following construction (of Li).

Construction 1.7. Let X a smooth hypersurface in a smooth projective variety Y. We denote by $P = \mathbb{P}(N_{X/Y}^\vee \oplus O_X)$ the projective closure of the dual normal bundle of Y in X. It comes equipped with a natural $\mathbb{C}^*$ action that rescales the fibers by acting with weights 1 respectively 0 on $\mathbb{P}(N_{X/Y})$ and O_X respectively. The fixed point locus of this $\mathbb{C}^*$ action consists of two components: the zero section $X_0 := \mathbb{P}(0 \oplus O_X) \backsimeq X \subset P$ and the infinity section $Y_\infty := P = \mathbb{P}(N_{X/Y}^\vee \oplus 0) \backsimeq X \subset P$.

For any $k \geq 0$ we define a normal crossing scheme Y_k, called the **k-th degeneration of Y**, as follows. It consists of $k+1$ irreducible components that will also be called levels. Level 0 is isomorphic to Y, whereas all the others are isomorphic to P. These components are glued transversally as follows:

• we glue $X \subset Y$ in level 0 to $X_0 \subset P$ in level 1;

• we glue $X_\infty \subset P$ in level i to $X_0 \subset P$in level $i+1$ for $i = 1, \ldots, k-1$.

For every k, there is a projection morphism $\pi : Y_k \to Y$, by collapsing the fibers in all levels greater than 0. Moreover, the $\mathbb{C}^*$ action on the k copies of P makes the group $(\mathbb{C}^*)^k$ into a group of automorphisms of X_k. We will call these automorphisms the **allowed automorphisms** of X_k.

Definition 1.8. An n-pointed **pre-stable map** to Y relative X is an n-pointed prestable map $(C, x_1, \ldots, x_n, f)$ to some degeneration Y_k such that:

• No irreducible component of C maps entirely to $X_\infty \subset Y_k$or to the singular locus of Y_k.

• Every point that maps to $X_\infty \subset Y_k$ is a marked point.

• Every point that maps to the singular locus of Y_k is a node with the property that the two local branches around the node map to the two different local components of Y_k with the same orders of contact to the singular locus on both sides.

A morphism $(C, x_1, \ldots, x_n, f) \to (C', x'_1, \ldots, x'_n, f')$ of n-pointed pre-stable relative maps of the same level k is a pair $(\varphi, \widetilde{\varphi})$, where

• $\varphi : C \to C'$ is a morphism of the underlying curves,

• $\widetilde{\varphi} : X_k \to X_k$ is an allowed automorphism , such that $\widetilde{\varphi} \circ f \circ \varphi = f'$.

A pre-stable relative map is called **stable** if its group of automorphisms is finite. The **class** of a pre-stable relative map is defined to be the element $\pi_* f_* [C] \in H_2^+(Y)$.

Definition 1.9. Let $(C, x_1, \ldots, x_n, f)$ be a pre-stable relative map of level k to Y relative X. For $i = 1, \ldots, n$ we define the **multiplicity** α_i of the ith marked point x_i to be the multiplicity of the point x_i in the divisor $f^* X_\infty$. The collection of these multiplicities will be denoted $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Definition 1.10. For any given n, $\beta \in H_2^+(X)$ and a collection of non-negative integers $\alpha = (\alpha_1, \ldots \alpha_n)$ such that $\sum_i \alpha_i = X \cdot \beta$, we denote by $\mathcal{M}_\alpha^X(Y, \beta)$ the set of isomorphism classes of all stable maps of any level to Y of class β and whose multiplicities are α.

Theorem 1.11. *The spaces $\mathcal{M}_\alpha^X(Y, \beta)$ are separated and proper Deligne-Mumford stacks of expected dimension*

$$\mathrm{vdim} \mathcal{M}_\alpha^X(Y, \beta) = \mathrm{vdim} M_n(Y, \beta) - \sum \alpha_i.$$

and there is a naturally defined virtual fundamental class $[\mathcal{M}_\alpha^X(Y, \beta]^{\mathrm{virt}} \in H_*(\mathcal{M}_\alpha^X(Y, \beta))$ of this dimension.

Proof. See [9], [10]. $\square$

Definition 1.12. The projection maps $\pi : Y_k \to Y$ give rise to morphisms

$$\pi_* : \mathcal{M}_\alpha^X(Y,\beta) \to M_n(Y,\beta)$$

that collapse all higher levels to the hypersurface X. We denote the image of this morphism by $M_\alpha^X(Y,\beta)$ and call this the moduli space of collapsed stable relative maps. Moreover, let $\alpha' = (\alpha, 1, \dots, 1)$ an n'-tuple of non-zero integers, and denote the composite map $\pi_{(n',n)} \circ \pi_*$ by $\pi_{(\alpha',\alpha)}$. This yields a map

$$\pi_{(\alpha',\alpha)} : \mathcal{M}_{\alpha'}^X(Y,\beta) \to M_n(Y,\beta)$$

that provides us with a well defined **moduli space of collapsed stable relative maps** denoted $M_\alpha^X(Y,\beta)$ and a virtual fundamental class on it, for every n-tuple of positive integers α.

PROPOSITION 1.13. *Definitions* 1.4 *and* 1.12 *of* $M_\alpha^X(Y,\beta)$ *agree, and* $\pi_*[\mathcal{M}_\alpha^X(Y,\beta)]$ *is a cycle in* $H_*(M_\alpha^X(Y,\beta))$ *in the expected dimension.*

Proof. See [4]. $\square$

Construction 1.14. Proposition 1.13 allows us to conclude $M_\alpha^X(Y,\beta)$ introduced in Definition 1.4 has the structure of a moduli space that comes equipped with a virtual fundamental class. In the next chapters we will always try to reduce ourselves to this definition and when there will be no risk of confusion we will simply call $M_\alpha^X(Y,\beta)$ the **moduli space of stable relative maps**.

Construction 1.15. On these spaces it makes sense to consider two kind of evaluation maps. We can of course consider the restrictions of the evaluation maps $ev_i : M_n(Y,\beta) \to X$ to $M_\alpha^X(Y,\beta) \subset M_n(Y,\beta)$, and moreover for any $\alpha_i > 0$ the maps

$$\begin{aligned} \widetilde{ev}_i : M_\alpha^X(Y,\beta) &\to X \\ (C, x_1, \dots, x_n, f) &\mapsto f(x_i). \end{aligned}$$

We will denote these second evaluation maps by $\widetilde{ev}_i$ or by $ev_{X,i}$, and even by ev_X when there is no risk of confusion.

Having now well defined virtual fundamental classes, we are ready to define the various types of Gromov-Witten invariants that we will deal with.

Notation 1.16. Fix an integer $1 \le i \le n$. By the ith **psi class**, denoted ψ_i we understand the first Chern class of the line bundle

$$L_i = \sigma_i^* \omega_{M_{n+1}(Y,\beta)/M_n(Y,\beta)},$$

where $\omega_{M_{n+1}(Y,\beta)/M_n(Y,\beta)}$ denotes the relative dualizing sheaf of the universal curve, and $\sigma_i : M_n(Y,\beta) \to M_{n+1}(Y,\beta)$ is the section corresponding to the

ith marked point (see [4]). It obviously makes sense to intersect with psi classes both the absolute and the relative moduli spaces $M_n(Y,\beta)$, respectively $M_\alpha^X(Y,\beta)$.

Definition 1.17. Let X be a hypersurface of Y. For any cohomology classes $\gamma_1,\dots\gamma_n \in H^*(Y)$, non-negative integers $k_1,\dots,k_n$ and $\beta \in H_2(X)$ we define the **absolute restricted Gromov-Witten invariant** on X of class β, namely,

$$\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle_\beta^X := \deg(\psi_1^{k_1}\cdot ev_1^*\gamma_1\cdots\psi_1^{k_n}\cdot ev_n^*\gamma_n\cdot[M_n(X,\beta)]^{\mathrm{virt}})\in\mathbb{Q}$$

where $\deg(\alpha)$ is 0 if $\alpha\in H_*(M_n(X,\beta))$ is not a 0-dimensional cycle (or equivalently when $\sum_i \mathrm{codim}\gamma_i + k_i \neq \mathrm{vdim}\ M_n(X,\beta)$) and the degree of α otherwise. The evaluation maps here, are the evaluations at Y. If $k_i = 0$ for some i we abbreviate $\tau_{k_i}(\gamma_i)$ to γ_i. A Gromov-Witten invariant is called **primary** if it does not contain any psi classes and **descendent** if it does.

Similarly, for any $\alpha = (\alpha_1,\dots,\alpha_n)$, any cohomology classes $\gamma_1,\dots\gamma_n\in H^*(Y)$ we define the **relative restricted Gromov-Witten invariant** of class $\delta\in H_2(Y)$, namely,

$$\langle \tau_{k_1}^{\alpha_1}(\gamma_1)\cdots\tau_{k_n}^{\alpha_n}(\gamma_n)\rangle_\delta^Y := \deg(\psi_1^{k_1}\cdot ev_1^*\gamma_1\cdots\psi_1^{k_n}\cdot ev_n^*\gamma_n\cdot[M_\alpha^X(Y,\delta)]^{\mathrm{virt}})\in\mathbb{Q}$$

where as before all the invariants with $\sum_i \mathrm{codim}\gamma_i + k_i \neq \mathrm{vdim}M_\alpha^X(Y,\beta)$ are considered to be 0. The evaluation maps are evaluations at Y and this is what motivates the name "restricted".

In the above notation let $M_n(X,\delta)$ be the disjoint union of all the moduli spaces $M_n(X,\beta)$ (with possibly different virtual dimensions) such that $i_*\beta=\delta$. Then we have an analogue Gromov-Witten invariant on the class

$$[M_n(X,\delta)]^{\mathrm{virt}} := \sum_{i_*\beta=\delta}[M_n(X,\beta)]^{\mathrm{virt}}$$

To distinguish the β-invariant from the δ-invariant of X, we will refer the absolute invariant of class $\beta\in H_2(X)$ as the **individual Gromov-Witten invariant**.

Definition 1.18. In the above notation, we consider a non-negative integer $p\le n$ and $\widetilde{\gamma}_1,\dots,\widetilde{\gamma}_p\in H^*(X)$. Then the **(unrestricted) absolute Gromov-Witten invariant** of X of class β is defined as

$$\langle \tau_{k_1}(\widetilde{\gamma}_1)\cdots\tau_{k_p}(\widetilde{\gamma}_p)\cdot\tau_{k_{p+1}}(\gamma_{p+1})\cdots\tau_{k_n}(\gamma_n)\rangle_\beta^X :=$$
$$:= \deg(\psi_1^{k_1}\cdot \widetilde{ev}_1^*\gamma_1\cdots\psi_p^{k_p}\cdot\widetilde{ev}_p^*\gamma_p\cdots\psi_n^{k_n}\cdot ev_n^*\gamma_n\cdot[M_n(X,\beta)]^{\mathrm{virt}}).$$

When $p=n$ we have the classical definition and we will call such an invariant the **Gromov-Witten invariant** of X. Analogously, if p is such that

$\alpha_1, \ldots \alpha_p > 0$, we define the **(unrestricted) relative Gromov-Witten invariant** of class δ

$$\langle \tau^{\alpha_1}_{k_1}(\widetilde{\gamma}_1) \cdots \tau^{\alpha_p}_{k_p}(\widetilde{\gamma}_p) \cdot \tau^{\alpha_{p+1}}_{k_{p+1}}(\gamma_{p+1}) \cdots \tau^{\alpha_n}_{k_n}(\gamma_n)\rangle^Y_\delta :=$$
$$:= \deg(\psi_1^{k_1} \cdot \widetilde{ev}_1^* \gamma_1 \cdots \psi_p^{k_p} \cdot \widetilde{ev}_p^* \gamma_p \cdots \psi_1^{k_n} \cdot ev_n^* \gamma_n \cdot [M^X_\alpha(Y, \delta)]^{\mathrm{virt}}).$$

When we will need to distinguish β and δ invariants on X we will call the β-invariants individual invariants.

Notation. When there is no risk of confusion we will omit the superscripts X or Y in the notation of the absolute respectively relative invariants.

2. THE MAIN THEOREM FOR NEF HYPERSURFACES

In this section we present our main technical result that provides an effective way of computing the Gromov-Witten invariants of a nef hypersurface from those of the ambient space. This will be done using Li's splitting theorem that reduces the invariants of a variety to invariants of an associated normal crossing scheme with two components glued along the divisor X. One of these components is the collapsed moduli space of relative stable maps $M^X_{\Gamma_1}(Y)$ and the other will be a the space denoted by $M^X_{\Gamma_2}(P)$, with P a projective line bundle over X. The main theorem will be obtained by projecting stable curves in P to X.

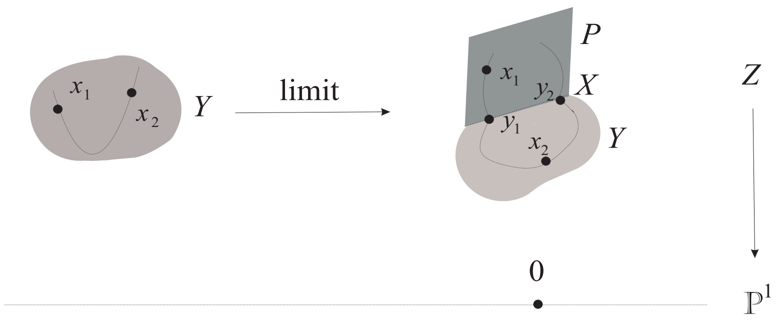
2.1. THE SPLITTING THEOREM AND VIRTUAL PUSH-FORWARD

Theorem 2.1.1 (Splitting Theorem). *Let Y be a variety and X a codimension-1 subvariety of Y. Let Z be the blow-up of $X \times \mathbb{P}^1$ in $Y \times \{0\}$, so that the general fiber of the projection $Z \to \mathbb{P}^1$ is isomorphic to X, whereas the fiber over zero is the normal crossing scheme $Y_1 := Y \cup_X P$. Let M be the moduli space of n-pointed stable maps to Z whose class in a general fiber of the morphism $Z \to \mathbb{P}^1$ is a fixed $\beta \in H_2(Y)$. Then M has a projection morphism to $\mathbb{P}^1$ and the general fiber of this morphism is $M_n(Y, \beta)$.*

The moduli space of stable maps to Y_1 is expressible as a product of moduli spaces of maps to Y and P and, moreover, the formula

$$[M_n(Y, \beta)]^{\mathrm{virt}} = \sum_{\Gamma_1, \Gamma_2} m(\Gamma_1 \Gamma_2) \cdot [M^X_{\Gamma_1}(Y)]^{\mathrm{virt}} \boxtimes [M^X_{\Gamma_2}(P)]^{\mathrm{virt}}$$

holds in $H_*(M)$, where we have used the following notation. The spaces $M^X_{\Gamma_1}(Y)$ (respectively $M^X_{\Gamma_2}(P)$) are moduli spaces of stable relative maps to X (respectively P) relative X, where Γ_1 (respectively Γ_2) denotes the collection of the



following data:

(i) *the number of connected components of the stable relative maps,*

(ii) *the (non-zero) homology class of all connected components,*

(iii) *for every connected component a subset of* $\{1, \ldots, n\}$ *of the marked points lying on it, where all these points have multiplicity* 0,

(iv) *for every connected component of class* δ *a collection of additional marked points* $\{y_i\}$ *lying on* X *with associated positive multiplicities* α_i*, such that* $\sum \alpha_i = \delta \cdot X$.

The sum in the above formula is taken over all pairs of data (Γ_1, Γ_2) *such that*

• *the glued stable map is connected and has the correct homology class, and*

• *the additional marked points* y_i *are labeled on both the* Y *and the* P *side by the same index set* $\{1, \ldots, r\}$ *for some* r*, and the multiplicities* α_i *associated to these points agree on both sides.*

The coefficient $m(\Gamma_1\Gamma_2)$ *is defined to be* $\frac{\alpha_1 \cdots \alpha_r}{r!}$*. The notation* $\boxtimes$ *means that we take the moduli spaces of collapsed stable relative maps on both sides and take their fiber product over the* r*-fold evaluation map to* X *at the points* y_i.

We will call the property of the moduli space $M_n(Y, \beta)$ expressed in the above formula the "splitting theorem" and we refer to [10] for the proof.

Definition 2.1.2. Let $p : M \to M'$ be a morphism of moduli spaces of stable maps. We say that p satisfies the virtual push-forward property if for any cohomology class $\gamma \in H^*(M)$ that is made up from evaluation classes, cotangent line classes and classes that are pulled-back from M' by p the following two conditions hold:

• if the dimension of the cycle $\gamma \cdot [M]^{\mathrm{virt}}$ is bigger than the *virtual* dimension of M', then $p_*(\gamma \cdot [M]^{\mathrm{virt}}) = 0$;

• if the dimension of the cycle $\gamma \cdot [M]^{\mathrm{virt}}$ is equal to the *virtual* dimension of M', then $p_*(\gamma \cdot [M]^{\mathrm{virt}}) = \lambda [M']^{\mathrm{virt}}$, for some $\lambda \in \mathbb{Q}$.

PROPOSITION 2.1.3. *Let L be a line bundle on a variety X and denote by $P = \mathbb{P}(L \oplus O_X)$ its closure. Let $\mathcal{M}_\alpha^X(P,\beta)$ a moduli space of stable maps relative $X_\infty = \mathbb{P}(L \oplus 0)$. Denote the marked points by $y_1, \dots y_N, x_1, \dots, x_n$, where y_i are the points with positive multiplicity and x_i are the points with zero multiplicity (i.e., $\alpha = (\alpha_1, \dots, \alpha_N, 0, \dots, 0)$). Let $p : \mathcal{M} \to M$ be the morphism that projects the maps in P to X forgets a given set of points and/or components and stabilizes the result. We assume P is well defined, i.e., that every component whose homology class β is a multiple of a fiber has at least* 3 *marked points that are not forgotten by p. Then p satisfies the push-forward property. If, moreover, L is nef then for any $0 \le t \le n$ and any t-tuple $m = (m_1, \dots, m_t)$ with $m_i \ge 0$,we have*

$$p_*\left(\prod_{k=1}^{t}\prod_{j=0}^{m_k-1}(ev_k^* X_\infty + j\psi_i)\cdot[\mathcal{M}]^{\mathrm{virt}}\right) =$$

$$= \begin{cases} 0 & \text{if } \sum_{i=1}^{t} m_i < X_\infty\cdot\beta + 1, \\ \left(\prod_{k=i_0}^{t}\prod_{j=m'_k}^{m_i-1}(j\psi_i + ev_k^* c_1(L))\right)\cdot[M]^{\mathrm{virt}} & \text{if } \sum_{i=1}^{t} m_i \ge X_\infty\cdot\beta + 1, \end{cases}$$

where i_0 is the minimum number such that $\sum_{i=1}^{i_0} m_i > X_\infty\cdot\beta$ and if $m'_{i_0} := X_\infty\cdot\beta - \sum_{i=1}^{i_0} m_i$, then m' is the t-tuple $(m_1, \dots, m_{i_0-1}, m'_{i_0}, 0, \dots, 0)$.

Proof. Analogous to Corollary 5.3.3 of [4] from Theorem 5.2.7 and Proposition 5.2.3. □

COROLLARY 2.1.4. *Under the hypothesis of the above theorem we consider $\mathcal{M}_\alpha^X(P)$ with possibly no marked points $x_1, \dots, x_n$, where $\alpha = (\alpha_1, 0, \dots, 0)$ and β is a multiple of a class of a fiber. Let $p : \mathcal{M} \to X$ be the obvious projection morphism. Then for all t-tuples $(m_1, \dots, m_t)$ as above we have*

$$p_*\left(\prod_{i=1}^{t}\prod_{k=0}^{m_i-1}(ev_i^* X_\infty + k\psi_i)\cdot[\mathcal{M}]^{\mathrm{virt}}\right) = \begin{cases} 0 & \text{if } \sum_{i=1}^{t} m \ne \alpha_1 - 1 + n, \\ \frac{1}{\alpha_1}[X] & \text{if } \sum_{i=1}^{t} m = \alpha_1 - 1 + n. \end{cases}$$

Proof. See [4] Proposition 5.2.3 and Corollary 5.3.4. □

2.2. THE MAIN THEOREM FOR NEF HYPERSURFACES

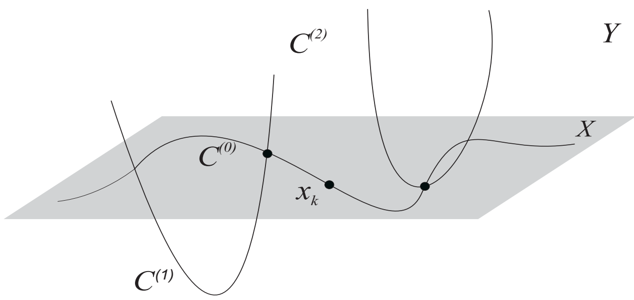
We begin with the description of the set-up. Let first X be a nef hypersurface of Y. Let α be an n-tuple of non-negative integers. We define $|\alpha| := n$

and $\sum \alpha := \sum \alpha_i$. For $1 \le k \le n$, we write $\alpha + e_k$ for $(\alpha_1, \dots, \alpha_k + 1, \dots, \alpha_n)$ $\alpha \cup \alpha'$ for $(\alpha_1, \dots, \alpha_n, \alpha'_1, \dots, \alpha'_m)$ and $\alpha + \alpha'_k$ for $(\alpha_1, \dots, \alpha_k + \alpha'_k, \dots, \alpha_n)$.

Definition 2.2.1. Consider $M^X_\alpha(Y), \beta$ and $1 \le k \le n$. Let t be a non-negative integer. Choose a partition $A = (\alpha^{(0)}_1, \dots, \alpha^{(0)}_n)$ of α such that $\alpha^{(0)}_k \in \alpha^{(0)}$. Let $B = (\beta^{(0)}, \dots, \beta^{(t)})$ be a $t+1$-tuple of homology classes with $\beta^{(0)} \in H_2(X)$ and $\beta^{(i)} \in H_2(Y) \setminus \{0\}$ for $i > 0$ such that $i_*\beta^{(0)} + \beta^{(1)} + \dots + \beta^{(t)} = \beta$, where $i : X \hookrightarrow Y$ is the inclusion. Finally, choose a t-tuple $M = (m^{(1)}, \dots, m^{(t)})$ of positive integers. With these notation we define the moduli space $D_k(Y, A, B, M)$ to be the fiber product

$$D_k(Y, A, B, M) := M_{|\alpha^{(0)}|+t}(X, \beta^{(0)}) \times_{X^t} \prod_{i=1}^{t} M_{\alpha^{(i)} \cup (m^{(i)})(X, \beta^{(i)})},$$

where the map from the first factor to X^r is the evaluation at the last t marked points, and the map from the second factor to X^r is the evaluation at the last marked point of each of its factors. We call $M_{|\alpha^{(0)}|+t}(X, \beta^{(0)})$ the *internal component* of $D_k(Y, A, B, M)$ and will usually denote it by $C^{(0)}$. Analogously we call all the other factors in $D_k(Y, A, B, M)$ *external components*, and denote them $C^{(i)}$ with $1 \le i \le t$. We define the virtual fundamental class to be $\frac{m^{(1)} \cdots m^{(t)}}{t!}$ times the class induced by the virtual fundamental class of each factor.



Definition 2.2.2. Let $D_{\alpha,k}(Y, \beta)$ be the disjoint union of the $D_k(Y, A, B, M)$ for all possible A, B and M satisfying

$$X \cdot i_*\beta^{(0)} + \sum_i m^{(i)} = \sum \alpha^{(0)}.$$

The virtual fundamental class of $D_{\alpha,k}(Y, \beta)$ is defined to be the sum of the virtual fundamental classes of its components $D_k(Y, A, B, M)$.

THEOREM 2.2.3 (Main theorem for nef hypersurfaces). *Let $\beta \in H_2(Y)$ be the class of a curve in Y and α be an n-tuple of non-negative integers such that $\sum \alpha \leq X \cdot \beta + 1$. With the notation as above we have*

$$\prod_{k=1}^{n} \prod_{j=0}^{\alpha_k - 1} (ev_k^* X + j\psi_k) \cdot [M_n^X(Y,\beta)]^{\mathrm{virt}} = [M_\alpha^X(Y,\beta)]^{\mathrm{virt}} +$$

$$+ \sum_{i=1}^{n} \sum_{\alpha'} \prod_{k=i}^{n} \prod_{j=\alpha'_k}^{\alpha_k - 1} (ev_k^* X + j\psi_k) \cdot [D_{\alpha',i}(Y,\beta)]^{\mathrm{virt}}$$

in the cohomology group of $M_\alpha^X(Y,\beta)$, where by $\sum_i \sum_{\alpha'}$ we mean a sum over i and over all the n-tuples $\alpha' := (\alpha_1, \ldots, \alpha_{i-1}, \alpha'_i, 0, \ldots, 0)$ with $0 \leq \alpha'_i < \alpha_i$.

Proof. The idea is to intersect the moduli space $[M_n(Y,\beta)]^{\mathrm{virt}}$ with the given number of evaluation and psi classes and express the product in terms of products of evaluation and psi-classes on the virtual classes of the moduli spaces of relative stable maps that appear in the splitting theorem. The result will then be obtained by projecting stable maps in P down to stable maps in X and establishing the topological type of the terms with the use of Proposition 2.1.3 and Corollary 2.1.4.

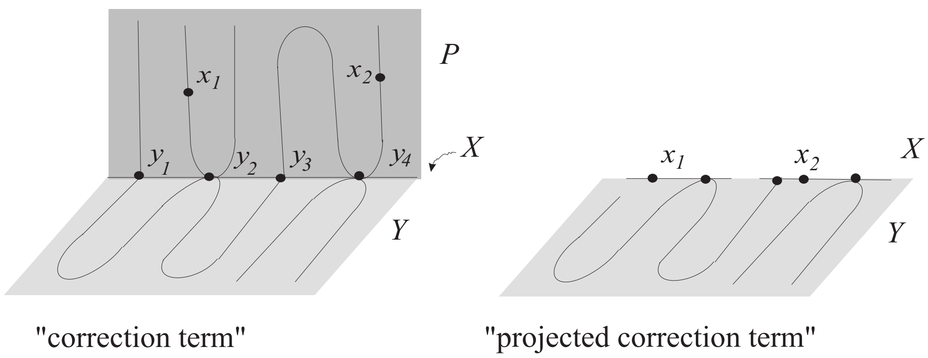
More precisely, intersecting $[M_n(Y,\beta)]^{\mathrm{virt}}$ with $\prod_{k=1}^{n} \prod_{j=0}^{\alpha_k-1} (ev_k^* X_\infty + j\psi_k)$ we obtain

$$\sum_{\Gamma_1,\Gamma_2} m(\Gamma_1\Gamma_2) \cdot [M_{\Gamma_1}^X(Y)]^{\mathrm{virt}} \boxtimes \left(\prod_{k=1}^{n} \prod_{j=0}^{\alpha_k-1} (ev_k^* X_\infty + j\psi_k) \cdot [M_{\Gamma_2}^X(P)]^{\mathrm{virt}} \right). \tag{1}$$

As a remark we see that both spaces $M_{\Gamma_1}^X(Y)$ and $M_{\Gamma_2}^X(P)$ can describe disconnected curves that we will view as products of irreducible curves (which is a non-trivial property proved in [4], Proposition 5.2.8).

Let us now consider the morphism $p : M_{\Gamma_2}^X(P) \to M_\Gamma(X)$ that projects the curves in P to X, where by Γ we denote the combinatorial data determined by $M_{\Gamma_2}^X(P)$. The projection obviously forgets fibers in P with no marked points. By Proposition 2.1.3 we get a non-zero result from such fibers of degree 1 while, by Corollary 2.1.4, p maps all multiple covers that have no marked points x_i to zero. In case of irreducible curves in $M_{\Gamma_2}^X(P)$ that have marked points x_i we distinguish two cases. If the considered irreducible component, say, $C_1^{(0)}$ has exactly one point x_i for some $1 \leq i \leq n$ and homology class $\widetilde{\beta}_1 \in H_2(P)$ that is a degree-d multiple cover of a fiber we apply Corollary 2.1.4 that gives (in terms of Definition 2.2.1) an external component intersecting X with multiplicity d. Note that by the same Corollary 2.1.4 the contribution of this invariant comes multiplied with a coefficient of 1. In all other cases, there is a well defined projection map that allows us to apply

Proposition 2.1.3. This gives a sum of connected curves with several internal irreducible components and several external irreducible components glued to them in such a way that the resulting curve has genus zero (i.e., without loops). We see that we can obtain such terms with more internal components only when any two of them are glued to the same external component that is also glued to at least one curve in $M^X_{\Gamma_2}(P)$ that is a fiber without marks. Indeed, looking at reducible maps in $M^X_{\Gamma_2}(P)$ of class $\widetilde{\beta}$ we see that either all marks lay on a unique internal component, either $\widetilde{\beta}$ contains degree-1 fibers. Let us assume for a contradiction that we have r components $C_1^{(0)}, \dots, C_r^{(0)}$ of classes $\widetilde{\beta}_1, \dots, \widetilde{\beta}_r \in H_2(P)$, each with at least one mark on it and $r > 1$. As the glued map has to be connected, we observe that the cancellation of one factor determines the cancellation of the whole term. But each $C^{(i)}$ that is not a multiple cover with one mark inquires a minimum number of $\widetilde{\beta}_i \cdot X_\infty + 1$ conditions, which leads to a total number $N := \sum_{i=1}^r \widetilde{\beta}_i \cdot X_\infty + r = \widetilde{\beta} \cdot X_\infty + r$ that obviously exceeds our number of conditions.



Actually we have a more precise statement. Let $C_1^{(0)}$ be an irreducible component of class $\widetilde{\beta}^{(0)}$ that projects down to a curve in X of class $\beta^{(0)} \neq 0$ and let us denote by $\sum \alpha_1^{(0)}$ the number of conditions corresponding to the marks lying on $C_1^{(0)}$. By Proposition 2.1.3 we see that the projected curve will have an internal component of class $\beta^{(0)}$ intersected with a number of $p := \sum \alpha_1^{(0)} - \widetilde{\beta}^{(0)} \cdot X_\infty - 1$ conditions corresponding to some $|\alpha_1^{(0)}|$-tuple $\alpha' := (\alpha_1, \dots, \alpha'_{i_0}, 0, \dots, 0)$. Let us now consider an additional curve with one component that is $C_1^{(0)}$ and a number of p degree-1 fibers each with one marked point on it with an assigned multiplicity at X_∞ of 1, in the $M^X_{\Gamma_2}(P)$ part and with corresponding curves in Y as prescribed in 2.1.1. If we denote the class of the projection of this curve by β_1, one can easily see that $X \cdot \beta_1 + 1 = \sum \alpha_1^{(0)}$.

From this we have that $X \cdot \beta^{(0)} = (\sum \alpha_1^{(0)} - p - 1) - \sum m_i$, or in the notations of 2.1.3, $X \cdot \beta^{(0)} = \sum \alpha' - \sum m_i$.

As it can be easily seen our current result differs slightly from the claim of the theorem. To see that the two statements agree we proceed by induction on the number n of marked points x_j.

For $n = 1$ we notice that there is always one internal component and there are no marked points on the external components. See [4], Remark 5.3.6 for a detailed proof.

For a general n, fixed p and $k > p$ and using the notation of 2.2.1 we have that any external component of $D_{\alpha',i}$ has a smaller number of marks x_j and hence we can apply the induction hypothesis to get

$$\prod_{j=0}^{\alpha_k-1} (ev_k^* X + j\psi_k) \cdot [D_{(\alpha_1,\dots,\alpha_i',0,\dots,0),i}(Y,\beta)]^{\mathrm{virt}} =$$
$$= [M_{|\alpha^{(0)}|+t}(X,\beta^{(0)})]^{\mathrm{virt}} \times_X [M_{(\alpha^{(1)}+\alpha_k)\cup(m^{(1)})}]^{\mathrm{virt}}$$
$$\times_{X^{t-1}} \prod_{i=2}^{t} [M_{\alpha^{(i)}\cup(m^{(i)})}(X,\beta^{(i)})]^{\mathrm{virt}} +$$
$$+ [M_{|\alpha^{(0)}|+t}(X,\beta^{(0)})]^{\mathrm{virt}} \times_X (D) \times_{X^{t-1}} \prod_{i=2}^{t} [M_{\alpha^{(i)}\cup(m^{(i)})}(X,\beta^{(i)})]^{\mathrm{virt}},$$

where we denoted by D the additional term of our statement that corresponds to $[M_{(\alpha^{(1)}+\alpha_k)\cup(m^{(1)})}]^{\mathrm{virt}}$. With this remark one could see that pushing 1 forward by p one gets precisely the terms in our claim. This completes the proof. □

Remark 2.2.4. It is proved in [4] that for a very ample hypersurface X we have the relation

$$(ev^* X + \alpha_k \psi_k) \cdot [M_\alpha(Y,\beta)]^{\mathrm{virt}} = [M_{\alpha+e_k}(Y,\beta)]^{\mathrm{virt}} + [D_{\alpha,k}(Y,\beta)]^{\mathrm{virt}} \tag{2}$$

for all $1 \le k \le n$. It is easily seen that repeated applications of 2 give precisely the statement of our main theorem. However, the proof of the converse is not so easy. We cannot deduce this formula directly from our theorem, as we must a-priori know that $[D_{\alpha,k}(Y,\beta)]^{\mathrm{virt}}$ have the right dimension. We will not insist as we do not need the result in this form and since the final results agree we may still think the main theorem in this form when convenient.

Remark 2.2.5. In our attempt of computing Gromov-Witten invariants of X in terms of Gromov-Witten invariants of Y, we will apply Theorem 2.2.3 with suitable α. The strategy is now obvious. We impose multiplicities $\alpha_1, \dots, \alpha_n$ at the marked points at X with α such that $\sum \alpha = X \cdot \beta + 1$. Then the moduli space $M_\alpha^X(Y,\beta)$ we obtain on the right hand side will be empty and we will be left with an invariant on Y on the left hand side, and with contributions from various $D_{\alpha',k}(Y,\beta)$ on the right hand side. It was proved in [4] (using the version of the main theorem of Section 2) that the terms coming from $D_{\alpha',k}(Y,\beta)$ are as follows: one term $M_{|\alpha|+t}(X,\beta^{(0)})$ with $i_*\beta^{(0)} = \beta$ and invariants on X and Y with smaller β or smaller number of points, where by smaller β we will mean smaller $d := i_*\beta$ (i is the inclusion of Y in $\mathbb{P}^r$). This will allow the computation of Gromov-Witten invariants of X in terms of invariants on Y and recursively known invariants on X. It can be easily seen that the same arguments are valid in our case, too. In fact, this is the reason why we preferred to state our main theorem without introducing new spaces with more internal components. We conclude that the main theorem 2.2.3 allows the computation of Gromov-Witten invariants of X supposing we know the invariants of Y. As before, on the right hand side we have a sum of terms one of which being the space we want $[M_{|\alpha|}(X,\beta^{(0)})]^{\mathrm{virt}}$. We will call *correction terms* all other terms is the sum.

3. COMPUTING GROMOV-WITTEN INVARIANTS

In this section we will try to apply the main theorem recursively to reconstruct the invariants of an s-codimensional subvariety X of a variety Z. In order to do this we need to overcome the following difficulty: not all the classes in $H^*(X)$ are pullbacks of classes in $H^*(Z)$, which essentially means that the main theorem does not provide *all* the unrestricted invariants we need. We will see that even if we wish to compute an invariant with all the cohomology classes being pullbacks of cohomolohy classes in Y, some new invariants will appear from the algorithm, having any number of such "unwanted" classes.

*From now on, we will consider any codimension-*1 *subvariety X of a variety Y respects the hypothesis of the main theorem. We will denote by i all the inclusion maps that will appear between different varieties.*

3.1. AN ALGORITHM

Definition 3.1.1. Let X be a projective variety. A subring $R \subseteq H^*(X)$ is called self-dual if the restriction of the cohomological Poincaré pairing to R is nondegenerate.

Construction 3.1.2. Let Z be a variety of complex dimension r, X a subvariety of Z and $i : X \hookrightarrow Z$ the inclusion morphism between them together with its pullback morphism induced in the cohomology rings $T = H^*Z$, $S = H^*X$

$$i^* : T \to S.$$

If $R = i^*T$, the image of T by i^* is self-dual in S, then we denote by $R^\perp$ the the orthogonal complement of R in S, with respect to the Poincaré pairing. Moreover, if $i_* : S \to T$ is the pushforward morphism, then we have $i_*\gamma = 0$, $\forall \gamma \in R^\perp$. Indeed, let $\gamma \in R^\perp \cap S^k$, where S^k is the graded part of k-codimensional cocycles in Z. If $\{b_1, \dots b_n\}$ is a basis of T^{r-k}, then

$$b_i \cdot (i_*(\gamma)) = i_*(i^*(b_i) \cdot \gamma) = 0, \quad \forall i = 1, \dots, n.$$

If $i_*(\gamma)$ is non-zero, then the intersection product on T would be degenerate, which contradicts the Poincaré duality theorem (see [6]).

Setting 3.1.3. Let (X, Y, Z) a triple of varieties with the following properties:

(i) $X \hookrightarrow Y \hookrightarrow Z \hookrightarrow \mathbb{P}^N$ with $\dim Z = \dim Y + 1 = \dim X + 2$ and X in Y respects the conditions of the main theorem;

(ii) $i^*H^*(Z)$ is self-dual in $H^*(X)$;

(iii) i_* induces an isomorphism between $H_2(Y)$ and $H_2(Z)$.

Let us now explain the terminology and notation. By *degree-d curves* in Z (and analogously in X or Y) we will mean curves in Z of class δ with δ such that $i_*\delta = dH^{N-1}$, where this time i is the inclusion of Z in $\mathbb{P}^N$ and H the class of a hyperplane in $\mathbb{P}^N$. Sometimes we will write this as $i_*\delta = d$ for short.

Remark 3.1.4. Before stating our main technical result, let us take a closer look at condition (iii) in the above setting. By Poincaré duality we have $H^2(X) \simeq H^2(Y)$ (as $\mathbb{Q}$-vector spaces). However, not any divisor on X is a pull-back of a divisor on Y. Let for example $X := \widetilde{\mathbb{P}}^2$ be $\mathbb{P}^2$ blown up in one point included in $Y := \mathbb{P}^2 \times \mathbb{P}^1$. Let us assume for a contradiction that E, the exceptional divisor on X is the pull-back of some divisor on Y. Then by the projection formula we have $i_*(i^*D \cdot E) = D \cdot i_*E$. But $E \cdot E = -1$ in $H^*(X)$ while $H^*(Y) \simeq \frac{K[s,t]}{(s^3,t^2)}$, and one can easily see that there are no negative intersection products of effective cycles.

For a converse statement we observe that in case $i^* : H^*(Y) \to H^*(X)$ is an isomorphism and X defines an ample line bundle generated by global sections on Y, any ample divisor on X is the complete intersection of X with some divisor on Y. This follows immediately from the Kodaira vanishing theorem and the exact sequence

$$0 \to H^0(X, i^*\mathcal{O}_Y(-X)) \to H^0(X, i^*\mathcal{O}_Y) \to H^0(X, \mathcal{O}_X) \to H^1(X, i^*\mathcal{O}_Y(-X)).$$

PROPOSITION 3.1.5. *Let X, Y, Z as in Setting 3.1.3 and $\alpha = (\alpha_1, \dots, \alpha_n)$ an n-tuple of positive integers such that $\alpha_1 > 0$.*

(i) *Assume $\widetilde{\gamma}_1$ is a cycle in the orthogonal complement $H^*(Z)^\perp$ of $i^*H^*(Z)$ in $H^*(X)$. If $X \in i^*H^2(Z)$ then the relative $\langle \tau_{k_1}^{\alpha_1}(\widetilde{\gamma}_1)\tau_{k_2}^{\alpha_2}(\gamma_2) \cdots \tau_{k_n}^{\alpha_n}(\gamma_n)\rangle_\delta^Y$ and* *the absolute* $\langle\tau_{k_1}(\widetilde{\gamma}_1)\tau_{k_2}(\gamma_2')\cdots\tau_{k_m}(\gamma_m')\rangle_{\delta_1}^X$ *Gromov-Witten invariants vanish for any* $\delta,\delta_1\in H_2(Z)$ *and any* $\gamma_2,\ldots,\gamma_n,\gamma_2',\ldots,\gamma_n'\in H^*(Z)$.

(ii) *If* $i^*H(Z)$ *is self-dual in* $H^*(Y)$ *and* $\widetilde{\gamma}_1$ *is a class in the orthogonal complement* $H^*(Z)^\perp$ *of* $i^*H^*(Z)$ *in* $H^*(Y)$, *then with the above notation the relative* $\langle\tau_{k_1}^{\alpha_1}(\widetilde{\gamma}_1)\tau_{k_2}^{\alpha_2}(\gamma_2)\cdots\tau_{k_n}^{\alpha_n}(\gamma_n)\rangle_\delta^Z$ *and the absolute* $\langle\tau_{k_1}(\widetilde{\gamma}_1)\tau_{k_2}(\gamma_2')\cdots\tau_{k_m}(\gamma_m')\rangle_{\delta_1}^Y$ *Gromov-Witten invariants vanish.*

Proof. We prove the first statement while the second part will follow by exactly the same arguments. The idea is to make induction on $d:=i_*\delta$ and n in that order for the relative invariant and on $d_1:=i_*\delta_1$ for the absolute invariant. More precisely, we assume the statement to be true for all relative invariants having

- smaller d or,
- the same d and smaller n,

and at the same time for all absolute invariants having

- smaller $d_1<d$.

For $d=1$, $n=1$, we compute the invariant by applying the main theorem twice. We start with the moduli space $M_1(Z,\delta)$ of 1-pointed curves of class δ in Z with $i_*\delta=1$, and increase the multiplicity s at Y until the space $M_s^Y(Z,\delta)$ becomes empty. Applying the theorem for $M_1(Y,\delta)$ as before, we obtain the class of the space $M_{\alpha_1}^X(Y,\delta)$ without correction terms as

$$\begin{aligned}(3)\qquad [M_{\alpha_1}^X(Y,\delta)]^{\mathrm{virt}} &= \prod_{j=0}^{\alpha_1}(\widetilde{ev}_1^*X+j\psi)[M_1(Y,\delta)]^{\mathrm{virt}}=\\ &=\prod_{j=0}^{\alpha_1}\prod_{i=0}^{s_1}(\widetilde{ev}_1^*X+j\psi)(ev_1^*Y+i\psi)[M_1(Z,\delta)]^{\mathrm{virt}},\end{aligned}$$

where we have denoted by $\widetilde{ev}$ the evaluation at Y. Intersecting with evaluations at all the γ's and psi-classes, we remark that all the evaluation maps at Y occurring in (3), except the first one, can by replaced by evaluations at Z as we are left only with evaluations $\widetilde{ev}^*X^k=ev^*\gamma$, for some $\gamma\in H^*(Z)$ and any k. What we have to do now is to intersect the whole equation with $ev_X^*\widetilde{\gamma}_1$ and the right hand side is zero as it contains

$$(4)\qquad ev^*Y\widetilde{ev}^*Xev_X^*\widetilde{\gamma}_1=ev^*Y\widetilde{ev}^*(i_*\widetilde{\gamma}_1)=ev^*(i_*\widetilde{\gamma}_1).$$

For $d_1=0$ we use

$$(5)\qquad [M_n(X,0)]^{\mathrm{virt}}=ev_1^*X[M_n(Z,0)]^{\mathrm{virt}}.$$

On the right hand side we have $ev_X^*\widetilde{\gamma}ev_1^*X=ev_1^*(i_*\widetilde{\gamma})$, and this vanishes as before.

For the general statement we proceed as before with the only difference that the correction terms will be nontrivial. Supposing we want to compute the invariant for the given α, the main theorem yields

$$\prod_{i=0}^{n}\prod_{j=0}^{\alpha_i-1}(\widetilde{ev}_i^*X+j\psi)[M_n(Y,\delta)]^{\mathrm{virt}}=[M_\alpha^X(Y,\delta)]^{\mathrm{virt}}+(\textit{correction } 1).$$

Consider first the case where the point x_1 lies in the internal component of the correction term, that is $M_r(X,\delta_1)$. Note that the left hand side will vanish as before by (4). The external components of this correction term have either smaller d, either smaller n. Using the induction hypothesis the whole invariant will be zero whenever the diagonals will contribute a class from $H^*(Z)^\perp$. Therefore the diagonals will not contribute classes $H^*(Z)^\perp$ to the internal component either, and what is left to prove is that the invariant on X with one class from $H^*(Z)^\perp$ vanishes.

If the point x_1 lies on one external component say C_1, then the only chance to have a non zero invariant is to have a class from $H^*(\mathbb{P}^r)^\perp$ at the diagonal. This turns again to the invariant on X with one class from $H^*(Z)^\perp$.

Let us now prove the statement for the invariants of X. But again by the main theorem, an invariant on $[M_r(X,\delta_1)]$ can be written as an invariant of Y minus a product of invariants coming from a correction term

$$(6)\qquad \prod_{j=0}^{s_2-1}(\widetilde{ev}_1^*X+j\psi)[M_r(Y,\delta_1)]^{\mathrm{virt}}=[M_r(X,\delta_1)]^{\mathrm{virt}}+(\textit{correction } 2).$$

Intersecting (6) with $\widetilde{\gamma}_1,\gamma_2,\ldots,\gamma_n$ and psi-classes the left hand side of equation (6) is zero by equation (4), while the correction term of (6) has x_1 on the internal component and external components in degree at most d_1. Hence, the invariant on X form (*correction* 2) will have exactly one class in $H^*(Z)^\perp$ and since its degree is $d_2<d_1$, it vanishes by induction. □

COROLLARY 3.1.6. *Assume* (i) *and* (ii) *in Proposition* 3.1.5. *Let s_Y the minimum intersection product of a curve in Z with Y and s_X the minimum intersection product of a curve in Y with X. Then the* (*absolute*) *Gromov-Witten invariants of X with at most $s:=\min(s_Y,s_X)$ evaluation classes from X can be computed recursively from the invariants of Z.*

Proof (Compare to [4], Proposition 2.5.9). We follow the arguments in the proof of Proposition 3.1.5. The only thing we need to remark is that there will be no "unwanted" classes from the diagonal. Let us examine again the equations, considering we have evaluations at X at the first s points, $m_1=Y\cdot\delta+2-s$, $m_2=X\cdot\beta+2-s$, where β is the class of a degree-δ curve in Y, i.e., $i_*\beta=\delta$. In both cases we have $m_1,m_2\geq 0$, by our assumption. So,

$$(7)\qquad ev_2^*Y\cdots ev_s^*Y\prod_{j=0}^{m_1}ev_1^*(Y+j\psi)=[M_r(Z,\delta_1)]^{\mathrm{virt}}$$

$$\widetilde{ev}_2^* X \cdots \widetilde{ev}_s^* X \prod_{j=0}^{m_2} (\widetilde{ev}_1^* X + j\psi)[M_r(Y,\delta)]^{\mathrm{virt}} = \tag{8}$$
$$= [M_r(X,\delta)]^{\mathrm{virt}} + (\textit{correction } 2).$$

Equation (8) is obtained from (7) by intersecting it with evaluations at X. This is allowed, as in (*correction* 1) all the first s points are imposed to lie in Y. In the same way, in (*correction* 2) the first s marks belong to X and it makes sense to intersect (8) with evaluations at X. Now, putting (7) and (8) together, the left hand side will be known by the same (4). In both (*correction* 1) and (*correction* 2) there will be no class from $H^*(Z)^\perp$ by Proposition 3.1.5 and therefore all the invariants will be known recursively. Indeed, all the external components have at most s classes from X because we must have at least one point from the first s on the internal component and this leaves us with a number of $s-1$ points plus one from the diagonal. The invariant from the internal component cannot have more than s evaluations at classes from $H^*(Z)^\perp$ either. The case where these additional classes could arise could only be from the diagonal with an external component having none of the first s marks, but then it vanishes.
This completes the proof. □

Remark 3.1.7. By [11] the condition $i^*H^*(Z)$ is self-dual in H^*Y is satisfied if Y is a very ample hypersurface of Z.

Definition 3.1.8. Let X, Z satisfy condition (ii) of Setting 3.1.3. We say that a sequence of varieties $Y_1, \dots Y_n$ such that $X \hookrightarrow Y_1 \cdots Y_n \hookrightarrow Z$ with each variety having codimension 1 in the one sitting after it in the above row, has the property ($\star$) if $H_2(Y_1) = \cdots = H_2(Y_n) = H_2(Z)$, $i^*H^*(Z)$ is self-dual in $H^*(Y_i)$ and $Y_i \in i^*H^*(Z) \subseteq H^*(Y_{i+1})$, for any $i = 1, \dots, n$.

Corollary 3.1.9. *Let a sequence of varieties $Y_1, \dots, Y_n$ with ($\star$), and Y_i satisfy* (i) *and* (ii) *of Proposition* 3.1.5 *for every $i = 1, \dots, n$. If s_X, s_{Y_i} are defined as before, then the invariants of X with at most $s := \min(s_X, s_{Y_1}, \dots, s_{Y_n})$ classes in $H^*(Z)^\perp$ can be computed recursively from those of Z.*

Proof. The proof is analogous to the proofs of Proposition 3.1.5 and Corollary 3.1.6. However, we cannot deduce the statement for Y_j from that of Y_{j+1}, because we did not impose the condition $i^*H^*(Y_{j+1})$ self-dual in $H^*(Y_j)$. □

Remark 3.1.10. As we did not impose $H^2(X)$ to be $i^*H^*(Z)$ the invariant that we compute will actually be a sum of invariants of X. More precisely, $M(X,\delta) = \sum_{i_*\beta=\delta} M(X,\beta)$. In a number of cases the individual invariant will turn out to be a known invariant, and it will be analyzed explicitly in next section.

3.1.1. **Reconstructing Gromov-Witten invariants from those of $\mathbb{P}^r$.** The most natural question when reconstructing Gromov-Witten invariants of a variety X from a bigger variety is "When can we reconstruct the invariants of X from those of $\mathbb{P}^r$?". We will see that $\mathbb{P}^r$ behaves nicely and has the self-duality property for any subvariety, but the condition ($\star$) on the intermediary varieties Y_i will turn to be strong enough to allow only computations of invariants of complete intersections.

LEMMA 3.1.11. *Let $X \hookrightarrow \mathbb{P}^r$ be any s-codimensional variety in the projective space and consider the ring $R = i^*H^*(\mathbb{P}^r) \subseteq H^*(X)$ induced in cohomology by the pullback of the inclusion map. Then R is self-dual and it has exactly one generator in each* (*even*) *dimension.*

Proof. Let $\gamma \in R$ a cocycle. Then there exists a cocycle in $H^*(\mathbb{P}^r)$ such that γ is its pullback. Let us assume for simplicity that $\gamma = i^*(H^k)$ for some $0 < 2k \leq 2(r-s)$, where $H \in H^2(\mathbb{P}^r)$ is the class of a hyperplane in $\mathbb{P}^r$. We have by the definition of the intersection product that $i_*(i^*(H^{r-s-k})) = H^{r-s-k}X$ is a non-zero cocycle in $H^*(\mathbb{P}^r)$, which means that $i^*(H^{r-s-k})$ has to be nonzero in $H^*(X)$. Intersecting γ with $i^*(H^{r-s-k})$ and pushing forward we get again a non-zero cocycle by the formula

$$i_*(i^*(H^k) \cdot i^*(H^{r-s-k})) = H^k i_*(i^*(H^{r-s-k})) = H^k \cdot H^{r-s-k}X.$$

This proves the first part of the statement. The second is just as easy: $i^*(H^k)$ is a non-zero cocycle in $H^k(X)$, for any $0 \leq 2k \leq 2(r-s)$. $\square$

Remark 3.1.12. Condition ($\star$) reads $H^2(Y_1) = \cdots = H^2(Y_n) = H^2(\mathbb{P}^r) = \mathbb{Q}$. This implies that the class of Y_i on Y_{i+1} will always be a pull-back of a class of a divisor of $\mathbb{P}^r$ and the same for X in Y_1. By Corollary 3.1.9 we will be able to compute the invariants of any complete intersection of dimension at least 2. Conversely, any Y_1 in $\mathbb{P}^r$ for which we can find a sequence with ($\star$) is a complete intersection (see [8]).

Remark 3.1.13. For complete intersections there is a very short proof using the Lefschetz Hyperplane Theorem. Let us argue again in codimension 2, as the general proof will work exactly the same. The only "new" cohomology that we can encounter is in the *middle* dimension, but it is immediate from [4], Lemma 2.5.5 that we cannot have non-zero invariants with exactly one such class in (*correction* 2). The $m := middle + 1$-cohomology will not be a problem either, because $H^m(X) = \mathbb{Q}$ by Poincaré duality, and by Lemma 3.1.11, $H^m(X) = i^*H^m(\mathbb{P}^r)$. The drawback of this proof in Chow groups is that there is no analogous Lefschetz theorem (unless we use identification theorems between Chow groups and cohomology); $A^1(X) \simeq A^1(Y)$, when X is a hyperplane section of Y, is a theorem of S. Lefschetz and A. Grothendieck, but no analogous statement for Chow groups has been proved yet in full generality.

3.2. THE GENERAL CASE

So far we have only considered the case $H^2(Y) = H^2(Z)$. In general, things get quickly complicated: we will need to compute individual invariants and moreover, we will have "unwanted" evaluation classes in the invariant coming from the correction term. In the end, both situations will lead to the same difficulty, that we will not overcome completely, even under stronger conditions. However, we will try along this section to compute the invariants in low degree. Some cases of varieties X, for which some of the Gromov-Witten invariants can be reconstructed from those of Z, will be analyzed further, though no closed-form result can be stated.

Proposition 3.2.1. *Let a sequence of varieties $Y := Y_0 \hookrightarrow Y_1 \cdots Y_n \hookrightarrow Z$ such that the sequence $Y_1, \ldots, Y_n$ have property $(\star)$ and the class of Y in Y_1 be the pull-back of a divisor class in Z. Suppose $X \hookrightarrow Y$ is the intersection of Y with a hypersurface in Z. Then the δ-invariants of X can be computed for any degree d curves δ in $H^2(Z)$ such that $d \leq s_X$.*

Proof. By Corollary 3.1.9 the invariants of Y can be computed and using the hypothesis $d \leq s_X$ no more than s_X classes from $H^*(Z)^\perp$ will appear in *correction* 2 from diagonals. Let us now prove that the individual invariants will do no harm in this case. For a fixed $\delta \in H^2(Z)$, let $\{\beta^1, \ldots, \beta^m\}$ the set of all homology classes in $H_2(Y)$ such that $i_*(\beta^i) = \delta$, for all $i \in \{1, \ldots, m\}$. By our assumption we can write $X = i^*(\gamma)$, for some $\gamma \in H^{2r-2}(Z)$ and by the projection formula we have

$$i_*(i^*(\gamma) \cdot \beta) = \gamma \cdot i_*\beta = \gamma \cdot \delta.$$

This means that the curves of class β^i intersect X with the same multiplicity m, independent of i. By the adjunction formula, K_Y is a pull-back of a divisor in Z, which yields $K_Y \cdot \beta$ is independent of i. This means that the dimension of $M_{|\alpha|}(Y, \beta^i)$ is also independent of i, hence for a fixed α we have $\dim[M^X_\alpha(Y, \beta^i)]^{\mathrm{virt}} = \dim[M_{|\alpha|}(Y, \beta^i)]^{\mathrm{virt}} + |\alpha| - \sum \alpha$, which is independent of i, too.

Supposing we want to compute an n-point invariant the next step is to apply as usual the main theorem, starting with the moduli space of n-pointed class-δ curves in Z, $M_n(Z, \delta)$. Thus,

$$\prod_i \prod_j (ev_i^* Y + j\psi)[M_n(\mathbb{P}^r, d)]^{\mathrm{virt}} = \sum_{i_*(\beta)=d} [M_n(Y, \beta)]^{\mathrm{virt}} + (\mathit{correction}) \tag{9}$$

$$\prod_i \prod_j (ev_i^* X + j\psi)[M_n(Y, \beta)]^{\mathrm{virt}} = [M_n(X, \beta)]^{\mathrm{virt}} + (\mathit{correction}\ \beta) \tag{10}$$

Let us now look at *correction* β, having in mind that we have a decomposition $\beta = \beta^{(0)} + \beta^{(1)} + \cdots + \beta^{(t)}$, where $\beta^{(0)}$ is the class of the internal component and $\beta^{(1)}, \ldots, \beta^{(t)}$ those of the external ones. Each $\beta^{(i)}$ comes equipped with a $\delta^{(i)}$, that is the degree of its push-forward in Z, as before. Observing that in (9) we have a sum over all β such that $i_*\beta = \delta$, and for each β we have in (*correction* β) all the possible decompositions, we can conclude that the final correction term that we get from (9) and (10) is a sum of products of invariants of class $\delta^{(i)}$. This is because for a given tuple $(\delta^{(0)}, \ldots, \delta^{(t)})$ we are allowed to take the same r-tuple $M = (m^{(1)}, \ldots, m^{(t)})$, and therefore the factor $\frac{m^{(1)} \cdots m^{(1)}}{t!}$ that appears in every (*correction* β) will be the same for all β corresponding to $(\delta^{(0)}, \ldots, \delta^{(t)})$. By the first part of the proof, in the invariant coming from any of the components of the correction term, there will be classes from the diagonal, independent of $\beta^{(i)}$ for the given $\delta^{(i)}$. To conclude, the δ invariant can be computed. $\square$

LEMMA 3.2.2. *Let Y be a variety with $H_2(Y) \neq H_2(Z)$. Then for any $\beta \in H_2(Y)$ the individual β-invariants can be reconstructed from the δ-invariants of Y.*

Proof. Let $\{l_1, \ldots, l_n\}$ a basis of $H_2(Y)$ and consider the set $\{\beta_1, \ldots \beta_m\}$, of all the distinct homology classes of Y such that $i_*\beta_i = d$, for a given $\delta \in H_2(Z)$. As these sets are finite, it makes sense to consider the number $t(\delta) := [\frac{m}{n-1}]$.

Now, to compute the individual invariants, we proceed as follows: add t points and impose them to belong to divisor classes, that we denote $D_1, \ldots, D_t$. In the equation we obtain, the left hand side will be the d-invariant, and on the right hand side we will have a sum of invariants with various β, namely,

$$
\begin{aligned}
(11) \qquad & \langle D_1 \cdots D_t \tau_{k_1}(\gamma_1)\tau_{k_2}(\gamma_2) \cdots \tau_{k_n}(\gamma_n)\rangle_\delta^Y = \\
&= \sum_{i_*\beta=\delta} D_1 \cdot \beta \cdots D_t \cdot \beta \langle \tau_{k_1}(\gamma_1)\tau_{k_2}(\gamma_2) \cdots \tau_{k_n}(\gamma_n)\rangle_\beta^Y .
\end{aligned}
$$

Finding divisors that each cancels a number of $n-1$ invariants in the sum, we will be left for the last divisor D_t with at most n terms in our equation. These terms cannot be pairwise linearly dependent, since the push-forward of any β_i is imposed to have degree d. This shows that D_t can be chosen to cancel all the terms but the one we wish. $\square$

3.2.1. **A few reconstruction cases.** We are now ready to compute Gromov-Witten invariants of a variety $X \hookrightarrow Y \hookrightarrow Z$ from the invariants of Z, provided of course that the invariants of Y can be reconstructed from those of Z as prescribed in Corollary 3.1.9 and Lemma 3.2.2. For this, we must take care of a few aspects. This time, the class of X on Y will no longer be a pullback of a cycle in $\mathbb{P}^r$ and, therefore, each evaluation at $\widetilde{ev}_i^* X$ will inquire an evaluation

ev_i^*Y. We will try here to reduce the number of evaluations at classes of X, that will appear from what we called "*correction* 2", but dropping the condition $H_2(Y) = H_2(Z)$, we will not be able to compute *all* the invariants of X, even if we consider more special cases. For simplicity, we will analyze in detail only the 1-point invariants of X.

Case 3.2.3. If $H^*(X)$ is generated by divisors, then we can apply the reconstruction theorem with a few decorations (see [11]). First, we must assure that the algorithm can also be applied for the β-invariants, β being this time a class of Y. Let us briefly recall how the proof works and, at the same time, notice the differences.

∘ Reduce all the descendant n-point invariants to primary or 2-point invariants, whenever $n \geq 3$, using the topological recursion relations. A δ-invariant $\delta \in H_2(X)$ of X will be on the left hand side with coefficient 1, and all we have to do is to sum up over all δ such that $i_*\delta = \beta$.

∘ Reduce all the primary n-point invariants with $n \geq 3$ to two point invariants by the WDVV equations. As before, we just have to sum up. We can formally write

$$\sum_{i_*\delta=\beta} \sum_{\delta_1+\delta_2=\delta} = \sum_{\substack{i_*\delta_1=\beta_1 \\ i_*\delta_2=\beta_2 \\ \beta_1+\beta_2=\beta}} .$$

∘ Compute the descendent 2-point invariants by adding a point and using the divisor equation. This will lead to primary invariants. The δ-invariant we want is multiplied with $D \cdot \delta$. We can compute the β-invariant by choosing the divisor D to be the hyperplane section.

∘ Compute the primary δ 2-point invariants from the δ 1-point invariants by adding a point and using again the WVDD equations. However, the β-invariants with 2 divisor classes, cannot be reduced to β-invariants with one point.

To conclude, we can only reduce to 2-point invariants.

Case 3.2.4. Another idea could be to assume $-K_X \geq 0$. All conditions coming from the diagonals will have to be divisor classes (see [4]). This means that it will be sufficient to compute the 1-point invariants.

Case 3.2.5. In full generality, we need to reconstruct the individual invariants. Let us assume we need an n-pointed invariant of class δ, and by this we mean an invariant with n classes in $H(Z)^{\perp}$. We say $\delta_1 \leq \delta$ if the degree of δ_1 is smaller than that one of δ. Now, if the finite set of positive integers $\{t(\delta_1) | \ \delta_1 \leq \delta\}$ is bounded by N, then we can compute the invariants having $N + n \leq \min(s_X, s_Y)$.

3.3. ENUMERATIVE APPLICATIONS

We will now apply our methods to compute Gromov-Witten invariants of 2-codimensional subvarieties of $\mathbb{P}^r$ by reducing them to invariants of $\mathbb{P}^r$ that we can compute. As mentioned before, the computations may take place both in cohomology and Chow groups. We will use Chow groups, as we will not be using any special results from cohomology theories; everything will work identically in cohomology.

In all the computations below, we use the computer program GROWI (see [5]) to evaluate the invariants of $\mathbb{P}^r$ or the invariants of hypersurfaces in $\mathbb{P}^r$ with all the evaluations at $\mathbb{P}^r$.

Example 3.3.1. Let X be the degree-9 threefold in $\mathbb{P}^5$, that is the complete intersection of two hypersurfaces Y_1, Y_2 in $\mathbb{P}^5$, each of degree 3. We have $K_X = 0$, and $\dim[M_{0,0}(X,d)]^{\mathrm{virt}} = 0$. Let us first count lines. To compute the invariants in degree 1 we start with the moduli space $M_1(\mathbb{P}^5, 1)$ of 1-pointed lines in $\mathbb{P}^5$ and raise the multiplicities in two steps, first using the main theorem for Y_1 in $\mathbb{P}^5$, and then for X in Y_1. Note that there can be no correction terms from reducible curves in any of our cases. Every external component has positive degree, which means that the internal component must be in degree 0, and that there is only one external component. But this cannot happen unless there are at least two marks on the contracted component. This shows that there will be no correction terms in our computation. The number of lines in X is now given by raising the multiplicity at the mark point from 0 to 4 that brings us to the number of lines in Y_1, and then from 0 to 4, to get $[M_1(X,1)]^{\mathrm{virt}}$. What we have to do next is to fix the mark and intersect with a hyperplane class. So,

$$\begin{aligned}
\langle H\rangle_1^X &= \deg\left(ev_X^* i^* H \cdot \prod_{i=0}^{3}(ev_{Y_1}^* X + i\psi)\cdot [M_1(Y_1,1)]^{\mathrm{virt}}\right) = \\
&= \deg\left(ev^* H \prod_{i=0}^{3}(ev^* Y_2 + i\psi)\cdot\left(\prod_{i=0}^{3}(ev^* Y_1 + i\psi)\cdot [M_1(\mathbb{P}^5,1)]^{\mathrm{virt}}\right)\right) = \\
&= \deg\left(ev^* H \prod_{i=0}^{3}(ev^* 3H + i\psi)\cdot \prod_{i=0}^{3}(ev^* 3H + i\psi)\cdot [M_1(\mathbb{P}^5,1)]^{\mathrm{virt}}\right) = \\
&= 1053.
\end{aligned}$$

Let us now turn to conics. We start with the 15 dimensional moduli space $M_1(\mathbb{P}^5, 2)$. To reach the space $M_1(Y_1, 2)$, we have to increase the multiplicity from 0 to 7. As before, we impose multiplicities at X until $M_1^X(Y,2)$ becomes empty or, more precisely, as $\deg(X \cdot i^*(2H)) = 6$, we have to raise the multiplicity at the marked point x_1 from 0 to 7. First, we decide which correction terms will appear. In our first step we may have correction terms with one or two external components of degree one, i.e. a moduli space $M_m^{Y_1}(\mathbb{P}^5, 1)$. The class $[M_1(\mathbb{P}^5, 1)]^{\text{virt}}$ has dimension 9, and we have that $H^6 = 0$ in $\mathbb{P}^5$. This means that m must be at least 4. But if we have a degree 0 internal component, it means that the multiplicity at Y_1 of this component must be at least 9, and if we have a degree-1 internal component, the multiplicity should again be at least 8. This shows us that we will have no correction term from this first step.

Increasing the multiplicity at X on Y_1, we will have as before external components $M_m^X(Y_1, 1)$ of virtual dimension $-K_{Y_1} \cdot i^*H + (4-3) + 1 - m = 3 + 1 + 1 - m = 5 - m$. This shows that we cannot have $m = 1$, because this will mean to have 4 conditions from the diagonal, which is impossible, X being of dimension 3. Thus we can only have correction terms in the last three steps. As proved in our algorithm, these correction terms will be known recursively. Indeed, the degree-0 internal component will be computable as an intersection product on X, the degree-1 internal component will be reduced using the divisors, and the fundamental class axioms to the degree-1 Gromov-Witten invariant, while the external components will be 1-pointed lines that can be obtained without correction terms. The left hand side of the main theorem leads to

$$\deg\left(ev^*H \cdot \prod_{i=0}^{6}(ev^*Y_2 + i\psi) \cdot \prod_{i=0}^{6}(ev^*Y_1 + i\psi)[M_1(\mathbb{P}^5, 2)]^{\text{virt}}\right) = \frac{1916541}{4}.$$

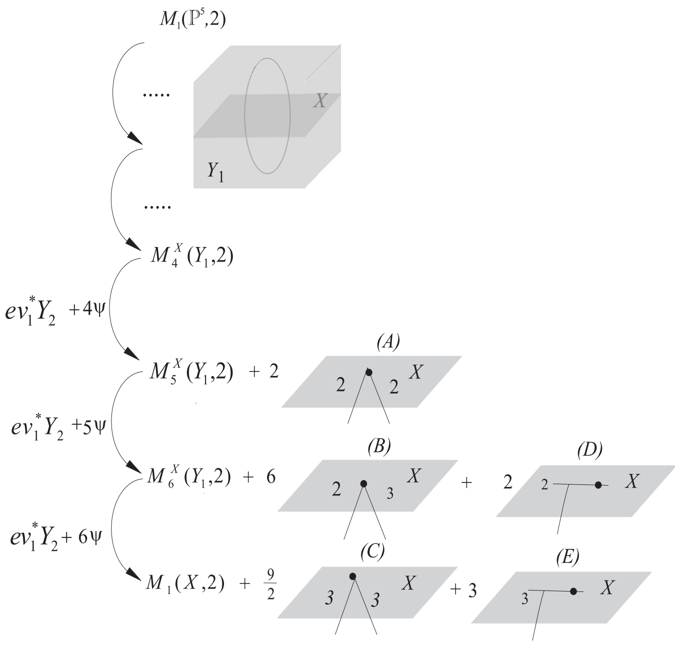
Let us evaluate the correction term (A). By Proposition 3.1.5, we will have no classes from $H^*(\mathbb{P}^5)^{\perp}$. The contribution from the external invariants is $\langle\tau^2(H^3)\rangle_1^{Y_1} \cdot \langle\tau^2(H^3)\rangle_1^{Y_1}$. Analyzing one more closely we have

$$\begin{aligned}\langle\tau^2(H^3)\rangle_1^{Y_1} &= \deg(ev_1^*H^3 ev_1^*Y_2(ev_1^*Y_2 + \psi)[M_1(Y_1, 1)]^{\text{virt}}) = \\ &= 9 \cdot \langle H^5\rangle_1^{Y_1} + 3 \cdot \langle H^4\psi\rangle_1^{Y_1} = 54.\end{aligned}$$

The internal component is a moduli space $M_3(X, 0)$ that receives $\frac{1}{9}$ times the fundamental class of X from the diagonal at both node points. Therefore the invariant given by the internal component of this correction term is

$$\begin{aligned}I &= \frac{4}{2} \cdot \frac{1}{9^2} \cdot \deg(ev_1^*H \cdot (ev_1^*Y_2 + 5\psi) \cdot (ev_1^*Y_2 + 6\psi)[M_3(X, 0)]^{\text{virt}}) = \\ &= \frac{4}{2} \cdot \frac{1}{9^2} \cdot \deg(ev_1^*H \cdot (ev_1^*Y_2 + 5\psi) \cdot (ev_1^*Y_2 + 6\psi)ev_1^*X[M_3(\mathbb{P}^5, 0)]^{\text{virt}}) = \\ &= \deg\left(\frac{2}{81} \cdot H \cdot 3H \cdot 3H \cdot X\right) = \frac{2}{81} \cdot \deg(H \cdot 3H \cdot 3H \cdot 9H^2) = 2.\end{aligned}$$

In the same way, we have $(B) = 46656$, $(C) = 93312$, $(D) = \frac{4}{3} \cdot 1053$,



(In the picture we have symbolically represented Y_1 by a 3-dimensional space and X by a plane. The numbers written next to each component lying in Y are the multiplicities at X. The marked point on a node is in fact a degree-0 component with 3 marked points mapping to X.)

$(E) = \frac{1}{3} \cdot 1053$, that lead to a Gromov-Witten invariant

$$\langle H \rangle_2^X = \frac{1916541}{4} - (A) - (B) - (C) - (D) - (E) = \frac{423549}{4}.$$

Now, we apply a more general formula (see [13]) that relates $n_{0,d} := \langle\ \rangle_d^X$, the Gromov-Witten invariant with no marked points of the Calabi-Yau manifold X, and the number $N_{0,d}$ of degree-d curves lying in X, namely,

$$n_{0,d} = \sum_{k|d} \frac{N_{0,d/k}}{k^3}.$$

The number we get in this way is $N_{0,2} = \frac{423549}{8} - \frac{1053}{8} = 52812$.

Example 3.3.2. With the notation above, let us look now at an example with $H_2(Y) \neq \mathbb{Q}$. Let Y be the quadric surface in $\mathbb{P}^3$, X the twisted cubic (that defines a very ample divisor on Y) and let us compute the number of lines in Y that are tangent to X at one point. This is one of the easiest examples where one can see how to compute individual invariants and how to reduce unrestricted invariants to restricted ones, so let us do the computations in detail. More precisely, $A^1(Y) \backsimeq \mathbb{Q}L_1 \oplus \mathbb{Q}L_2$ where we denoted by L_1, L_2

the classes of the two lines generating $A^1(Y)$. By [7], $K_Y = -2L_1 - 2L_2$, hence $\dim\,[M_1(Y,L_1)]^{\mathrm{virt}} = \dim\,[M_1(Y,L_2)]^{\mathrm{virt}} = 2$. This yields

$$\deg([M_2^X(Y,1)]^{\mathrm{virt}}) = \deg([M_2^X(Y,L_1)]^{\mathrm{virt}}) + \deg([M_2^X(Y,L_2)]^{\mathrm{virt}}). \tag{12}$$

Let us compute $I := \deg([M_2^X(Y,L_1)]^{\mathrm{virt}})$, the first term of (12) without correction terms, noting that $[X] = 2L_1 + L_2$:

$$\begin{aligned} I &= \deg(\widetilde{ev}_1^* X(\widetilde{ev}_1^* X + \psi)[M_1(Y,L_1)]^{\mathrm{virt}}) \\ &= \deg(\widetilde{ev}_2^* L_2 \widetilde{ev}_1^* X(\widetilde{ev}_1^* X + \psi)[M_2(Y,1)]^{\mathrm{virt}}) \\ &= \deg(\widetilde{ev}_2^* L_2(\widetilde{ev}_1^* 4L_1 \cdot L_2 + \widetilde{ev}_1^*(2L_1 + L_2)\psi)[M_2(Y,1)]^{\mathrm{virt}}) \\ &= \langle (4L_1 \cdot L_2)L_2\rangle_1^Y + \langle \tau_1(2L_1 + L_2)L_2\rangle_1^Y \end{aligned}$$

We now apply the main theorem starting with $M_2(\mathbb{P}^3, 1)$ and raising the multiplicities m_1, m_2 at both marked points until $m_1 + m_2 = 3$. We will get $[M_2(Y,1)]^{\mathrm{virt}}$ without correction terms, namely,

$$[M_2(Y,1)]^{\mathrm{virt}} = ev_1^* Y(ev_1^* Y + \psi) ev_2^* Y [M_2(\mathbb{P}^3,1)]^{\mathrm{virt}}.$$

Altogether,

$$\begin{aligned} I &= \deg((\widetilde{ev}_1^* 2i^* H^2 + \widetilde{ev}_1^*(2L_1 + L_2)\psi)\widetilde{ev}_2^* L_2 ev_1^* Y(ev_1^* Y + \psi) ev_2^* Y[M_2(\mathbb{P}^3,1)]^{\mathrm{virt}}) \\ &= \deg((ev_1^* 4H^3 + ev_1^* 3H^2\psi)(ev_1^* Y + \psi) ev_2^* H^2 [M_2(\mathbb{P}^3,1)]^{\mathrm{virt}}) \\ &= 10\langle \tau_1(H^3)H^2\rangle_1^{\mathbb{P}^3} + 3\langle \tau_2(H^2)H^2\rangle_1^{\mathbb{P}^3} = 4. \end{aligned}$$

In the same way, we see that

$$J := \deg([M_2^X(Y,L_2)]^{\mathrm{virt}} = \langle (4L_1 \cdot L_2)L_1\rangle_1^Y + \langle \tau_1(2L_1 + L_2)L_1\rangle_1^Y = I$$

and this yields $\deg([M_2^X(Y,1)]^{\mathrm{virt}}) = 2I = 8$.

Going further with this example, we will compute the number of conics in Y intersecting X with multiplicity 4. We have $\deg\,[M_1(Y,2L_1)]^{\mathrm{virt}} = \dim\,[M_1(Y,2L_2)]^{\mathrm{virt}} = \dim\,[M_1(Y,L_1+L_2)]^{\mathrm{virt}} = 4$. As before, the left hand side of the theorem yields

$$(L) := \prod_{i=0}^{3}(\widetilde{ev}_1^* X + i\psi)\prod_{j=0}^{4}(ev_1^* Y + j\psi)\cdot [M_1(\mathbb{P}^3,2)]^{\mathrm{virt}} = \frac{33}{2}. \tag{13}$$

We now look at the correction terms appearing at both steps. We see that $M_1(Y,2)$ is obtained without correction terms by raising the multiplicity at Y from 0 to 5. Applying the main theorem for $M_1(Y,2)$, we get two correction terms (A), (B) with a contracted component and two external components, each intersecting X with multiplicity 1, and respectively 1 and 2. Even if X is not a hyperplane section of Y, we have only degree-0 internal components, and thus it is not needed to compute individual Gromov-Witten invariants. Therefore, from the correction term we have the invariant

$$(A) := \frac{1}{2}\cdot\frac{1}{9}\langle\widetilde{X}\cdot\widetilde{i^*H}\rangle_1^Y\langle 1i^*X1\rangle_0^X\langle\widetilde{X}\cdot\widetilde{i^*H}\rangle_1^Y = \frac{1}{2}\langle\widetilde{L_1\cdot L_2}\rangle_1^Y\langle 1i^*X1\rangle_0^X\langle\widetilde{L_1\cdot L_2}\rangle_1^Y.$$

The internal component gives

$$\begin{aligned}\langle 1i^*X1\rangle_0^X &= \deg(ev_X^* i^*X[M_3(X,0)]^{\mathrm{virt}}) = \deg(ev_Y^*Xev_Y^*X[M_3(Y,0)]^{\mathrm{virt}})\\ &= \deg(ev_Y^*4L_1\cdot L_2[M_3(Y,0)]^{\mathrm{virt}}) = 4,\end{aligned}$$

hence the first correction term is

$$(A) = \frac{1}{2}\cdot\langle\widetilde{L_1\cdot L_2}\rangle_1^Y\cdot 4\cdot\langle\widetilde{L_1\cdot L_2}\rangle_1^Y = 4\cdot 2 = 8. \tag{14}$$

In the same way, the external component of (B) intersecting X with multiplicity 2 is $E := \widetilde{ev}_1X(\widetilde{ev}_1^*X+\psi)\prod_{j=0}^2(ev_1^*Y+j\psi)\cdot[M_1(\mathbb{P}^3,1)]^{\mathrm{virt}}$. This gives the term (B) as

$$(B) := 2\cdot E\cdot\langle H11\rangle_0^X\cdot\langle\widetilde{L_1\cdot L_2}\rangle_1^Y = 2\cdot 2\cdot 1\cdot 2 = 8.$$

Then by (13) and (14) we have the desired invariant $(L)-(A)-(B) = \frac{1}{2}$. As in the previous example, the invariant is non-enumerative because of multiple covers.

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