JUMP DIFFUSION OPTIONS WITH TRANSACTION COSTS

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We develop a method for pricing long and short positions in European options modeled by jump diffusion process (where the jump component of the stock return represents "non-systematic" risk) inclusive of transaction costs. We compute the total transaction costs and the turnover for different option types, transaction cost regimes, and revision interval lengths.

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1. **INTRODUCTION**

The classical option pricing theory developed by Black and Scholes assumes perfect markets. It relays on the arbitrage argument by which investors can use a replicating portfolio consisting of (in the case of a call option) a long position in the risky asset and a short position in bonds to exactly reproduce the return structure of the option. But this portfolio must be continuously adjusted, meaning that the weights on the portfolio must be continuously changed in order to eliminate all the risk from the total position (short a call option, long in risky asset and short in bonds).

There are two problems with this model. First, it assumes a perfect market, and therefore that the rebalancing is costless; if we consider transaction costs, the constant rebalancing used in the Black–Scholes setup will be infinitely costly (no matter how small the transaction costs are) since the diffusion processes have infinite variation. Second, for the arbitrage theory to be true, Black and Scholes assumed that the underlying stock price obeys a stochastic process with continuous paths. In many cases (jump-diffusion, Markovian diffusion, stochastic volatility-stochastic interest rate) we cannot apply their technique.

A few researchers have addressed the first problem (see, for example, Leland [7], Merton [9], and Boyle and Vorst [2]), providing replicating strategies that provide finite prices in the case when the costs are proportional to the amount of the transactions.

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Other researchers have addressed the second problem by proposing models for the price of the underlying security different from (or more general than) Brownian motion. In particular, Merton [9] introduced the so-called *jump diffusion* processes, which include random jumps in price occurring at random intervals.

In this paper we study the combination of both approaches, developing a scheme for pricing long and short positions in European options modeled by jump diffusion process (where the jump component of the stock return represents "non-systematic" risk) inclusive of transaction costs.

The paper is organized as follows. Section 2 reviews Merton's results about option pricing when the returns follow a jump diffusion model, without transaction costs. Section 3 presents the option pricing scheme using a jump diffusion model inclusive of transaction costs. Section 4 presents some empirical results about the total transaction costs and turnover, Section 5 includes some remarks regarding the hedging errors while Section 6 presents the conclusions of the paper.

2. OPTION PRICING WHEN THE UNDERLYING STOCK PRICE RETURNS ARE DISCONTINUOUS: A REVIEW

Merton [9] studied option pricing in the case that the stock price of the underlying asset can be described as a jump diffusion process. The underlying stock price returns are a mixture of continuous time processes and a Poisson process; it can be described by

$$\frac{\mathrm{d}S}{S} = (\alpha - \lambda k)\mathrm{d}t + \sigma \mathrm{d}Z + \mathrm{d}q, \tag{1}$$

where α and σ^2 are the instantaneous expected return on the stock respectively the instantaneous variance of the returns, conditional on no arrivals of important new information (i.e., the Poisson event does not occur); $q(t)$ is an independent Poisson process; q and Z are also assumed to be independent; λ is the mean number of arrivals per unit of time, $k \equiv E_y(Y-1)$, where $Y-1$ is the random percentage change in the stock price if the Poisson event occurs; E_y is the expectation operator over the random variable Y.

Equation (1) can be written in a more explicit form as

$$\frac{\mathrm{d}S}{S} = (\alpha - \lambda k)\mathrm{d}t + \sigma \mathrm{d}Z,$$

if the Poisson event does not occur, and

$$\frac{\mathrm{d}S}{S} = (\alpha - \lambda k)\mathrm{d}t + \sigma \mathrm{d}Z + Y - 1$$

if the Poisson event does occur.

Merton assumed that the jump represents pure non-systematic risk, and proved that the differential equation that is verified by the option price, when the dynamics of the stock price is as given above, is

$$\frac{1}{2}\sigma^2 S^2 C_{SS} + (r - \lambda k)SC_S + C_\tau - rC + \lambda E(C(SY,\tau) - C(S,\tau)) = 0 \tag{2}$$

subject to the boundary conditions

$$C(0,\tau) = 0, \quad C(S,0) = \max[0, S - P],$$

where $C(S,\tau)$ is the option price written as a function of the time until expiration and S, P is the exercise price, r is the risk free rate, and the subscripts represent partial derivatives with respect to the corresponding variable.

It is important to note that even though the jumps represent non-systematic risk, the jump component does affect the equilibrium option price.

Define $W(S,\tau;P,\sigma^2,r)$ to be the Black–Scholes formula for the no-jump (continuous) case. Then

$$C(S,\tau) = \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\lambda\tau}(\lambda\tau)^n}{n!} E_n \left[W(SX_n \mathrm{e}^{-\lambda k Z}, \tau; P, r) \right], \tag{3}$$

where $X_0 \equiv 1$, X_n is a random variable which has the same distribution as the product of n independent random variables distributed identically to Y defined in (1) and E_n is the expectation operator over the random variable X_n.

3. OPTION PRICING WITH POSITIVE TRANSACTION COSTS FOR JUMP DIFFUSION PROCESSES

Consider a portfolio formed with one option and $-N$ shares of stock. The assumptions of our model are as follows.

- The portfolio is revised every Δt units of time, where Δt is a non-infinitesimal, fixed time-step.
- The stock price S obeys the equation

  $$\mathrm{d}S = (\alpha - \lambda k)S\,\mathrm{d}t + \sigma S Z \sqrt{\mathrm{d}t} + S\,\mathrm{d}q,$$

  where α, λ, k, σ, q are as in equation (1).
- Transaction costs for buying or selling the asset are proportional to the value of the transaction. For example, if ν shares are bought or sold at price S, then the transaction costs are $\rho|\nu|S$, where ρ is a constant depending on the individual investor.
- The source of the jump is a firm- (or even industry-) specific information, hence the jump component will represent “non-systematic” risk, i.e. the jump component is uncorrelated with the market.

THEOREM 1. *A long position in an European option on jump diffusion inclusive of transaction costs can be priced using Merton's formula with a modified variance*

$$\hat{\sigma}^2 = \sigma^2 + \frac{2}{\Delta t}\rho E\left(\left|\frac{\Delta S}{S}\right|\right).$$

Proof. Since S obeys equation (1), the option price C will satisfy an equation of the same type (when there are jumps in S there are jumps in C), but with different parameters:

$$\frac{\mathrm{d}C}{C} = (\alpha_C - \lambda k_C)\mathrm{d}t + \sigma_C\,\mathrm{d}Z + \mathrm{d}q_C, \tag{4}$$

where α_C, k_C, and σ_C are defined as in (1) but for the option price C, i.e. α_C is the instantaneous expected return on the option; σ_C^2 is the instantaneous variance of the return, conditional on the Poisson event not occurring); $q_C(t)$ is the independent Poisson process with parameter λ, $k_C \equiv E(Y_C - 1)$, where $Y_C - 1$ is the random percentage change in the option price if the Poisson event occurs.

First, observe that if the percentage increase in S is Y, then the percentage increase in C is $Y_C = \frac{C(SY,t)}{C(S,t)}$. Hence

$$k_C = E_{Y_C}\left[\frac{C(SY,t) - C(S,t)}{C(S,t)}\right] = \frac{E_Y[C(SY,t) - C(S,\tau)]}{C(S,t)}. \tag{5}$$

Then looking at C as a function $C(S,\tau)$ of S and time to expiration τ, since the Poisson process has bounded variation, we can apply Ito's formula and obtain

$$\mathrm{d}C = C_\tau \mathrm{d}\tau + C_S \mathrm{d}S + \frac{1}{2}C_{SS}(\mathrm{d}S)^2. \tag{6}$$

Substituting (5) and (6) into (4) yields

$$\alpha_C C = C_t + (\alpha - \lambda k)SC_S + \frac{1}{2}S^2 C_{SS}\sigma^2 + \lambda E_Y[C(SY,t) - C(S,\tau)] \tag{7}$$

and

$$\sigma_C C = SC_S\sigma.$$

Return now to our hedge portfolio. The only source of uncertainty in the return is the jump component of the stock. By hypothesis, such components represent non-systematic risk, hence they are independent of the market, hence the "beta" of the portfolio is 0. Therefore, the expected return on the portfolio is equal to the risk free rate of return. If α_C is the rate of return on the option, over the small interval of time Δt we have

$$(\alpha_C C)\Delta t - (N\alpha S)\Delta t = r(C - NS)\Delta t.$$

If we introduce transaction costs, then we have

$$(8)\qquad (\alpha_C C)\Delta t - [(N\alpha S)\Delta t - E(\rho S|\nu|)] = r(C - NS)\Delta t.$$

Substituting (7) into (8) yields

$$(9)\qquad (C_\tau + (\alpha - \lambda k)SC_S + \frac{1}{2}S^2 C_{SS}\sigma^2 + \lambda E_Y[C(SY,t) - C(S,\tau)])\Delta t - \\ - (N\alpha S)\Delta t + E(\rho S|\nu|) = r(C - NS)\Delta t.$$

In the continuous case, Black and Scholes derive the number of shares of stock that will create a risk-less hedge. In the jump case, there is no such risk-less mix. However, we create a mix which eliminates all systematic risk, and in that sense, is a hedge. In Lemma 1 bellow we show that the number of shares for this hedge is equal to C_S. Observe that in both cases (the continuous and the jump case), the number N of shares is equal to the derivative of the option pricing function with respect to the stock price, but the formulas for the number of shares are different since the formulas for the option pricing are different.

Substituting $N = C_S$ into (9) and dividing by Δt yield

$$(10)\qquad C_t + (r - \lambda k)SC_S + \frac{1}{2}s^2 C_{SS}\sigma^2 + \lambda E_Y[C(SY,t) - C(S,\tau)] - rC + \\ + \frac{1}{\Delta t}E(\rho S|\nu|) = 0.$$

Let us look now at the term $\frac{1}{\Delta t}\rho S|\nu|$. We have

$$\nu = C_S(S + \Delta S, \tau + \Delta t) - C_S(S,\tau) = \\ = C_{SS}(S,\tau)\Delta S + C_{St}(S,\tau)\Delta t + \frac{1}{2}C_{SSS}(S,\tau)\Delta S^2 + C_{StS}(S,\tau)\Delta S\Delta t + \cdots .$$

The dominant term is $C_{SS}(S,\tau)\Delta S$. This way,

$$\nu = C_{SS}(S,\tau)\Delta S + O(\Delta t^{3/2}) \approx C_{SS}(S,\tau)\Delta S$$

and

$$(11)\qquad \frac{1}{\Delta t}\rho S|C_{SS}(S,\tau)\Delta S| = \frac{1}{\Delta t}\rho\left|C_{SS}(S,\tau)S^2\frac{\Delta S}{S}\right| = \frac{1}{\Delta t}\rho C_{SS}(S,\tau)S^2\left|\frac{\Delta S}{S}\right|.$$

To see that the term $C_{SS}(S,\tau)S^2$ is positive one should keep in mind that, as in Merton's case, the formula for $C(S,\tau)$ involves the sum of infinite Black Scholes formulas for which we know that the second derivative with respect to S is positive.

Substituting (11) into (10) we obtain the differential equation that C must satisfy, inclusive of transaction costs, namely,

$$\frac{1}{2}C_{SS}S^2\left(\sigma^2+\frac{2}{\Delta t}\rho E\left(\left|\frac{\Delta S}{S}\right|\right)\right)+C_t+(r-\lambda k)SC_S-rC+ \\ +\lambda E_Y[C(SY,\tau)-C(S,\tau)]=0 \tag{12}$$

with the boundary conditions

$$C(0,\tau)=0,\quad C(S,0)=\text{Max}[0,S-P].$$

With the notation

$$\hat{\sigma}^2=\sigma^2+\frac{2}{\Delta t}\rho E\left(\left|\frac{\Delta S}{S}\right|\right), \tag{13}$$

the equation for the value of the option is identical to Merton's value with the exception that the actual variance σ^2 is replaced by the modified variance $\hat{\sigma}^2$. □

Lemma 1. *With the above notation, the hedge strategy that eliminates all systematic risk is* $N=C_S$.

Proof. The value of our portfolio is $\Pi=C-NS$, hence $\Delta\Pi=\Delta C-N\Delta S$. If we substitute (6) in the equation for $\Delta\Pi$, we obtain

$$\Delta\Pi=\Delta t\left(\frac{1}{2}\sigma^2S^2C_{SS}+C_\tau\right)+S(\alpha-\lambda k)(C_S-N)\Delta t+ \\ +\sigma S(C_S-N)\Delta Z+S(C_S-N)\Delta q.$$

Now, observe that if we take $N=C_S$, then we eliminate the random component in the random walk, therefore our strategy must be $N=C_S$. □

If we do the same analysis for a short option, but we change all the signs with the exception of the transaction cost term, which must always be subtracted from the return on the portfolio, we obtain the following result.

Corollary 1. *A short position in an European option on jump diffusion inclusive of transaction costs can be priced using Merton's formula with modified variance*

$$\hat{\sigma}^2=\sigma^2-\frac{2}{\Delta t}\rho E\left(\left|\frac{\Delta S}{S}\right|\right). \tag{14}$$

Note that $\hat{\sigma}$ is influenced by the size of $\frac{1}{\Delta t}E\left(\left|\frac{\Delta S}{S}\right|\right)$. If it is very high, then the transaction costs term will overtake the initial variance. This implies that the costs of rebalancing are too high and that the chosen Δt is too small for the initial variance (in a short position one would obtain a negative variance if this term is too big!). The portfolio is rehedged too often. But if the term

$\frac{1}{\Delta t}E\left(\left|\frac{\Delta S}{S}\right|\right)$ is too small, then the transaction costs will not have a large effect on $\hat{\sigma}$. That happens when Δt is too large, hence when we do not rebalance the portfolio enough; Δt should be decreased in order to minimize the risk. A difficult question to answer seems to be how to choose Δt in order to have an optimal hedge.

4. ESTIMATING TURNOVER AND TRANSACTION COSTS OF REPLICATING STRATEGIES

We saw in the previous sections that the strategy $N = C_S$ with initial cost $C[S_0; P, r, \sigma, \lambda, k, T]$ eliminates the systematic risk when there are no transaction costs, and that the strategy $N = \hat{C}_S$ with initial cost $\hat{C}[S_0; P, r, \hat{\sigma}, \lambda, k, T]$ eliminates the systematic risk inclusive of transaction costs. Therefore, the difference $Z = |\hat{C}_0 - C_0|$ between the two initial option prices, where C_0 is option price exclusive of transaction costs, is a measure of the total transaction costs associated with the hedge strategy. Using an expension of Taylor type we can approximate Z by

$$Z = |\frac{\delta C}{\delta \sigma}(\sigma - \hat{\sigma}) + \cdots| \approx \frac{2\rho}{\sqrt{2\pi}} S_0 N'(d_1)\sqrt{T}\frac{1}{\Delta t}E\left(\left|\frac{\Delta S}{S}\right|\right).$$

The percentage change is given by

$$\text{Percent} = \frac{Z}{C_0} \times 100\%,$$

while the turnover estimates are given by

$$\text{Turnover} = \frac{\mathrm{Z}}{\rho \mathrm{S}_0}.$$

Note that since Z is given by the difference of two Merton formulas with different variances, it will be greatly influenced by the quantities influencing the variances, i.e., the level of the transaction costs ρ as well as the revision interval environment parameters $(r, \sigma^2, k, \lambda)$. Note again the importance of the size of the term $\frac{1}{\Delta t}E\left(\left|\frac{\Delta S}{S}\right|\right)$ in evaluating the size of the transaction costs.

For preparing the tables and charts below we used simulated data with $S_0 = 45.41$, and $T = 1$ (1 year to maturity), $r = 4.5\%$, $\sigma = 0.1717$, $\lambda = 0.1$, $k = 0.05$, $\delta^2 = .1$, $\alpha = .0037$. The reason we chose these numbers was because they model a real stock price of an asset traded on NASDAQ. But since we could not find a good method for extracting the volatility (in the underlying asset) which comes from the jumps (the δ) we decided to come up with a simulation model based on the total volatility.

As in the above formulas, $\hat{C}_0$ is the initial option price when the underlying stock price follows a jump diffusion, inclusive of transaction costs while C_0

is the option price when the underlying stock price follows a jump diffusion, ignoring the transaction costs.

Table 1

Option prices with transaction costs for a short option

P	(Δt)	ρ	C_0	$\hat{C_0}$	Z	%	Turnover
43.0000	0.0192	0.0025	5.8754	5.6677	0.2076	3.5336	1.8288
		0.0050		5.4468	0.4285	7.2936	1.8874
		0.0100		4.9599	0.9155	15.5819	2.0161
	0.0385	0.0025		5.7305	0.1449	2.4658	1.2762
		0.0050		5.5793	0.2961	5.0391	1.3040
		0.0100		5.2553	0.6200	10.5528	1.3654
	0.0577	0.0025		5.7583	0.1171	1.9932	1.0315
		0.0050		5.6371	0.2383	4.0557	1.0495
		0.0100		5.3811	0.4943	8.4132	1.0885
45.0000	0.0192	0.0025	4.6844	4.4455	0.2389	5.0992	2.1041
		0.0050		4.1846	0.4998	10.6703	2.2015
		0.0100		3.5598	1.1246	24.0064	2.4765
	0.0385	0.0025		4.5183	0.1661	3.5466	1.4634
		0.0050		4.3420	0.3424	7.3087	1.5079
		0.0100		3.9499	0.7345	15.6793	1.6174
	0.0577	0.0025		4.5503	0.1341	2.8628	1.1813
		0.0050		4.4098	0.2746	5.8626	1.2095
		0.0100		4.1050	0.5794	12.3681	1.2759
48.0000	0.0192	0.0025	3.2400	2.9833	0.2568	7.9247	2.2617
		0.0050		2.7002	0.5398	16.6594	2.3773
		0.0100		2.0036	1.2364	38.1607	2.7228
	0.0385	0.0025		3.0616	0.1784	5.5058	1.5714
		0.0050		2.8714	0.3686	11.3778	1.6236
		0.0100		2.4427	0.7973	24.6089	1.7558
	0.0577	0.0025		3.0961	0.1439	4.4421	1.2678
		0.0050		2.9446	0.2954	9.1163	1.3009
		0.0100		2.6133	0.6267	19.3418	1.3800

In Table 1 we present the option prices for short options whose underlying stock price obeys a jump diffusion Z, and the turnover for a variety of options, transaction costs, and revision period assumptions. We used short options since their behavior is a bit more interesting. If the transaction costs are too high, the transaction costs term will overtake the variance of the continuous component. As for the environment, we use the special case when the random variable Y has a log-normal distribution. If δ^2 is the variance of the logarithm of Y, then

$$C(S,\tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!} f_n(S,\tau),$$

where $\lambda' = \lambda(1+k)$ and $f_n(S,\tau)$ is the value option, conditional on knowing that exactly n Poisson jumps will occur during the life of the option. In our simulation, the time to maturity is one year, and we use three striking prices $P = 43,\ 45$ and 48.

In Table 2 we present the same computations as in Table 1 but for a long option.

Table 2

Option prices with transaction costs for a long option

P	(Δt)	ρ	C_0	$\hat{C}_0$	Z	%	Turnover
43.0000	0.0192	0.0025	5.8754	6.0715	0.1962	3.3385	1.7278
		0.0050		6.2578	0.3824	6.5087	1.6843
		0.0100		6.6055	0.7302	12.4280	1.6080
	0.0385	0.0025		6.0146	0.1392	2.3692	1.2262
		0.0050		6.1487	0.2733	4.6515	1.2037
		0.0100		6.4034	0.5280	8.9874	1.1628
	0.0577	0.0025		5.9887	0.1134	1.9296	0.9986
		0.0050		6.0987	0.2233	3.8008	0.9835
		0.0100		6.3093	0.4340	7.3861	0.9556
45.0000	0.0192	0.0025	4.6844	4.9060	0.2216	4.7311	1.9522
		0.0050		5.1137	0.4293	9.1645	1.8908
		0.0100		5.4961	0.8117	17.3275	1.7875
	0.0385	0.0025		4.8420	0.1576	3.3648	1.3884
		0.0050		4.9923	0.3079	6.5734	1.3562
		0.0100		5.2746	0.5902	12.5987	1.2997
	0.0577	0.0025		4.8129	0.1285	2.7432	1.1319
		0.0050		4.9365	0.2521	5.3808	1.1101
		0.0100		5.1708	0.4864	10.3830	1.0711
48.000	0.0192	0.0025	3.2400	3.4768	0.2368	7.3084	2.0858
		0.0050		3.6977	0.4577	14.1270	2.0159
		0.0100		4.1026	0.8626	26.6234	1.8996
	0.0385	0.0025		3.4085	0.1685	5.2016	1.4845
		0.0050		3.5687	0.3287	10.1451	1.4477
		0.0100		3.8683	0.6283	19.3924	1.3837
	0.0577	0.0025		3.3775	0.1374	4.2421	1.2107
		0.0050		3.5092	0.2692	8.3093	1.1857
		0.0100		3.7583	0.5183	15.9968	1.1414

Table 3 shows the difference between the option price in the Black-Scholes model (denoted by B) and jump diffusion model (denoted by C), with transaction costs (the notation will have a hat) and without transaction costs. In order to account for the influence of the variance on the option price, we consider the stock price variance in the Black Scholes model equal to the total variance, from the pure diffusion and from the jump of the stock price in the jump diffusion model, i.e., equal to $\sigma^2 + \lambda\delta^2$.

Table 3
The difference between the Black-Scholes model (B)
and the jump diffusion model (C)

λ	(Δt)	ρ	$B-C$	$\hat{B}-\hat{C}$	$\%price$	$\widehat{Spread}$	$\widetilde{Spread}$	$\%Spread$
0.0500	0.0192	0.0025	0.0766	0.0697	−9.0518	0.5079	0.5224	2.8453
		0.0030		0.0684	−10.7003	0.6105	0.6277	2.8261
		0.0050		0.0641	−16.3733	1.0269	1.0547	2.7065
		0.0100		0.0667	−12.9356	2.1644	2.2050	1.8754
	0.0385	0.0025		0.0712	−7.0535	0.3578	0.3688	3.0882
		0.0030		0.0702	−8.4056	0.4297	0.4429	3.0791
		0.0050		0.0663	−13.5211	0.7193	0.7411	3.0248
		0.0100		0.0595	−22.3436	1.4717	1.5118	2.7221
0.1000	0.0192	0.0025	0.1386	0.1249	−9.9353	0.4936	0.5220	5.7577
		0.0030		0.1223	−11.7837	0.5932	0.6271	5.7245
		0.0050		0.1132	−18.3817	0.9975	1.0525	5.5190
		0.0100		0.1111	−19.8950	2.0990	2.1858	4.1332
	0.0385	0.0025		0.1280	−7.6977	0.3469	0.3686	6.2500
		0.0030		0.1259	−9.1890	0.4166	0.4426	6.2344
		0.0050		0.1180	−14.9089	0.6973	0.7402	6.1410
		0.0100		0.1028	−25.8480	1.4256	1.5059	5.6269
0.1500	0.0192	0.0025	0.1894	0.1689	−10.8081	0.4797	0.5216	8.7347
		0.0030		0.1650	−12.8517	0.5764	0.6265	8.6911
		0.0050		0.1508	−20.3364	0.9691	1.0507	8.4221
		0.0100		0.1399	−26.1200	2.0360	2.1713	6.6477
	0.0385	0.0025		0.1736	−8.3353	0.3365	0.3684	9.4855
		0.0030		0.1705	−9.9635	0.4040	0.4423	9.4651
		0.0050		0.1585	−16.2716	0.6762	0.7394	9.3429
		0.0100		0.1341	−29.1853	1.3813	1.5012	8.6765
0.2000	0.0192	0.0025		0.2033	−11.7147	0.4663	0.5212	11.7733
		0.0030		0.1982	−13.9577	0.5603	0.6260	11.7219
		0.0050		0.1789	−22.3313	0.9418	1.0492	11.4051
		0.0100		0.1567	−31.9741	1.9752	2.1599	9.3517
	0.0385	0.0025		0.2096	−8.9998	0.3264	0.3681	12.7932
		0.0030		0.2055	−10.7691	0.3919	0.4420	12.7692
		0.0050		0.1896	−17.6781	0.6558	0.7387	12.6255
		0.0100		0.1554	−32.5176	1.3387	1.4973	11.8477

First, we observed that all the price percentage changes are negative, where

$$\%\text{price} = [\hat{C} - \hat{B} - (C - B)]/(C - B),$$

which means that the difference between the option prices is higher when the transaction costs are ignored than when they are not ignored. Then we observe a decrease in $\hat{B} - \hat{C}$ as well as an increase in the absolute value of the the percentage change, with the increase of transaction costs. That means that the transaction costs effect is higher in the jump diffusion case than in the Black-Scholes case. We observe that both $\hat{B} - \hat{C}$ and $B - C$ increase dramatically with λ, while the percentage change is decreasing with λ, which means that the transaction costs accentuates the influence of λ on the option prices. Table 3 also gives us the size of the spread of the option prices (the price of a long option minus the price of a short option) in both models Black-Scholes and the jump diffusion.

Now, we present several plots in which are described the behavior of the option prices with transaction costs in the jump diffusion case as well as the behavior of the total transaction costs-spread (the difference between the price of a long option and the price of a short option) in the same case, as well as the difference between the option prices in the jump diffusion case and in the Black-Scholes model, with the change of different parameters. In Figure 1 we plotted the price of a short option in the jump diffusion model as a function of the transaction costs ρ for different adjustment times Δt. We see

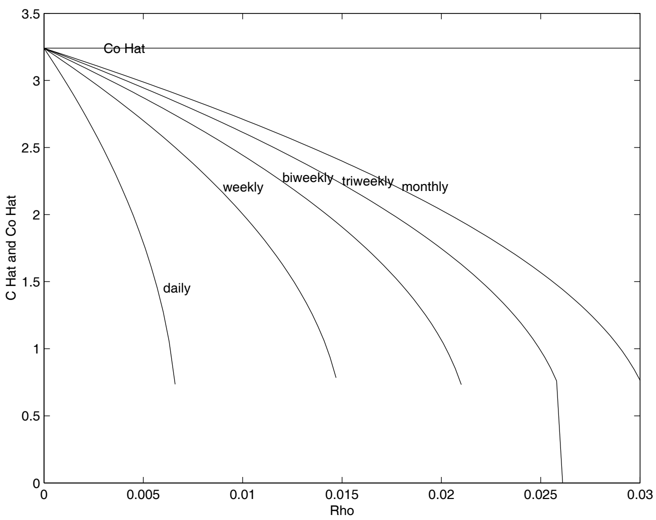

Fig. 1. $\hat{C}$ versus ρ for different (Δt). Co Hat stands for the option price with no transaction costs.

a rapid decrease in $\hat{C}$ with the increase in ρ. Moreover, we observe that as Δt decreases, the level of the transaction costs for which the option price cannot be computed (because $\hat{\sigma}$ would be negative) is decreasing. For example, if we readjust the portfolio monthly we can allow the transaction costs to be even larger than 3%, while if we readjust the portfolio daily the transaction costs cannot be higher than .6%. In Figure 2 we present the change in the spread of the option price, with the level of the transaction costs. In contrast with

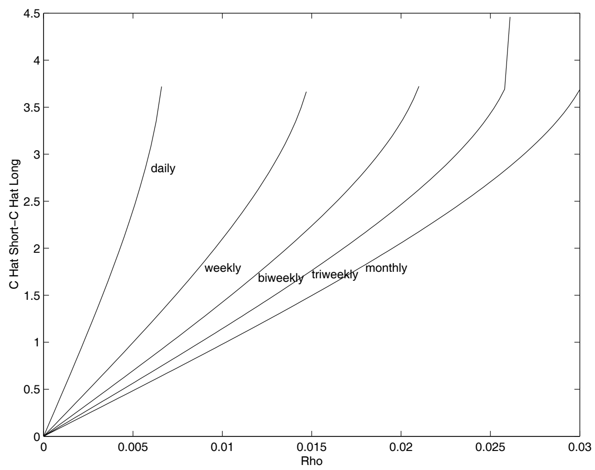

Fig. 2. The spread (difference between the price on a long position and the price in a short position) of an option on jump diffusion, versus ρ for different (Δt).

Figure 1, here the spread increase sharply with the transaction costs. The highest increase we see in the spread if the portfolio is adjusted daily.

In Figure 3 we kept all the parameters of the option constant and we vary the interval Δt of adjustment for the replicating portfolio, and we observe an increase in the option price and a decrease in spread as Δt increases.

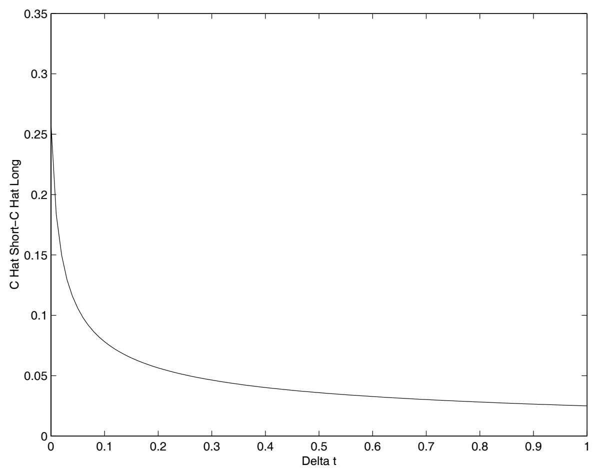

Fig. 3. The spread (difference between the price on a long position and the price in a short position) of an option on jump diffusion, versus Δt.

Figure 4 shows a decrease with ρ in the difference between the option prices in the two models, but an increase with Δt. We also observe that there is a value for ρ which, if reached, the differences increase dramatically and are not monotone anymore.

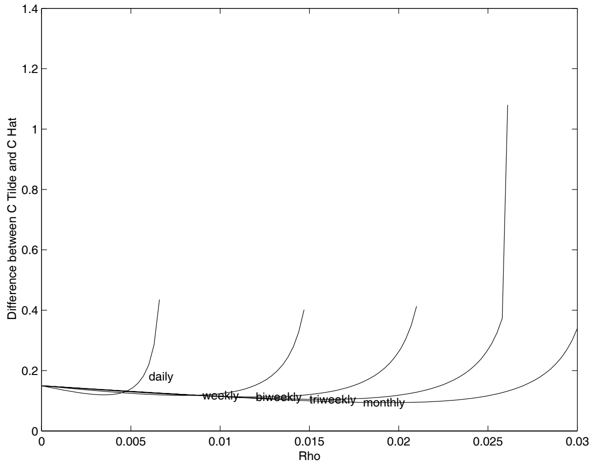

Fig. 4. The difference between $\hat{C}$ and $\tilde{C}$ versus ρ, for different Δt.Hat stands for the jump diffusion model while tilde for the Black-Scholes model. The subscript o stands for no transaction costs.

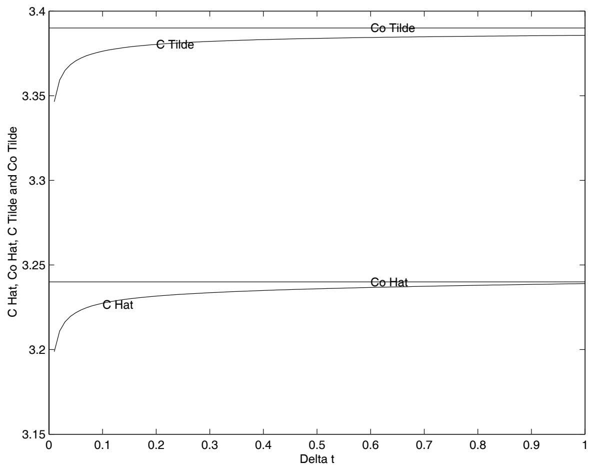

Fig. 5. $\hat{C}$ and $\tilde{C}$ versus Δt. Hat stands for the jump diffusion model while tilde for the Black-Scholes model. The subscript o stands for no transaction costs.

In Figure 5 we plotted $\hat{C}$ and $\tilde{C}$ against the adjustment interval Δt for the replicating portfolio, and observe that $\hat{C}$ converges to $\hat{C}_0$ faster than $\tilde{C}$ converges to $\tilde{C}_0$ as Δt increases, which means that the transaction costs have a stronger effect on the option prices in the jump diffusion model.

5. HEDGING ERRORS

We prove that errors in hedging, after the transaction costs, are in expectation of order $O((\Delta t)^{1/2})$, meaning that for very small Δt they could be ignored.

THEOREM 2. *Let $\hat{C}$ be Merton's price using the modified variance $\hat{\sigma}^2$. The replicating strategy $N = \hat{C}_S$ and $M = \hat{C} - \hat{C}_S S$ will lead, in expectation, to errors in hedging of order $O((\Delta t)^{1/2})$, inclusive of transaction costs.*

Proof. First we compute the error in hedging ΔE over the interval Δt:

$$\Delta E = \Delta\hat{C} - \Delta P - TC,$$

where

$$\Delta P = N\Delta S + Mr\Delta t + O((\Delta t)^2),$$

is the change in the portfolio value. The transaction costs are

$$TC = \rho|\Delta N(S + \Delta S)|,$$

and the change in the option price is

$$\Delta\hat{C} = \hat{C}(S + \Delta S, \tau + \Delta t) - \hat{C}(S, \tau).$$

Substituting $N = \hat{C}_S$ and $M = \hat{C} - \hat{C}_S S$ we obtain

$$\Delta P = \hat{C}_S S \left(\frac{\Delta S}{S}\right) + (\hat{C} - \hat{C}_S S)r\Delta t + O((\Delta t)^2),$$

and also

$$\begin{aligned} TC &= \frac{1}{2}\rho|(\hat{C}_S(S + \Delta S, \tau + \Delta t) - \hat{C}_S(S, \tau))(S + \Delta S)| = \\ &= \frac{1}{2}\rho\hat{C}_{SS}S^2\left|\frac{\Delta S}{S}\right| + O((\Delta t)^{3/2}). \end{aligned}$$

We obtained the last equality by applying (11) and the fact that $\hat{C}_{SS}S^2$ as well as $\hat{C}_{SSs}$, $\hat{C}_{St}$ and ΔS are $O((\Delta t)^{1/2})$ (see Lemma 3). Also, using the Taylor expansion we obtain

$$\Delta\hat{C} = \hat{C}_S S \left(\frac{\Delta S}{S}\right) + \frac{1}{2}\hat{C}_{SS}S^2\left(\frac{\Delta S}{S}\right)^2 + \hat{C}_t + O((\Delta t)^{3/2}).$$

Substituting the last three equations into the expression of ΔE and using equation (12) yield

$$\frac{1}{2}\hat{C}_{SS}S^2\left(\frac{\Delta S}{S}\right)^2 - (\hat{C} - \hat{C}_S S)r\Delta t + \hat{C}_t\Delta t - \frac{1}{2}\rho\hat{C}_{SS}S^2\left|\frac{\Delta S}{S}\right| + \\ +O((\Delta t)^{3/2}) = \frac{1}{2}\hat{C}_{SS}S^2\left[\sigma^2\Delta t - \left(\frac{\Delta S}{S}\right)^2\right] + \lambda k S C_S \Delta t - \\ - \lambda E_Y[C(SY,\tau) - C(S,\tau)] + O((\Delta t)^{3/2}).$$

Taking the expectation and observing that $E[(\frac{\Delta S}{S})^2]$ is $O(\Delta t)$ and that $E\big(\lambda kSC_S\Delta t - \lambda E_Y[C(SY,\tau) - C(S,\tau)]\big) = 0$, we obtain $E(\Delta E) = O(\Delta^{3/2})$. Since the errors in hedging are not correlated over intervals of time we conclude that the edging error over the whole period of time is $O((\Delta t)^{1/2})$.

We also observe that $\mathrm{var}(\Delta E) = O((\Delta t)^2)$ and, similarly, we conclude that the total variance is $O(\Delta t)$. □

Lemma 2. *The terms $\hat{C}_{SS}S^2$, $\hat{C}_{SSS}$, and $\hat{C}_{St}$ are $O((\Delta t)^{1/2})$.*

Proof. We start with showing that $\hat{\sigma}^2 - \sigma^2$ is $O((\Delta t)^{-1/2})$. Indeed, computing $E\left(\left|\frac{\Delta S}{S}\right|\right)$ we obtain

$$\begin{aligned} E\left(\left|\frac{\Delta S}{S}\right|\right) &= \sum_{n=0}^{\infty} E\left(\left|(\alpha - \lambda k)\Delta t + \sigma\Delta Z + n\Delta t\right|\right) P(q(t) = n) \\ &= \sum_{n=0}^{\infty} E\left(\left|(\alpha - \lambda k)\Delta t + \sigma\Delta Z + n\Delta t)\right|\right) \frac{\mathrm{e}^{-\lambda\Delta t}(\lambda\Delta t)^n}{n!} \\ &= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\lambda\Delta t}(\lambda\Delta t)^n}{n!}\int_{-\infty}^{\infty}|\sigma\sqrt{\Delta t}x + (n - \lambda k)\Delta t|\mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x. \end{aligned}$$

Hence

$$\lim_{\Delta t\to 0}\left[\frac{1}{\Delta t}E\left(\left|\frac{\Delta S}{S}\right|\right)\cdot((\Delta t))^{\frac{1}{2}}\right] =$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{\Delta t\to 0}\frac{1}{\sqrt{\Delta t}}\left(\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\lambda\Delta t}(\lambda\Delta t)^n}{n!}\int_{-\infty}^{\infty}|\sigma\sqrt{\Delta t}x + (n-\lambda k)\Delta t|\mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x\right) =$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{\Delta t\to 0}\left(\frac{\mathrm{e}^{-\lambda\Delta t}}{\sqrt{\Delta t}}\int_{-\infty}^{\infty}|\sigma\sqrt{\Delta t}x - \lambda k\Delta t|\mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x + \right.$$

$$\left. + \frac{1}{\sqrt{\Delta t}}\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-\lambda\Delta t}(\lambda\Delta t)^n}{n!}\int_{-\infty}^{\infty}|\sigma\sqrt{\Delta t}x + (n-\lambda k)\Delta t|\mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x\right).$$

We observe that both terms of the limit above are finite. In fact,

$$\frac{1}{\sqrt{2\pi}}\lim_{\Delta t\to 0}\frac{\mathrm{e}^{-\lambda\Delta t}}{\sqrt{\Delta t}}\int_{-\infty}^{\infty}|\sigma\sqrt{\Delta t}x-\lambda k\Delta t|\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x=$$

$$=\frac{1}{\sqrt{2\pi}}\lim_{\Delta t\to 0}\int_{-\infty}^{\infty}|\sigma x-\lambda k\sqrt{\Delta t}|\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x=\frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|x|\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x<\infty.$$

On the other hand, we have

$$\frac{1}{\sqrt{2\pi}}\lim_{\Delta t\to 0}\frac{1}{\sqrt{\Delta t}}\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-\lambda\Delta t}(\lambda\Delta t)^n}{n!}\int_{-\infty}^{\infty}|\sigma\sqrt{\Delta t}x+(n-\lambda k)\Delta t|\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x=$$

$$=\frac{1}{\sqrt{2\pi}}\lim_{\Delta t\to 0}\mathrm{e}^{-\lambda\Delta t}\sum_{n=1}^{\infty}\frac{(\lambda\Delta t)^n}{n!}\int_{-\infty}^{\infty}|\sigma x+(n-\lambda k)\sqrt{\Delta t}|\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x.$$

Then we observe that

$$\lim_{\Delta t\to 0}\sum_{n=1}^{\infty}\frac{(\lambda\Delta t)^n}{n!}\int_{-\infty}^{\infty}|\sigma x+(n-\lambda k)\sqrt{\Delta t}|\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x$$

is bounded, hence the above limit equals 0 and we conclude that $\hat{\sigma}^2-\sigma^2$ is $O(\Delta t^{-1/2})$. To prove that the previous term is bounded, observe that

$$\lim_{\Delta t\to 0}\sum_{n=1}^{\infty}\frac{(\lambda\Delta t)^n}{n!}\int_{-\infty}^{\infty}|\sigma x+(n-\lambda k)\sqrt{\Delta t}|\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x\le$$

$$\le\lim_{\Delta t\to 0}\sum_{n=1}^{\infty}\frac{(\lambda\Delta t)^n}{n!}\int_{-\infty}^{\infty}\sigma|x|\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x+$$

$$+\lim_{\Delta t\to 0}\sum_{n=1}^{\infty}\frac{(\lambda\Delta t)^n}{n!}\int_{-\infty}^{\infty}\lambda k\sqrt{\Delta t}\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x+$$

$$+\lim_{\Delta t\to 0}\sum_{n=1}^{\infty}\frac{(\lambda\Delta t)^n}{n!}n\sqrt{\Delta t}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x,$$

and each of the three terms of the sum is bounded.

Next, observe that if we denote by $V_n=SX_n\mathrm{e}^{-\lambda k\tau}$, then we have

$$S^2\hat{C}_{SS}(S,\tau)=\sum_{n=0}^{\infty}P_n(\tau)E_n\left\{V_n^2W_{V_nV_n}(V_n,\tau;P,\hat{\sigma}^2,r)\right\}.$$

This is because C verifies the same SPDE as in Merton's model, but with the modified $\hat{\sigma}$ instead of σ. So, we replace σ by $\hat{\sigma}$ in (3) and obtain the above equation. But, as

$$V_n^2W_{V_nV_n}(V_n,\tau;P,\hat{\sigma}^2,r)=\frac{V_nN'(\hat{d}_1)}{\hat{\sigma}(T-\tau)^{1/2}}=\frac{V_n\mathrm{e}^{-\frac{1}{2}\hat{d}_1^2}}{\sqrt{2\pi}\hat{\sigma}(T-\tau)},$$

it is enough to prove that each of the $V_n^2 W_{V_n V_n}$ is $O((\Delta t)^{1/2})$ in order to conclude that $\hat{C}_{SS}S^2$ is $O((\Delta t)^{1/2})$. But

$$N'(\hat{d}_1) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-1/2\hat{d}_1^2},$$

and

$$\hat{d}_1 = \frac{\ln \frac{V_n}{P} + (r + 1/2\hat{\sigma}^2)(T - \tau)}{\hat{\sigma}(T - \tau)^{1/2}}.$$

Therefore $\hat{d}_1$ is $O(\hat{\sigma})$ hence $O((\Delta t)^{-1/2})$ and $N'(\hat{d}_1)$ is $O(\exp(-1/2(\Delta t)^{-1}))$. It follows that $\hat{C}_{SS}S^2$ is $O(\exp(-1/2(\Delta t)^{-1})(\Delta t)^{1/2})$, which is $O((\Delta t)^{1/2})$.

The proof of $\hat{C}_{SSS}$ and $\hat{C}_{St}$ being $O((\Delta t)^{1/2})$ is similar. □

6. CONCLUDING REMARKS

We developed a method for computing the price of an option when the underlying asset returns follow a jump-diffusion model, inclusive of transaction costs. The formula holds when the jump component of the model represents non-systematic risk. The strategy depends on the level of transaction costs, the revision interval of the replicating portfolio and all the parameters of the jump diffusion. The zero-transaction costs option price lies between the long option and short option price inclusive of transaction costs. Observe that our model is consistent with the other models. If ρ is 0 (no transaction costs), we get the same equation Merton obtained for pricing an option whose underlying stock price obeys a jump diffusion process. In the case of $\lambda = 0$, but $\rho \neq 0$ (no jumps, but the transaction costs are not ignored), we obtain the same equation Leland obtained when he priced an option, inclusive of transaction costs, since the infinite sum from the solution reduces to one term that does not depend on λ. When both λ and ρ are 0 (no jumps and no transaction costs), we obtain Black-Scholes formula. Moreover, we observed that keeping all the parameters constant, just by adding some jumps in the underlying stock price the effect of transaction costs is higher.

It would be interesting to see which one of the two models (the Black-Scholes or the jump diffusion), inclusive of transaction costs, is a better model. For that we would need to look at data on option prices over the years and see which model fits better the data.

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